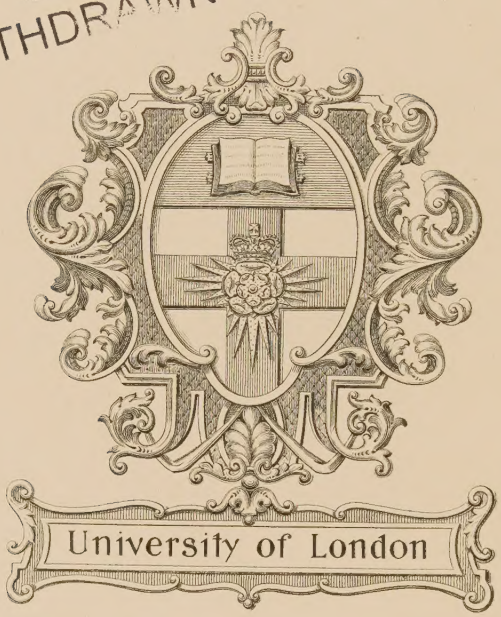
(RLL. PRIVATE Historical Society)
DEPOSITORY

SHL
WITHDRAWN



PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 44

LONDON:
PRINTED AND PUBLISHED FOR THE SOCIETY BY
C. F. HODGSON & SON, LTD.,
2 NEWTON STREET, KINGSWAY, W.C.2.

1938

BIBL.
LONDIN.
UNIV.

CONTENTS.

PAGE

Bhatnagar, S. P. A local property of the allied series of a Fourier series 315

Busbridge, I. W. Dual integral equations 115

Cherry, T. M. Analytic quasi-periodic curves of discontinuous type on a torus... 175

Davenport, H. On the product of three homogeneous linear forms (II) 412

Edge, W. L. Notes on a net of quadric surfaces : III. The scroll of trisecants of the Jacobian curve 466

Estermann, T. On Goldbach's theorem : proof that almost all even positive integers are sums of two primes 307

Hirsch, K. A. On infinite soluble groups (I) 53

———— On infinite soluble groups (II) 336

Hodge, W. V. D. An extension of Poincaré's theorem on defective integrals 216

———— Algebraic correspondences between surfaces 226

James, R. D. On a diophantine equation of the fourth degree ... 140

Kober, H. Involutorische Transformationen und selbstreziproke Funktionen 453

Korn, A. Ueber die Entwicklung der Lösung der Telegraphengleichung nach Bessel'schen Funktionen 352

Kuttner, B. The relation between De la Vallée Poussin and Abel summability 92

Levinson, N. General gap Tauberian theorems (I) 289

Love, E. R. Fractional integration and almost periodic functions... 363

Love, E. R., and Young, L. C. On fractional integration by parts 1

Milne, W. P. The bicircular quintic curve 130

———— A triad of quadrinodal cubic surfaces containing a quadricubic curve 345

Oppenheim, A. The continued fractions associated with chains of quadratic forms 323

Pitt, H. R. General Tauberian theorems 243

Rado, R. A theorem on general measure functions 61

Roth, L. Some threefolds of genera zero 36

Schiffer, M. A method of variation within the family of simple functions 432

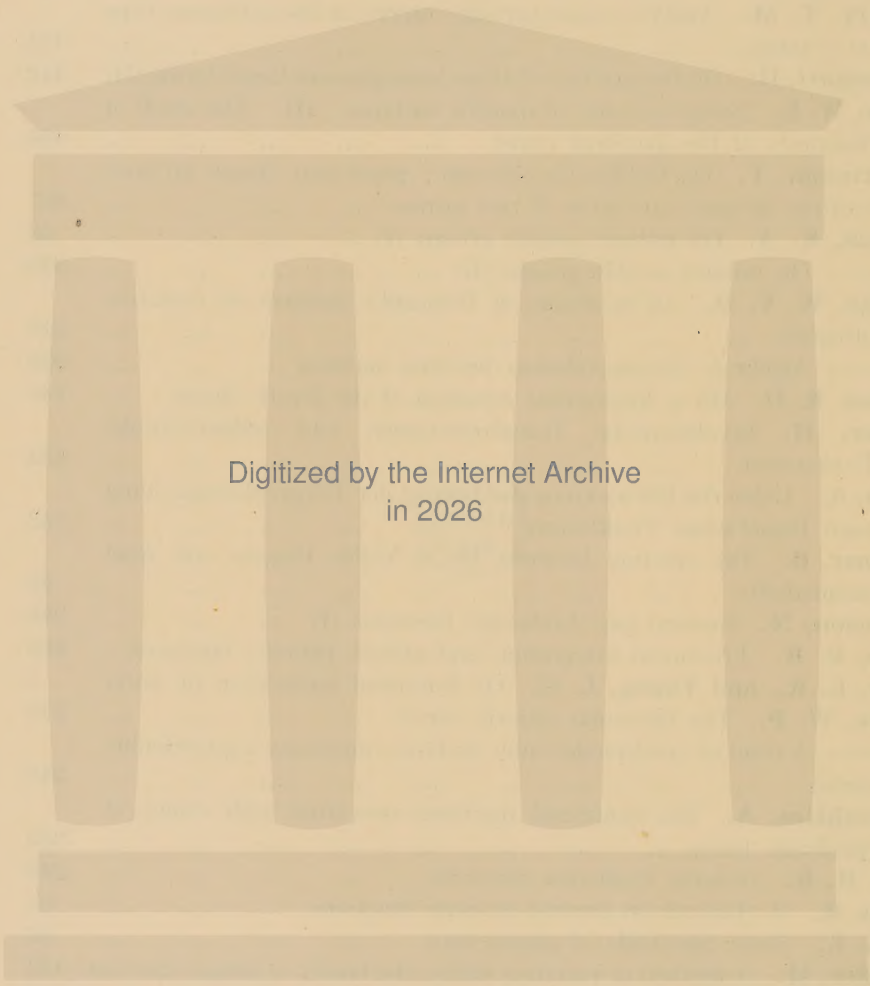
———— On the coefficients of simple functions 450

Semple, J. G. Singularities of space algebraic curves 149

Steen, S. W. P. An introduction to the theory of operators : III. Complex operators 398

Wolf, F. Approximation by trigonometrical polynomials and almost periodicity 100

Young, L. C., and Love, E. R. On fractional integration by parts 1



Digitized by the Internet Archive
in 2026

https://archive.org/details/london-mathematical-society-proceedings_1938_44

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON FRACTIONAL INTEGRATION BY PARTS

By E. R. Love *and* L. C. Young.

[Received 26 April, 1937.—Read 20 May, 1937.]

1. *Introduction.* The "fractional" generalization of integration by parts obtained in this note depends on fractional integrals and derivatives in the sense of Riemann-Liouville or in that of Weyl. It belongs to the order of ideas of our recent work on Stieltjes integrals and to that of a well-known inequality of Hardy and Littlewood [**2**, Theorem 3].

We establish the fractional formula in three principal cases: (i) for certain Lebesgue integrals, (ii) for certain Riemann-Stieltjes integrals, (iii) for generalized Stieltjes integrals of the type recently defined by means of binary derivation. The discussion of (i) [*vide* (C) below] is an easy deduction from results of Hardy and Littlewood; that of (ii) (*vide* Theorems 2 and 3 below) is based on a theorem for linear functionals (Theorem 1) and leads to various incidental results, thus: *a function f of the class* Lip (α, p), *where* $\alpha > 1/p$, *necessarily has bounded p-th power variation* (*vide* Theorems 4, 5, 6, and, further, Theorem 14; *cf.* also Young [**10**] for slightly more precise results). Finally, the discussion of (iii) is based on a simple special case (C_1) of both (i) and (ii), the general formula being an immediate consequence when the methods of binary derivation are used.

The interest of fractional integration by parts may be illustrated by the fact that, when combined with a preliminary change of variable, it enables us to transform a *non-absolutely convergent Riemann-Stieltjes integral* of a pair of continuous functions which have bounded p-th and q-th

power variation respectively, where $1/p+1/q>1$, into an *elementary integral* of a product of continuous functions. This brings us very close to the order of ideas of the elementary theory of non-absolutely convergent series, as conceived by Abel, Dirichlet, and others, the main difference being that partial summation has here been replaced by a fractional analogue.

2. *Notation.* We follow the Hardy-Littlewood definitions [**2**, 565–566] of the classes L, L^p, Lip k, Lip (k, p) of functions, and restrict our discussion to real variables and real functions†.

Given f of L, $\alpha>0$, and a finite interval (a, b), we define a *right-handed α-th integral with origin a* and a *left-handed α-th integral with origin b* (following Riemann and Liouville, Hardy and Littlewood) formally, in $a\leqslant x\leqslant b$, as follows:

$$f_\alpha^+(a, x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t)(x-t)^{\alpha-1}dt,$$

$$f_\alpha^-(x, b)=\frac{1}{\Gamma(\alpha)}\int_x^b f(t)(t-x)^{\alpha-1}dt.$$

When we are concerned with a fixed finite interval (a, b) we may use simply f_α^+ and f_α^- to denote the functions whose values are

$$f_\alpha^+(x)=f_\alpha^+(a, x),\quad f_\alpha^-(x)=f_\alpha^-(x, b).$$

To these we add the further definitions:

$$f_0^+(a, x)=f_0^+(x)=f(x),\quad f_0^-(x, b)=f_0^-(x)=f(x).$$

These fractional integrals have the following fundamental properties [**7**, 398, examples 23 and 25], to which we shall refer as A:

THEOREMS A. *If f is L and $\alpha>0$, then $f_\alpha^+(x)$ and $f_\alpha^-(x)$ exist for almost all x, and f_α^+ and f_α^- are L.*

If f is L, $\alpha>0$ and $\beta>0$, then

$$(f_\alpha^+)_\beta^+(x)=f_{\alpha+\beta}^+(x),\quad (f_\alpha^-)_\beta^-(x)=f_{\alpha+\beta}^-(x),$$

for all x for which the right-hand sides exist.

† The work remains valid, and the detail is substantially unchanged, if our functions are *complex* functions of a real variable, and the order of integration α is replaced by a complex number $\alpha+i\theta$ whose *real part* α satisfies the inequalities that occur in the definitions and the various hypotheses.

For derivatives of order α, where $0 \leqslant \alpha \leqslant 1$, we make the following formal definitions:

$$f^+_{-\alpha}(a, x) = \frac{d}{dx} f^+_{1-\alpha}(a, x), \quad f^-_{-\alpha}(x, b) = -\frac{d}{dx} f^-_{1-\alpha}(x, b).$$

Here the case $\alpha = 0$ is consistent with the definitions already made. We use abbreviations of notation for derivatives corresponding to those for integrals.

Weyl [8] defined α-th integrals with infinite origins, for periodic functions whose mean value over a period is zero. For such a function f these are defined formally as $f_\alpha{}^+(-\infty, x)$ and $f_\alpha{}^-(x, +\infty)$. Part of this paper is devoted to extending our theorems to apply to these integrals. In Hardy and Littlewood's work [2] these extensions are trivial; in fact Weyl's integral is easier to work with *ab initio* and more appropriate to their purposes. But in the principal part of the present discussion the most natural arguments depend on approximation by non-periodic step-functions and apply only to the Riemann-Liouville case.

We repeatedly use the following theorem of Hardy and Littlewood [2, Theorem 23], referring to it as B:

THEOREM B. *If*

$$p \geqslant 1, \quad 0 < \alpha < k \leqslant 1,$$

f is summable on a finite interval (a, b) and is zero everywhere outside it, and

$$\int_a^{b+h} |f(x)-f(x-h)|^p \, dx = O(h^{kp}) \quad (h > 0),$$

then $f^+_{-\alpha}(a, x)$ and $f^-_{-\alpha}(x, b)$ exist for almost all x and belong to L^p in (a, b); and f is equivalent to $(f^+_{-\alpha})_\alpha{}^+$ and to $(f^-_{-\alpha})_\alpha{}^-$. Further, if l.i.m. *denotes "limit in mean, with index p, over (a, b)", then*

$$f^+_{-\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \operatorname*{l.i.m.}_{\epsilon \to 0} \phi_\epsilon, \quad f^-_{-\alpha} = \frac{\alpha}{\Gamma(1-\alpha)} \operatorname*{l.i.m.}_{\epsilon \to 0} \psi_\epsilon,$$

where, for $\epsilon > 0$,

$$\phi_\epsilon(x) = \int_{-\infty}^{x-\epsilon} \big(f(x)-f(t)\big)(x-t)^{-\alpha-1} \, dt, \quad \psi_\epsilon(x) = \int_{x+\epsilon}^{\infty} \big(f(x)-f(t)\big)(t-x)^{-\alpha-1} \, dt.$$

Hardy and Littlewood's enunciation and proof of this theorem are for periodic functions f and Weyl integrals. But their proof uses periodicity only in order to ensure the convergence of the infinite integrals involved, and is applicable without change to prove B [Weyl integrals with origin $-\infty$ becoming integrals with origin a on account of our hypothesis $f(x)=0$ $(x<a)$]. In applying the proof to left-handed integrals, some care is needed in the five lines following on: "Hence $f_{1-\beta}$ is the integral of a g belonging to L^p"; we propose the following modification:

Since $f^-_{1-\beta}(x)$ vanishes at $x=b$,

$$f^-_{1-\beta}(x)=\int_b^x g(t)\,dt \quad (a\leqslant x\leqslant b),$$

so $f^-_{-\beta}(x)=-g(x)$ almost everywhere in (a, b). Also

$$f^-_{1-\beta}(x)=-\int_x^b g(t)\,dt=-g_1{}^-(x, b) \quad (a\leqslant x\leqslant b),$$

whence
$$f^-_{1-\beta}=-g_1{}^-=-(g_\beta{}^-)^-_{1-\beta}; \tag{A}$$

therefore f is equivalent to $-g_\beta{}^-$ and so to $(f^-_{-\beta})_\beta{}^-$.

(We have used α in place of Hardy and Littlewood's β. Further we have stated in our enunciation of B the explicit expressions for $f^+_{-\alpha}$ and $f^-_{-\alpha}$ which Hardy and Littlewood obtain in the course of their proof.)

In all matters concerning functions of bounded higher variation and Riemann-Stieltjes integrals of these, a general reference is made to **9**. We have, however, slightly altered the terminology: a function of the Wiener class W_p will now be said to be *of bounded p-th power variation*. In addition we use the classes

$$W_{p+0}=\prod_{q>p} W_q \ (p\geqslant 1), \quad W_{p-0}=\sum_{q<p} W_q \ (p>1).$$

Frequent use is made of the notation

$$p'=\frac{p}{p-1} \quad (p>1).$$

Occasionally we require $|X|_+=\max(X, 0)$, $|X|_-=-\min(X, 0)$. These "semi-moduli" of X are non-negative and satisfy

$$|X|_+-|X|_-=X, \quad |X|_++|X|_-=|X|.$$

We indicate that we have used the classical inequalities of Hölder and Minkowski (in any of their various forms) by quoting the name ("Hölder" or "Minkowski") at the appropriate place†.

3. *The Hardy-Littlewood form of fractional integration by parts.* The genesis of "fractional integration by parts" is the following deduction, which we call C, from a theorem of Hardy and Littlewood [**2**, Theorem 3]:

THEOREM C. *If*

$$p>1, \quad q>1, \quad 0<\alpha<1, \quad \frac{1}{p}+\frac{1}{q}-1\leqslant\alpha,$$

and $$f \text{ is } L^p, \quad g \text{ is } L^q,$$

then $$\int_a^b f_a{}^+ g\,dx=\int_a^b f g_b{}^-\,dx.$$

In proving this we may take $a=0$, and define $f(x)=0=g(x)$ $(x>b)$. Write

$$\mu=\left(\alpha+1-\frac{1}{p}-\frac{1}{q}\right)\Big/\left(\frac{1}{p'}+\frac{1}{q'}\right), \qquad \frac{1}{r}=\frac{1}{p}+\frac{\mu}{p'}, \quad \frac{1}{s}=\frac{1}{q}+\frac{\mu}{q'}.$$

Then $$0\leqslant\mu<1, \; p\geqslant r>1, \quad q\geqslant s>1;$$

so f is L^r and g is L^s, since the interval of integration is finite, and

$$\frac{1}{r}+\frac{1}{s}-1=\alpha.$$

Now $$\iint f(x)\,g(y)\,|x-y|^{\alpha-1}\,dx\,dy$$

is absolutely convergent over $(x\geqslant 0, \; y\geqslant 0)$ and therefore over $(0\leqslant x\leqslant y\leqslant b)$ [**2**, Theorem 3, or **5**, Theorem 383], whence

$$\int_a^b f_a{}^+(a,y)\,g(y)\,dy=\frac{1}{\Gamma(\alpha)}\int_0^b g(y)\,dy\int_0^y f(x)\,(y-x)^{\alpha-1}\,dx$$

$$=\frac{1}{\Gamma(\alpha)}\int_0^b f(x)\,dx\int_x^b g(y)(y-x)^{\alpha-1}\,dy=\int_a^b f(x)\,g_a{}^-(x,b)\,dx.$$

† It is easily seen that only trivial alterations are required in our notation to make it apply to the *complex* case (foot-note †, p. 2 above). The principal changes involved (other than those referred to at the time) are:

(i) The conclusions of Theorem 1 and 20 are not expressible in terms of lower and upper integrals, but in a form clearly indicated by the proofs.

(ii) Different constants in the lemmas.

(iii) Alteration of α to $\alpha+i\theta$ where α occurs as an exponent, but not where the modulus of an $(\alpha+i\theta)$-th power is taken.

(iv) In C_1, and throughout § 11, $0<\alpha<1$.

For later purposes we mention particularly the following special case of C, obtained by writing $g_\alpha^-(x) = -h(x)$ and by supposing that the function $h(x)$ has the form $-\int_x^b h'(t)\,dt$, where h' is L^μ:

THEOREM C_1. *If*

$$\lambda > 1, \quad \mu > 1, \quad 0 \leqslant \alpha \leqslant 1,$$

and f *is* L^λ, h' *is* L^μ,

$$h(x) = -\int_x^b h'(t)\,dt,$$

then

$$\int_a^b f_\alpha{}^+ h_{-\alpha}^-\,dx = \int_a^b f\,h\,dx.$$

Here we may suppose that $0 < \alpha < 1$; we write $\lambda = p$, and we then need verify only that the function $g = (h')_{1-\alpha}^- \equiv -h_{-\alpha}^-$ belongs to L^q for some $q > 1$ such that $(1/p)+(1/q)-1 < \alpha$.

If $1-\alpha < 1/\mu$, then g is L^q where $q = \mu\Big/\big(1-\mu(1-\alpha)\big)$ [**2**, Theorem 4]; also

$$\frac{1}{q} = \frac{1}{\mu} - (1-\alpha) < 1, \qquad \frac{1}{p} + \frac{1}{q} - 1 = \left(\frac{1}{\lambda} - 1\right) + \left(\frac{1}{\mu} - 1\right) + \alpha < \alpha.$$

Similarly, if $1-\alpha = 1/\mu$, then g is L^q for every $q > 1$.

If $1-\alpha > 1/\mu$, then g is continuous [**7**, 398, example 24].

4. *A theorem on linear functionals.* Before passing on to the other forms of fractional integration by parts, we prove a slight extension of our theorem [**6**, 248] expressing certain general linear functionals as Stieltjes integrals. For a further extension of this theorem, not required for our present purposes, see §12.

We need two lemmas:

LEMMA 1. *If*

$$p \geqslant 1, \quad a = c_0 < c_1 < \ldots c_{n-1} < c_n = b,$$

and

$$f(c_\nu) = 0 \quad (0 < \nu < n),$$

then

$$V_p(f;\ a, b)^p \leqslant 2^p \sum_{\nu=1}^{n} V_p(f;\ c_{\nu-1}, c_\nu)^p.$$

With the additional hypothesis $f(x) \geqslant 0$ $(a \leqslant x \leqslant b)$ it is easy to show that

$$V_p(f;\ a, b)^p \leqslant \sum_{\nu=1}^{n} V_p(f;\ c_{\nu-1}, c_\nu)^p.$$

Without any additional hypothesis,

$$\begin{aligned} V_p(f)^p &= V_p(|f|_+ - |f|_-)^p \\ &\leqslant \Big(V_p(|f|_+) + V_p(|f|_-)\Big)^p && [\mathbf{9},\ 260] \\ &\leqslant 2^{p-1}\Big(V_p(|f|_+)^p + V_p(|f|_-)^p\Big) && [\text{Hölder}] \\ &\leqslant 2^{p-1}\sum_{\nu=1}^{n}\Big(V_p(|f|_+;\ c_{\nu-1},\ c_\nu)^p + V_p(|f|_-;\ c_{\nu-1},\ c_\nu)^p\Big) \\ &\leqslant 2^{p-1}\sum_{\nu=1}^{n}\Big(V_p(f;\ c_{\nu-1},\ c_\nu)^p + V_p(f;\ c_{\nu-1},\ c_\nu)^p\Big). \end{aligned}$$

LEMMA 2. *If $q > p \geqslant 1$ and f is W_p, then, given $\epsilon > 0$, there is a subdivision, $a = x_0 < \xi_1 < x_1 < \xi_2 < x_2 < \ldots < x_{k-1} < \xi_k < x_k = b$, such that*

$$V_q(f-\phi) < \epsilon, \quad \max_{i=1}^{k}(x_i - x_{i-1}) < \epsilon,$$

where ϕ is the step-function defined as follows:

$$\phi(x) = f(\xi_i) \quad (x_{i-1} < x < x_i) \quad (0 < i \leqslant k),$$

$$\phi(x_i) = f(x_i) \qquad (0 \leqslant i \leqslant k).$$

Given $\epsilon > 0$, define $\lambda > 0$ by $2^{2q-p+2} V_p(f)^p \lambda^{q-p} = \epsilon^q$.

The set of discontinuities of f at which the oscillation of f is not less than λ is finite; therefore there is a subdivision, $a = y_0 < y_1 < \ldots < y_{h-1} < y_h = b$, such that this set of discontinuities is included in (y_j) and also

$$\max_{j=1}^{h}(y_j - y_{j-1}) < \epsilon.$$

For each j, the oscillation of f at any point interior to (y_{j-1}, y_j) is less than λ, and $f(y_{j-1}+0)$, $f(y_j-0)$ exist; therefore there is a subdivision of (y_{j-1}, y_j) in each open interval of which f has oscillation less than 2λ. Therefore there is a subdivision (which includes all points y_j),

$$a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b,$$

such that f has oscillation less than 2λ in each open interval (x_{i-1}, x_i) and

$$\max_{i=1}^{k}(x_i - x_{i-1}) < \epsilon.$$

Choose ξ_i such that $x_{i-1} < \xi_i < x_i$ $(0 < i \leqslant k)$ and define ϕ as in the enunciation. Then

$$V_q(f-\phi)^q \leqslant 2^q \sum_{i=1}^{k} V_q(f-\phi;\ x_{i-1}, x_i)^q \qquad \text{(Lemma 1)}$$

$$\leqslant 2^q \sum_{i=1}^{k} (2\lambda)^{q-p} V_p(f-\phi;\ x_{i-1}, x_i)^p, \qquad [9, 259]$$

since $f-\phi$ has oscillation less than 2λ in each *closed* interval (x_{i-1}, x_i). To estimate $V_p(f-\phi;\ x_{i-1}, x_i)^p$, write $f(x)-\phi(x) = \psi(x)$; given any subdivision $x_{i-1} = t_0 < t_1 < \ldots < t_{n-1} < t_n = x_i$ for which $n > 1$,

$$\sum_{\nu=1}^{n} |\psi(t_\nu)-\psi(t_{\nu-1})|^p = |\psi(t_1)|^p + \sum_{\nu=2}^{n-1} |\psi(t_\nu)-\psi(t_{\nu-1})|^p + |\psi(t_{n-1})|^p$$

$$= |f(t_1)-f(\xi_i)|^p + \sum_{\nu=2}^{n-1} |f(t_\nu)-f(t_{\nu-1})|^p + |f(t_{n-1})-f(\xi_i)|^p$$

$$\leqslant 3V_p(f;\ x_{i-1}, x_i)^p,$$

and so $$V_p(f-\phi;\ x_{i-1}, x_i)^p \leqslant 3V_p(f;\ x_{i-1}, x_i)^p.$$

Therefore

$$V_q(f-\phi)^q \leqslant 2^q(2\lambda)^{q-p} \sum_{i=1}^{k} 3V_p(f;\ x_{i-1}, x_i)^p$$

$$\leqslant 2^q(2\lambda)^{q-p}\, 3\, V_p(f;\ a, b)^p < \epsilon^q;$$

this completes the proof of the lemma.

We can now prove the theorem on linear functionals in the following form:

THEOREM 1. *If $q > p \geqslant 1$ and L is a functional defined in W_q satisfying*

$$L\left(\sum_{i=1}^{k} c_i f_i\right) = \sum_{i=1}^{k} c_i L(f_i) \qquad [c_i \text{ constants}, f_i \text{ in } W_q],$$

$$|L(f)| \leqslant M\left(|f(a)| + V_q(f;\ a, b)\right) \qquad [f \text{ in } W_q],$$

$$L(f) = 0 \qquad [\text{if } f(t) = 0 \text{ for all but one value of } t],$$

then there is a g of $W_{q'}$ such that, for all f of W_p,

$$\underline{\int} f\,dg \leqslant L(f) \leqslant \overline{\int} f\,dg.$$

Further $g(x) = -L(\bar{x})$, where $\bar{x}(t) = 0$ $(t \leqslant x)$ and $\bar{x}(t) = 1$ $(t > x)$.

Define g in the manner stated, and suppose that g is not $W_{q'}$. Then there is a subdivision $a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$ such that

$$\sum_{i=1}^{k} |g(x_i) - g(x_{i-1})|^{q'} > 2^{q'+1} M^{q'},$$

and therefore there is a sequence (c_i) such that

$$S_q(c_i) \leqslant 1, \quad \sum_{0<i\leqslant j\leqslant k} c_i \big(g(x_j) - g(x_{j-1})\big) > M \qquad [6, 249].$$

Putting $C_j = \sum_{i=1}^{j} c_i$, this gives

$$\begin{aligned} M &< \sum_{j=1}^{k} C_j \big(g(x_j) - g(x_{j-1})\big) \\ &= L\Big(\sum_{j=1}^{k} C_j(\bar{x}_{j-1} - \bar{x}_j)\Big) \\ &\leqslant M V_q\Big(\sum_{j=1}^{k} C_j(\bar{x}_{j-1} - \bar{x}_j)\Big) = M S_q(c_i) \leqslant M, \end{aligned}$$

a contradiction. Therefore g is $W_{q'}$.

Given f of W_p and $\epsilon > 0$, there is a step-function ϕ, as in Lemma 2:

$$\phi = \sum_{i=1}^{k} f(\xi_i)(\bar{x}_{i-1} - \bar{x}_i - \dot{x}_i) + \sum_{i=0}^{k} f(x_i)\dot{x}_i \dagger,$$

where $\dot{x}(t) = 1$ $(t = x)$ and $\dot{x}(t) = 0$ $(t \neq x)$, such that

$$V_q(f - \phi) < \epsilon/M, \quad \max_{i=1}^{k} (x_i - x_{i-1}) < \epsilon/M.$$

Since $L(\dot{x}) = 0$ for all x,

$$\begin{aligned} L(\phi) &= \sum_{i=1}^{k} f(\xi_i)\, L(\bar{x}_{i-1} - \bar{x}_i) \\ &= \sum_{i=1}^{k} f(\xi_i)\big(g(x_i) - g(x_{i-1})\big); \end{aligned}$$

therefore

$$\begin{aligned} \Big| L(f) - \sum_{i=1}^{k} f(\xi_i)\big(g(x_i) - g(x_{i-1})\big) \Big| &= |L(f - \phi)| \\ &\leqslant M\big(|f(a) - \phi(a)| + V_q(f - \phi)\big) \\ &= M V_q(f - \phi) < \epsilon. \end{aligned}$$

† A set of x_i different from that occurring earlier in the proof.

Taking a sequence of numbers $\epsilon \to 0$, we obtain in this way an everywhere dense sequence of subdivisions and an approximative sum to $\int f dg$ (in the Riemann-Stieltjes sense) on each subdivision such that the approximative sums tend to $L(f)$. Hence the conclusion.

We shall use the term "linear", of a functional L in a field C, to indicate the property

$$L\left(\sum_{i=1}^{k} c_i f_i\right) = \sum_{i=1}^{k} c_i L(f_i) \qquad (c_i \text{ constants}, f_i \text{ in } C).$$

5. *Fractional integration by parts for Stieltjes integrals.* We are now in a position to establish the second form of fractional integration by parts in terms of Riemann-Liouville fractional integrals and Riemann-Stieltjes integrals. Two lemmas are needed:

LEMMA 3. *If $1/q < \alpha < 1$ and g is L^q, then $g_\alpha^+(x)$ and $g_\alpha^-(x)$ exist and are continuous throughout $a \leqslant x \leqslant b$.*

Compare **7**, 398, example 24. The proof is elementary.

LEMMA 4. *Under the hypotheses of B,*

$$\left(\int_a^b |f_{-\alpha}^+(x)|^p dx\right)^{1/p} \leqslant \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha-1} \left(\int_a^b |f(x)-f(x-u)|^p dx\right)^{1/p} du,$$

$$\left(\int_a^b |f_{-\alpha}^-(x)|^p dx\right)^{1/p} \leqslant \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty u^{-\alpha-1} \left(\int_a^b |f(x)-f(x+u)|^p dx\right)^{1/p} du.$$

For, given $\epsilon > 0$,

$$\left(\int_a^b |\phi_\epsilon(x)|^p dx\right)^{1/p} \leqslant \left\{\int_a^b \left(\int_\epsilon^\infty |f(x)-f(x-u)|\, u^{-\alpha-1} du\right)^p dx\right\}^{1/p}$$

$$\leqslant \int_\epsilon^\infty \left(\int_a^b |f(x)-f(x-u)|^p u^{-(\alpha+1)p} dx\right)^{1/p} du$$

[Minkowski]

$$\leqslant \int_\epsilon^\infty u^{-\alpha-1} \left(\int_a^b |f(x)-f(x-u)|^p dx\right)^{1/p} du,$$

and our assertion regarding $f_{-\alpha}^+$ follows by mean convergence of ϕ_ϵ. Similarly for $f_{-\alpha}^-$.

Our theorem is:

THEOREM 2. *If $1/q < \alpha < 1$, f is $W_{q'}$ and g is L^q, then $g_\alpha^{\pm}$ are W_{q-0} and the following equations between Riemann-Stieltjes and Lebesgue integrals hold:*

$$\int_a^b f\,dg_\alpha^- = -\int_a^b f_{\alpha-1}^+ g\,dx, \qquad \int_a^b f\,dg_\alpha^+ = +\int_a^b f_{\alpha-1}^- g\,dx.$$

Choose r such that $1/q < 1/r < \alpha$. Then $r' > q'$, and g is L^r.

Given ϕ of $W_{r'}$, define $\phi(x) = 0$ $(x < a,\ x > b)$; then ϕ has bounded r'-th power variation on the whole real axis, and so

$$\int_a^{b+h} |\phi(x) - \phi(x-h)|^{r'}\,dx = O(h) \quad (h > 0) \qquad [\mathbf{9}, 259].$$

Further $0 < 1-\alpha < 1/r' < 1$. So $\phi_{\alpha-1}^+$ exists almost everywhere and is L^r in (a, b) (B). Therefore the Lebesgue integral

$$-\int_a^b \phi_{\alpha-1}^+(t)\,g(t)\,dt$$

exists; further, it is a linear functional of ϕ throughout $W_{r'}$, zero if ϕ is almost everywhere zero, and (as we shall verify in a moment) satisfies a "boundedness" condition

$$\left| \int_a^b \phi_{\alpha-1}^+ g\,dt \right| \leqslant M\Big(|\phi(a)| + V_{r'}(\phi)\Big),$$

in which M is independent of ϕ, throughout $W_{r'}$. Therefore, by Theorem 1, the function

$$\psi(x) = \int_a^b \bar{x}_{\alpha-1}^+(t)\,g(t)\,dt$$

is W_r; and we have, for every ϕ of $W_{q'}$,

$$\underline{\int_a^b} \phi\,d\psi \leqslant -\int_a^b \phi_{\alpha-1}^+ g\,dt \leqslant \overline{\int_a^b} \phi\,d\psi.$$

Now $\qquad \bar{x}(t) = 0 \quad (a \leqslant t \leqslant x), \qquad \bar{x}(t) = 1 \quad (x < t \leqslant b),$

$$\bar{x}_\alpha^+(t) = 0 \quad (t \leqslant x), \qquad \Gamma(\alpha)\,\bar{x}_\alpha^+(t) = \int_x^t (t-s)^{\alpha-1}\,ds = (t-x)^\alpha/\alpha \quad (t > x),$$

$$\bar{x}_{\alpha-1}^+(t) = 0 \quad (t < x), \qquad \Gamma(\alpha)\,\bar{x}_{\alpha-1}^+(t) = (t-x)^{\alpha-1} \quad (t > x);$$

therefore $\qquad \psi(x) = \dfrac{1}{\Gamma(\alpha)} \int_x^b g(t)\,(t-x)^{\alpha-1}\,dt = g_\alpha^-(x).$

So g_a^- is W_r, and therefore W_{q-0}. And, since f is $W_{q'}$,

$$\underline{\int}_a^b f\,dg_a^- \leqslant -\int_a^b f_{a-1}^+ g\,dt \leqslant \bar{\int}_a^b f\,dg_a^-.$$

But $1/q'+1/r>1$, and g_a^- is continuous (Lemma 3); therefore the first Riemann-Stieltjes integral of our conclusion exists. The second part of the conclusion is proved similarly.

To complete the proof, it remains to verify the boundedness condition. Given ϕ of $W_{r'}$, define† $\Phi(x)=V_{r'}(\phi;\ -\infty,\ x)^{r'}$; this satisfies

$$|\phi(x)-\phi(x-u)|^{r'} \leqslant \Phi(x)-\Phi(x-u) \quad (u>0),$$

so

$$\begin{aligned}\int_a^b |\phi(x)-\phi(x-u)|^{r'}\,dx &\leqslant \int_a^b \Phi(x)\,dx-\int_{a-u}^{b-u}\Phi(x)\,dx \quad (u>0)\\ &= \int_{b-u}^{b}\Phi(x)\,dx-\int_{a-u}^{a}\Phi(x)\,dx\\ &\leqslant u\Phi(b),\end{aligned}$$

while

$$\int_a^b |\phi(x)-\phi(x-u)|^{r'}\,dx = \int_a^b |\phi(x)|^{r'}\,dx \quad (u>b-a).$$

By Lemma 4,

$$\begin{aligned}&\frac{\Gamma(\alpha)}{1-\alpha}\left(\int_a^b |\phi_{a-1}^+(x)|^{r'}\,dx\right)^{1/r'}\\ &\leqslant \int_0^\infty u^{\alpha-2}\left(\int_a^b |\phi(x)-\phi(x-u)|^{r'}\,dx\right)^{1/r'}du\\ &\leqslant \int_0^{b-a} u^{\alpha-1-1/r}\Phi(b)^{1/r'}\,du+\int_{b-a}^\infty u^{\alpha-2}\left(\int_a^b |\phi(x)|^{r'}\,dx\right)^{1/r'}du\\ &= \frac{(b-a)^{\alpha-1/r}}{\alpha-1/r}\Phi(b)^{1/r'}+\frac{(b-a)^{\alpha-1}}{1-\alpha}\left(\int_a^b |\phi(x)|^{r'}\,dx\right)^{1/r'}.\end{aligned}$$

Now

$$\begin{aligned}|\phi(x)| &\leqslant |\phi(a)|+|\phi(x)-\phi(a)|\\ &\leqslant |\phi(a)|+V_{r'}(\phi;\ a,\ b) \quad (a\leqslant x\leqslant b);\end{aligned}$$

therefore

$$\left(\int_a^b |\phi(x)|^{r'}\,dx\right)^{1/r'} \leqslant (b-a)^{1/r'}\left(|\phi(a)|+V_{r'}(\phi)\right).$$

† Defining $\phi(x)=0$ outside $(a,\ b)$, as before.

To estimate $\Phi(b)$, let $\psi(x)=\phi(a)$ $(x<a)$ and $\psi(x)=\phi(x)$ $(x\geqslant a)$. Then

$$\begin{aligned}\Phi(b)^{1/r'} &\leqslant V_{r'}(\phi;\ -\infty,\ b)\\ &\leqslant V_{r'}(\phi-\psi;\ -\infty,\ b)+V_{r'}(\psi;-\infty,\ b) \qquad [\mathbf{9},\ 260]\\ &= |\phi(a)|+V_{r'}(\phi;\ a,\ b).\end{aligned}$$

Thus

$$\left(\int_a^b |\phi_{\alpha-1}^+(x)|^{r'}\,dx\right)^{1/r'} \leqslant \frac{1-\alpha}{\Gamma(\alpha)}\,(b-a)^{\alpha-1/r}\left(\frac{1}{\alpha-1/r}+\frac{1}{1-\alpha}\right)\left(|\phi(a)|+V_{r'}(\phi)\right).$$

But, since g is L^r,

$$\left|\int_a^b \phi_{\alpha-1}^+(t)\,g(t)\,dt\right| \leqslant \left(\int_a^b |\phi_{\alpha-1}^+(t)|^{r'}\,dt\right)^{1/r'}\left(\int_a^b |g(t)|^r\,dt\right)^{1/r},$$

from which the required condition follows.

The hypothesis $1/q<\alpha$ is necessary in Theorem 2. For, if $1/q=\alpha<1$, there is a function g of L^q such that $g_\alpha^{\pm}$ are unbounded [**2**, 578] and so not integrable in the Riemann-Stieltjes sense. This function g also belongs to L^r if $1/r>\alpha$.

From Theorem 2 we deduce a symmetrical theorem on fractional integration by parts for Stieltjes integrals (compare C):

THEOREM 3. *If*

$$p>1,\quad q>1,\quad 0<\alpha<1,\quad 1-1/p-1/q<\alpha,$$

and f *is* W_p, g *is* W_q,

then

$$\int_a^b f\,dg_\alpha^- + \int_a^b g\,df_\alpha^+ = 0.$$

We may suppose, by increasing q if necessary, that $1-1/p-1/q\geqslant 0$. There is a β such that $\max(\alpha,\ 1/p')<\beta<\min(\alpha+1/q,\ 1)$; then

$$0<\beta-\alpha<1/q\leqslant 1/p'<\beta<1.$$

Therefore, since g is $\mathrm{Lip}(1/q,\ q)$, $g_{\alpha-\beta}^-$ exists almost everywhere and is L^q, and $(g_{\alpha-\beta}^-)_{\beta-\alpha}^-$ is equivalent to g (B). So

$$g_{\alpha-\beta}^- \text{ is } L^{p'},\quad g_\alpha^- = \left((g_{\alpha-\beta}^-)_{\beta-\alpha}^-\right)_\alpha^- = (g_{\alpha-\beta}^-)_\beta^-, \qquad \text{(A)},$$

and

$$\int f\,dg_\alpha^- = \int f\,d(g_{\alpha-\beta}^-)_\beta^- = -\int f_{\beta-1}^+ g_{\alpha-\beta}^-\,dx \qquad \text{(Theorem 2)}.$$

Again, put $\gamma = 1+\alpha-\beta$, so that

$$0 < \gamma-\alpha < 1/p \leqslant 1/q' < \gamma < 1.$$

Then $f^+_{\alpha-\gamma}$ exists almost everywhere and is L^p, and $(f^+_{\alpha-\gamma})^+_{\gamma-\alpha}$ is equivalent to f (B), whence

$$f^+_{\alpha-\gamma} \text{ is } L^{q'}, \quad f_\alpha{}^+ = \left((f^+_{\alpha-\gamma})^+_{\gamma-\alpha}\right)_\alpha{}^+ = (f^+_{\alpha-\gamma})_\gamma{}^+ \qquad \text{(A)},$$

and

$$\int g\, df_\alpha{}^+ = \int g\, d(f^+_{\alpha\ \gamma})_\gamma{}^+ = \int g^-_{\gamma-1} f^+_{\alpha-\gamma}\, dx \qquad \text{(Theorem 2)}.$$

Our conclusion follows, since $\beta+\gamma = 1+\alpha$†.

In § 9 we shall show that this theorem fails in the limiting case

$$1-1/p-1/q = \alpha.$$

6. *Some consequences.* From Theorem 2 and (B) we obtain three corollaries:

THEOREM 4. *If* $1/q < \alpha = 1/p < 1$ *and* g *is* L^q, *then* $g_\alpha{}^\pm$ *are* W_{p+0}.

For, if $p < r < q$, then $1/r < \alpha < 1$ and g is L^r, so $g_\alpha{}^\pm$ are W_r (Theorem 2).

THEOREM 5. *If* $1/q < 1/q+\alpha = 1/p \leqslant 1$ *and* g *is* W_q, *then* $g_\alpha{}^\pm$ *are* W_{p+0}.

Given r such that $p < r < q$, there is a β satisfying

$$\max(\alpha,\ 1/r) < \beta < 1/p,$$

and so

$$0 < \beta-\alpha < 1/q < 1/r < \beta < 1.$$

Now g is Lip $(1/q,\ q)$; so $g_{\alpha-\beta}$ exists almost everywhere and is L^q in $(a,\ b)$, and $(g_{\alpha-\beta})_{\beta-\alpha}$ is equivalent to g (B). Therefore $g_{\alpha-\beta}$ is L^r and $g_\alpha = \left((g_{\alpha-\beta})_{\beta-\alpha}\right)_\alpha = (g_{\alpha-\beta})_\beta$; whence g_α is W_r (Theorem 2).

THEOREM 6. *If* $q > p \geqslant 1$ *and* g *is* Lip $(1/p,\ q)$, *then there is a* g_0 *equivalent to* g *such that* g_0 *is* W_{p+0}.

Choose r_n such that $p < r_n < q$ and $r_n \to p$, and choose α_n such that $1/r_n < \alpha_n < 1/p$. Then $g_{-\alpha_n}$ exists almost everywhere and is L^q, and g is equivalent to $(g_{-\alpha_n})_{\alpha_n}$ (B). So $g_{-\alpha_n}$ is L^{r_n}, whence the function $g_n = (g_{-\alpha_n})_{\alpha_n}$ is

† In the *complex* extension of Theorem 3, a few simple modifications of the proof are required, such as replacing the equation

$$g_\alpha{}^- = (g^-_{\alpha-\beta})_\beta{}^- \quad \text{by} \quad g^-_{\alpha+i\theta} = (g^-_{\alpha-\beta+\frac{1}{2}i\theta})^-_{\beta+\frac{1}{2}i\theta}.$$

Similar changes are needed in the complex extensions of Theorems 12 and 13, and in that of C in § 6.

W_{r_n} (Theorem 2); and g_n is continuous (Lemma 3). Thus all the g_n, being continuous and equivalent to g, are identical with one continuous g_0, which belongs to all the classes W_{r_n} and so to W_{p+0}.

We have been unable to decide whether it is possible or not to replace W_{p+0} by W_p in the conclusions of the above three theorems.

From Theorem 2 we can also deduce an *alternative proof of the case* $1/p+1/q-1<\alpha$ *of* C (which certainly suffices to establish C_1):

There exist β and γ such that

$$1/p<\beta<1, \quad 1/q<\gamma<1, \quad \beta+\gamma-1=\alpha;$$

for instance, we may take

$$\beta=\frac{1}{p}+\frac{\mu}{p'}, \quad \gamma=\frac{1}{q}+\frac{\mu}{q'}, \quad \text{where} \quad \mu=\left(\alpha+1-\frac{1}{p}-\frac{1}{q}\right)\Big/\left(\frac{1}{p'}+\frac{1}{q'}\right).$$

Choose r and s such that

$$1/p<1/r<\beta, \quad 1/q<1/s<\gamma, \quad 1/r+1/s>1.$$

Then $$f \text{ is } L^r, \quad g \text{ is } L^s,$$

whence, by four applications of Theorem 2,

$$f_\beta^+ \text{ is } W_r \text{ and so } W_{s'}, \quad g_\gamma^- \text{ is } W_s \text{ and so } W_{r'},$$

$$\int f_\beta^+ dg_\gamma^- = -\int (f_\beta^+)_{\gamma-1}^+ g\,dx, \qquad \int g_\gamma^- df_\beta^+ = +\int (g_\gamma^-)_{\beta-1}^- f\,dx.$$

Now

$$(f_\beta^+)_\gamma^+ = (f_{\alpha+1-\gamma}^+)_\gamma^+ = \left((f_\alpha^+)_{1-\gamma}^+\right)_\gamma^+ = (f_\alpha^+)_1^+ \dagger, \qquad \text{(A)}$$

whence $$f(_\beta^+)_{\gamma-1}^+(x) = \frac{d}{dx}\int_a^x f_\alpha^+(t)\,dt = f_\alpha^+(x) \quad \text{for almost all } x.$$

Similarly

$$(g_\gamma^-)_\beta^- = (g_{\alpha+1-\beta}^-)_\beta^- = \left((g_\alpha^-)_{1-\beta}^-\right)_\beta^- = (g_\alpha^-)_1^- \dagger, \qquad \text{(A)}$$

and $$(g_\gamma^-)_{\beta-1}^-(x) = -\frac{d}{dx}\int_x^b g_\alpha^-(t)\,dt = g_\alpha^-(x) \quad \text{for almost all } x.$$

† These two lines are unnecessarily complicated for present purposes. But the same work will be used to prove Theorem 12 (§ 8); this complication is then necessary, since Weyl fractional integrals are not generally convergent for orders greater than 1.

Therefore

$$\int f g_\alpha^- dx - \int f_\alpha^+ g\, dx = \int g_\gamma^- df_\beta^+ + \int f_\beta^+ dg_\gamma^-$$

$$= f_\beta^+(b) g_\gamma^-(b) - f_\beta^+(a) g_\gamma^-(a) = 0,$$

using Stieltjes integration by parts and the facts $f_\beta^+(a) = 0 = g_\gamma^-(b)$.

[This proof is independent of that given in § 3. For Theorem 2 depends only on B and Stieltjes integral theory, while the proof in § 3 is mainly Hardy and Littlewood, **2**, Theorem 3; this latter is independent of Hardy and Littlewood's proof of B.]

From Theorem 3 we deduce the following inversion theorem, the Stieltjes analogue of the Lebesgue inversion theorem, implicit in **2**, Theorem 3, from which C was first obtained.

Theorem 7. *Under the hypotheses of Theorem* 3,

$$\int_a^b df(x) \int_a^x (x-y)^\alpha dg(y) = \int_a^b dg(y) \int_y^b (x-y)^\alpha df(x).$$

From Theorem 3,

$$0 = \int_a^b f\, dg_\alpha^+ + \int_a^b g\, df_\alpha^-$$

$$= f(b) g_\alpha^+(b) - \int_a^b g_\alpha^+ df - g(a) f_\alpha^-(a) - \int_a^b f_\alpha^- dg. \qquad \text{(i)}$$

But

$$\alpha\Gamma(\alpha) g_\alpha^+(x) = \alpha \int_a^x g(y)(x-y)^{\alpha-1} dy = -\int_a^x g(y)\, d_y (x-y)^\alpha$$

$$= g(a)(x-a)^\alpha + \int_a^x (x-y)^\alpha dg(y),$$

and

$$\alpha\Gamma(\alpha) f_\alpha^-(y) = \alpha \int_y^b f(x)(x-y)^{\alpha-1} dx = \int_y^b f(x)\, d_x (x-y)^\alpha$$

$$= f(b)(b-y)^\alpha - \int_y^b (x-y)^\alpha df(x).$$

Our conclusion follows at once by substituting these expressions in (i).

With Theorem 3, Theorem 7 fails in the case $1-1/p-1/q = \alpha$, since the above proof is reversible. Theorems 3 and 7 are, in fact, equivalent.

7. *Transformation of certain non-absolutely convergent integrals.* A special case of Theorem 2, combined with results of Hardy and Littlewood, gives at once

THEOREM 8. *Let f be* Lip $(1/p)$, *G be* Lip $(1/q)$, $p>1$, $q>1$, $1/p+1/q>1$. *Then there exist continuous functions F' (depending only on f, p, q) and g (depending only on G, p, q) such that*

$$\int_a^b \Big(f(x)-f(a)\Big)\,d\Big(G(b)-G(x)\Big) = \int_a^b F'(x)\,g(x)\,dx.$$

We may suppose that $f(a)=G(b)=0$, and define $f(x)=0$ $(x<a)$ and $G(x)=0$ $(x>b)$. Then we may adapt **2**, Theorem 19, in the same way as we adapted **2**, Theorem 23, to prove B; we find that the function $F'=f_{\alpha-1}^{+}$ exists and is continuous, and $f_\alpha{}^+$ is its integral, α being a number subject to $1/p'<\alpha<1/q$. Similarly $g=G_{-\alpha}^{-}$ exists and is continuous, and *a fortiori* $L^{p'}$; and $G=g_\alpha{}^-$. Also f is W_p. Hence, by Theorem 2,

$$-\int_a^b f\,dG = -\int_a^b f\,dg_\alpha^{-} = \int_a^b g(x)\,F'(x)\,dx.$$

By means of a preliminary change of variable, we can apply the above theorem to the case of a pair of continuous functions of the classes W_p, W_q respectively, where $p>1$, $q>1$, $1/p+1/q>1$.

We may suppose that the functions belong to $W_p{}^*$, $W_q{}^*$ by changing slightly the values of p and q if necessary. The change of variable

$$t = V_p(f;\ a,\ x)^p + V_q(G;\ a,\ x)^q + (x-a),$$

which is continuous and strictly increasing, then has the required effect.

8. *Weyl integrals.* We now turn to periodic functions, of which we define certain classes:

Given $\omega>0$, let $P(\omega)$ denote the class of functions of period ω; and let $P=\sum_{\omega>0} P(\omega)$ denote the class of all periodic functions.

Our standard classes of periodic functions are defined thus:

Given $p\geqslant 1$, L^p is the class of measurable periodic f such that $|f|^p$ is integrable on every finite interval. In particular $L=L^1$.

Lip k, W_p, Lip (k,p) are similarly the classes of periodic functions which have the appropriate property over every finite interval. (It is, of

course, necessary and sufficient that these properties hold over a period in order that they should hold over every finite interval.)

Given $\omega > 0$, let $P_0(\omega)$ denote the class of f of L. $P(\omega)$ such that

$$\int_0^\omega f(t)\,dt = 0.$$

And let $P_0 = \sum_{\omega>0} P_0(\omega)$. The significance of this class, of course, consists in the existence and properties of the Weyl fractional integrals of its members [**2**, 567, **8**].

We now use f_α^+ and f_α^- to denote the functions defined in the following way [*cf.* **8**]:

$$f_\alpha^+(x) = f_\alpha^+(-\infty, x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x f(t)(x-t)^{\alpha-1}\,dt,$$

$$f_\alpha^-(x) = f_\alpha^-(x, +\infty) = \frac{1}{\Gamma(\alpha)}\int_x^{+\infty} f(t)(t-x)^{\alpha-1}\,dt,$$

where f is P_0 and $0 < \alpha < 1$.

The fundamental properties of these integrals are:

THEOREMS A′. *If f is $P_0(\omega)$ and $0 < \alpha < 1$, then $f_\alpha^+(x)$ and $f_\alpha^-(x)$ exist for almost all x, and f_α^+ and f_α^- are $P_0(\omega)$.*

If f is P_0, $\alpha > 0$, $\beta > 0$, and $\alpha+\beta < 1$, then

$$(f_\alpha^+)_\beta^+(x) = f_{\alpha+\beta}^+(x), \quad (f_\alpha^-)_\beta^-(x) = f_{\alpha+\beta}^-(x),$$

for all x for which the right-hand sides exist.

If f is $P_0(\omega)$, it has just one indefinite integral f_1 belonging to $P_0(\omega)$; and, if $0 < \alpha < 1$, then, for all x,

$$+(f_\alpha^+)_{1-\alpha}^+(x) = f_1(x) = -(f_\alpha^-)_{1-\alpha}^-(x).$$

(The definitions of f_α^+ and f_α^- fail if $\alpha = 1$, since the improper integrals involved are not convergent.)

Derivatives are defined formally as before.

An elementary lemma is needed in extending our theorems:

LEMMA 5. *If f is $P_0(\omega)$ and $0 < \alpha < 1$, then*

$$F(x) = \int_0^\infty f(t)(x+t)^{\alpha-1}\,dt$$

exists and is differentiable for all $x>0$, *its derivative satisfying*

$$F'(x)=(\alpha-1)\int_0^\infty f(t)(x+t)^{\alpha-2}dt, \qquad |F'(x)|\leqslant 2\frac{1-\alpha}{x^{2-\alpha}}\int_0^\omega |f(t)|\,dt.$$

Consequently $$V_1(F;\ c,\ \infty)\leqslant \frac{2}{c^{1-\alpha}}\int_0^\omega |f(t)|\,dt \quad (c>0).$$

Let $$\phi(s)=\int_0^s f(t)\,dt,$$

a continuous function of period ω, satisfying, for all s,

$$|\phi(s)|\leqslant\int_0^\omega |f(t)|\,dt.$$

Given $\beta>0$, x and (a, b) such that $0\leqslant a<b<\infty$ and $x+a>0$,

$$\int_a^b f(t)(x+t)^{-\beta}dt=\phi(b)(x+b)^{-\beta}-\phi(a)(x+a)^{-\beta}+\beta\int_a^b\phi(t)(x+t)^{-\beta-1}dt,$$

so

$$\left|\int_a^b f(t)(x+t)^{-\beta}dt\right|\leqslant\int_0^\omega|f(s)|\,ds\left((x+b)^{-\beta}+(x+a)^{-\beta}+\beta\int_a^b(x+t)^{-\beta-1}dt\right)$$

$$=2(x+a)^{-\beta}\int_0^\omega|f(t)|\,dt.$$

Thus $$\int_0^\infty f(t)(x+t)^{-\beta}dt$$

is uniformly convergent in $x\geqslant c>0$; it is also continuous at each $x>0$, since each function

$$\int_0^{n\omega} f(t)(x+t)^{-\beta}dt$$

has this property.

It follows that, given numbers $\xi>c>0$,

$$\int_c^\xi dx\int_0^{n\omega}f(t)(x+t)^{\alpha-2}dt\to\int_c^\xi dx\int_0^\infty f(t)(x+t)^{\alpha-2}dt \quad (n\to\infty).$$

But

$$(1-\alpha)\int_c^\xi dx\int_0^{n\omega}f(t)(x+t)^{\alpha-2}dt=(1-\alpha)\int_0^{n\omega}f(t)\,dt\int_c^\xi(x+t)^{\alpha-2}dx$$

$$=\int_0^{n\omega}f(t)\left((c+t)^{\alpha-1}-(\xi+t)^{\alpha-1}\right)dt$$

$$\to F(c)-F(\xi) \quad (n\to\infty);$$

therefore

$$F(\xi)-F(c)=(\alpha-1)\int_c^\xi dx\int_0^\infty f(t)(x+t)^{\alpha-2}dt.$$

Thus $F'(\xi)$ exists and is the continuous function stated, throughout $\xi>0$.

Further $$\left|\int_0^\infty f(t)(x+t)^{\alpha-2}dt\right|\leqslant 2x^{\alpha-2}\int_0^\omega|f(t)|dt \quad (x>0),$$

and

$$V_1(F;\ c,\infty)\leqslant\int_c^\infty|F'(x)|dx\leqslant 2(1-\alpha)\int_0^\omega|f(t)|dt\int_c^\infty x^{\alpha-2}dx \quad (c>0).$$

We need also the form of B appropriate to Weyl integrals:

THEOREM B′. *If* $p\geqslant 1,\quad 0<\alpha<k\leqslant 1,\quad f$ *is* $P_0(\omega)$,

and $$\int_0^\omega|f(x)-f(x-h)|^p dx=0(h^{kp})\ .\ (0<h<\omega),$$

then $f_{-\alpha}^+$ *and* $f_{-\alpha}^-$ *exist almost everywhere and belong to* $L^p P_0(\omega)$; *and* f *is equivalent to* $(f_{-\alpha}^+)_\alpha^+$ *and to* $(f_{-\alpha}^+)_\alpha^-$. *Further, if "*l.i.m.*" denotes "limit in mean, with index* p, *over any given finite interval", then*

$$f_{-\alpha}^+=\frac{\alpha}{\Gamma(1-\alpha)}\operatorname*{l.i.m.}_{\epsilon\to 0}\phi_\epsilon,\qquad f_{-\alpha}^-=\frac{\alpha}{\Gamma(1-\alpha)}\operatorname*{l.i.m.}_{\epsilon\to 0}\psi_\epsilon,$$

where, for $\epsilon>0$,

$$\phi_\epsilon(x)=\int_{-\infty}^{x-\epsilon}\Big(f(x)-f(t)\Big)(x-t)^{-\alpha-1}dt,\qquad \psi_\epsilon(x)=\int_{x+\epsilon}^{\infty}\Big(f(x)-f(t)\Big)(t-x)^{-\alpha-1}dt.$$

This is precisely the theorem proved by Hardy and Littlewood [**2**, Theorem 23].

The extensions of Theorems 4 and 5 are immediate:

THEOREM 9. *If* $1/q<\alpha=1/p<1$ *and* g *is* $L^q P_0$, *then* $g_\alpha^{\pm}$ *are* W_{p+0}.

THEOREM 10. *If* $1/q<1/q+\alpha=1/p\leqslant 1$ *and* g *is* $W_q P_0$, *then* $g_\alpha^{\pm}$ *are* W_{p+0}.

For

$$\Gamma(\alpha)g_\alpha^+(x)=\int_0^\infty g(-t)(x+\omega+t)^{\alpha-1}dt+\Gamma(\alpha)g_\alpha^+(-\omega,x),$$

in which the second term on the right is W_{p+0} (Theorems 4 and 5) and the first is W_1 (Lemma 5) in $0\leqslant x\leqslant\omega$.

The extension of Theorem 2 takes the following form:

THEOREM 11. *If $1/q<\alpha<1$, f is $W_{q'}P_0(\omega)$ and g is $L^qP_0(\omega)$, then the following equations between Riemann-Stieltjes and Lebesgue integrals hold:*

$$\int_0^\omega f\,dg_\alpha^- = -\int_0^\omega f_{\alpha-1}^+ g\,dx, \qquad \int_0^\omega f\,dg_\alpha^+ = +\int_0^\omega f_{\alpha-1}^- g\,dx.$$

By Theorem 9, g_α^- is W_{q-0}; and, by the argument which establishes Theorem 9, using Lemmas 3 and 5, g_α^- is continuous. Therefore the Riemann-Stieltjes integral exists.

By B′, $f_{\alpha-1}^+$ is $L^{q'}$, since $0<1-\alpha<1/q'<1$ and f is Lip $(1/q',\ q')$. Therefore the Lebesgue integral exists.

For positive integral n we have, by periodicity,

$$\begin{aligned}
&2n\Gamma(\alpha)\left(\int_0^\omega f\,dg_\alpha^- + \int_0^\omega f_{\alpha-1}^+ g\,dx\right)\\
&\quad = \Gamma(\alpha)\left(\int_{-n\omega}^{n\omega} f\,dg_\alpha^- + \int_{-n\omega}^{n\omega} f_{\alpha-1}^+ g\,dx\right)\\
&\quad = \Gamma(\alpha)\left(\int_{-n\omega}^{n\omega} f(x)\,dg_\alpha^-(x,\ n\omega) + \int_{-n\omega}^{n\omega} f_{\alpha-1}^+(-n\omega,\ x)\,g(x)\,dx\right)\\
&\qquad + \int_{-n\omega}^{n\omega} f(x)\,d\left(\int_{n\omega}^{\infty} g(t)(t-x)^{\alpha-1}\,dt\right) + \int_{-n\omega}^{n\omega} g(x)\,\frac{d}{dx}\left(\int_{-\infty}^{-n\omega} f(t)(x-t)^{\alpha-1}\,dt\right)dx,
\end{aligned}$$

since the first two integrals of this last expression exist (Theorem 2).

Now

$$\begin{aligned}
J_1(n) &= \int_{-n\omega}^{n\omega} f(x)\,d\left(\int_{n\omega}^{\infty} g(t)(t-x)^{\alpha-1}\,dt\right)\\
&= \int_{-2n\omega}^{0} f(y)\,d\left(\int_0^\infty g(u)(\overline{u+n\omega}-\overline{y+n\omega})^{\alpha-1}\,du\right)\\
&= -\int_0^{2n\omega} f(-x)\,d\left(\int_0^\infty g(u)(u+x)^{\alpha-1}\,du\right)\\
&= -\int_0^\omega f(-x)\,d\left(\int_0^\infty g(t)(t+x)^{\alpha-1}\,dt\right) - \int_\omega^{2n\omega} f(-x)\,d\left(\int_0^\infty g(t)(t+x)^{\alpha-1}\,dt\right),
\end{aligned}$$

since the latter integral exists [the integrand being $W_{q'}$ and the integrator a continuous function of W_1 (Lemma 5)]. Further, if H is an upper bound of $|f(x)|$, we have

$$\left|\int_\omega^{2n\omega} f(-x)\,d\left(\int_0^\infty g(t)(t+x)^{\alpha-1}\,dt\right)\right| \leqslant HV_1\left(\int_0^\infty g(t)(t+x)^{\alpha-1}\,dt;\ \omega,\ \infty\right),$$

and this last expression is finite (Lemma 5), and independent of n. Therefore $J_1(n)$ is bounded.

Again

$$J_2(n) = \int_{-n\omega}^{n\omega} g(x)\,\frac{d}{dx}\left(\int_{-\infty}^{-n\omega} f(t)(x-t)^{\alpha-1}\,dt\right)dx$$

$$= \int_0^{2n\omega} g(y)\,\frac{d}{dy}\left(\int_0^{\infty} f(-u)(\overline{y-n\omega}+\overline{u+n\omega})^{\alpha-1}\,du\right)dy$$

$$= \int_0^{\omega} g(x)\,\frac{d}{dx}\left(\int_0^{\infty} f(-t)(x+t)^{\alpha-1}\,dt\right)dx + \int_{\omega}^{2n\omega} g(x)\,\frac{d}{dx}\left(\int_0^{\infty} f(-t)(x+t)^{\alpha-1}\,dt\right)dx,$$

the latter integral existing (and hence the former) since

$$\int_{\omega}^{2n\omega}\left|g(x)\,\frac{d}{dx}\left(\int_0^{\infty} f(-t)(x+t)^{\alpha-1}\,dt\right)\right|dx$$

$$\leqslant 2(1-\alpha)\int_0^{\omega}|f(t)|\,dt\int_{\omega}^{2n\omega}|g(x)|\,x^{\alpha-2}\,dx \qquad \text{(Lemma 5)}$$

$$\leqslant 2(1-\alpha)\int_0^{\omega}|f(t)|\,dt\sum_{\nu=1}^{2n-1}\int_{\nu\omega}^{(\nu+1)\omega}|g(x)|\,(\nu\omega)^{\alpha-2}\,dx$$

$$< 2\,\frac{1-\alpha}{\omega^{2-\alpha}}\int_0^{\omega}|f(t)|\,dt\int_0^{\omega}|g(x)|\,dx\sum_{\nu=1}^{\infty}\nu^{\alpha-2}.$$

a finite bound, independent of n. Therefore $J_2(n)$ is bounded.

Therefore

$$\int_0^{\omega} f\,dg_{\alpha}^{-} + \int_0^{\omega} f_{\alpha-1}^{+}\,g\,dx = 0 + O\left(\frac{1}{n}\right), \qquad \text{(Theorem 2)}$$

whence the first equation of our conclusion follows. Similarly the second.

The following two theorems are proved just like their analogues in §§ 6, 5:

THEOREM 12. *If*

$$p > 1, \quad q > 1, \quad 0 < \alpha < 1, \quad 1/p + 1/q - 1 < \alpha,$$

and f *is* $L^p P_0(\omega)$, g *is* $L^q P_0(\omega)$,

then

$$\int_0^{\omega} f_{\alpha}^{+}\,g\,dx = \int_0^{\omega} f\,g_{\alpha}^{-}\,dx.$$

As before, $(f_{\beta}^{+})_{\gamma}^{+}$ is an indefinite integral of f_{α}^{+}, and $(g_{\gamma}^{-})_{\beta}^{-}$ of $-g_{\alpha}^{-}$, so that $(f_{\beta}^{+})_{\gamma-1}^{+}$ is equivalent to f_{α}^{+} and $(g_{\gamma}^{-})_{\beta-1}^{-}$ to g_{α}^{-}.

Also, by periodicity,

$$\int_0^\omega g_\gamma^- df_\beta^+ + \int_0^\omega f_\beta^+ dg_\gamma^- = f_\beta^+(\omega) g_\gamma^-(\omega) - f_\beta^+(0) g_\gamma^-(0) = 0.$$

Otherwise there is no difference, except of interpretation, from the proof of the weakened form of C established in § 6.

THEOREM 13. *If*

$$p > 1, \quad q > 1, \quad 0 < \alpha < 1, \quad 1 - 1/p - 1/q < \alpha,$$

and $$f \text{ is } W_p P_0(\omega), \quad g \text{ is } W_q P_0(\omega),$$

then $$\int_0^\omega f dg_\alpha^- + \int_0^\omega g df_\alpha^+ = 0.$$

Apart from interpretation, the proof is identical with that of Theorem 3, using B′ instead of B, and Theorem 11 instead of Theorem 2.

9. *A "Gegenbeispiel".* We now demonstrate the failure of Theorems 3 and 13 in the case $1-1/p-1/q = \alpha$ by means of Weierstrass functions, the integrals of which can be evaluated explicitly.

Let $\omega = 2\pi$, let c be an integer greater than 1, and let

$$f(x) = \sum_{\nu=1}^{\infty} c^{-\nu/p} \cos(c^\nu x), \quad g(x) = \sum_{\nu=1}^{\infty} c^{-\nu/q} \sin(c^\nu x + \tfrac{1}{2}\pi\alpha).$$

These functions more than fulfil the hypotheses of both theorems, since f is Lip $(1/p)$ and g is Lip $(1/q)$ [1].

Now the α^+ integral of $\sin(nx+\lambda)$ is

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)} \int_0^\infty \sin(nx+\lambda-nu)\, u^{\alpha-1} du \\
&\quad = \frac{1}{\Gamma(\alpha)} \left(\sin(nx+\lambda) \int_0^\infty \cos v \frac{v^{\alpha-1}}{n^\alpha} dv - \cos(nx+\lambda) \int_0^\infty \sin v \frac{v^{\alpha-1}}{n^\alpha} dv \right) \\
&\quad = n^{-\alpha} \left(\sin(nx+\lambda) \cos \tfrac{1}{2}\pi\alpha - \cos(nx+\lambda) \sin \tfrac{1}{2}\pi\alpha \right) \\
&\quad = n^{-\alpha} \sin(nx+\lambda-\tfrac{1}{2}\pi\alpha),
\end{aligned}$$

by contour integration. Therefore

$$g_\alpha^+(x) = \sum_{\nu=1}^{\infty} c^{-\nu/q} \left(\sin(c^\nu x + \tfrac{1}{2}\pi\alpha) \right)_\alpha^+ = \sum_{\nu=1}^{\infty} c^{-\nu/q-\nu\alpha} \sin(c^\nu x) = \sum_{\nu=1}^{\infty} c^{-\nu/p'} \sin(c^\nu x),$$

with respect to which f is not Riemann-Stieltjes-integrable; so Theorem 13 fails.

For Theorem 3, the α-th integral concerned is

$$g_\alpha^+(a, x) = g_\alpha^+(-\infty, x) - \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{a} g(t)(x-t)^{\alpha-1} dt.$$

Now $$\int_{-\infty}^{a} g(t)\,(x-t)^{\alpha-1} dt = \int_{0}^{\infty} g(a-u)(x-a+u)^{\alpha-1} du,$$

a function of bounded variation in (a', b) for every a' such that $a < a' < b$ (Lemma 5). Also f is continuous, and Riemann-Stieltjes-integrable with respect to $g_\alpha^+(-\infty, x)$ on no interval (a', b). Therefore f is Riemann-Stieltjes-integrable with respect to $g_\alpha^+(a, x)$ on no interval (a', b); nor, therefore, on (a, b). So Theorem 3 also fails.

10. *An inequality connected with Theorem* 6. The preceding work assists us in proving an inequality from which we obtain the following corollary:

If $1/q < k \leqslant 1$ *and* (g_n) *is a sequence of continuous periodic functions having the same period and satisfying a* Lip (k, q) *condition uniformly, then* (g_n) *also satisfies a* W_q *condition uniformly.*

A uniform W_q condition on a sequence is required in a recent theorem on Stieltjes integration of sequences [**9**, 269]. The above corollary is designed to facilitate the verification of this rather complicated condition. Its application is, of course, limited by the fact that many W_q-functions are not Lip (k, q) for any $k > 1/q$. The restriction of continuity is comparatively trivial since any Lip (k, q)-function is equivalent to a continuous function if $kq > 1$.

The inequality which we shall prove is to be compared with **2**, Theorem 24; and also with

$$\int_{a+h}^{b} |g(x)-g(x-h)|^q dx \leqslant h V_q(g;\ a, b)^q \quad (g \text{ of } W_q, \quad h > 0),$$

which is easily deduced from the fundamental property

$$|g(x)-g(x-h)|^q \leqslant \phi(x)-\phi(x-h) \quad (h > 0),$$

where $\phi(x) = V_q(g;\ a, x)^q$, of W_q-functions.

First we require the periodic analogues of Lemmas 3 and 4. These are:

LEMMA 6. *If* $1/q < \alpha < 1$ *and* g *is* $L^q P_0$, *then* $g_\alpha^{\pm}(x)$ *exist and are continuous everywhere.*

This is trivial; see the beginning of the proof of Theorem 11.

LEMMA 7. *If* $p \geqslant 1, \quad 0 < \alpha < k \leqslant 1,$

and f *is* $P_0(\omega), \quad \int_0^\omega |f(x)-f(x-h)|^p\,dx \leqslant \mu^p h^{kp} \quad (0<h<\omega),$

then $$\left(\int_0^\omega |f_{-\alpha}^{\pm}(x)|^p\,dx\right)^{1/p} \leqslant \frac{\mu}{k-\alpha}\,\frac{k\omega^{k-\alpha}}{\Gamma(1-\alpha)}.$$

Using B′ in place of B we find, as in Lemma 4, that

$$\left(\int_0^\omega |f_{-\alpha}^{\pm}(x)|^p\,dx\right)^{1/p} \leqslant \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty u^{-\alpha-1}\left(\int_0^\omega |f(x)-f(x-u)|^p\,dx\right)^{1/p} du.$$

The same is true of $f_{-\alpha}^{-}$, since, by periodicity,

$$\int_0^\omega |f(x)-f(x-u)|^p\,dx = \int_0^\omega |f(x+u)-f(x)|^p\,dx.$$

So

$$\begin{aligned}
\frac{\Gamma(1-\alpha)}{\alpha}\left(\int_0^\omega |f_{-\alpha}^{\pm}(x)|^p\,dx\right)^{1/p} &\leqslant \sum_{\nu=0}^{\infty}\int_0^\omega (\nu\omega+t)^{-\alpha-1}\left(\int_0^\omega |f(x)-f(x-t)|^p\,dx\right)^{1/p} dt \\
&\leqslant \sum_{\nu=0}^{\infty}\int_0^\omega (\nu\omega+t)^{-\alpha-1}\,\mu t^k\,dt \\
&\leqslant \mu\left(\int_0^\omega t^{k-\alpha-1}\,dt + \sum_{\nu=1}^{\infty}(\nu\omega)^{-\alpha-1}\int_0^\omega t^k\,dt\right) \\
&= \mu\omega^{k-\alpha}\left(\frac{1}{k-\alpha}+\frac{1}{k+1}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{1+\alpha}}\right) \\
&\leqslant \mu\omega^{k-\alpha}\left(\frac{1}{k-\alpha}+\frac{1}{k+1}\left(1+\int_1^\infty \frac{dx}{x^{1+\alpha}}\right)\right) \\
&\leqslant \mu\omega^{k-\alpha}\left(\frac{1}{k-\alpha}+\frac{1}{\alpha}\right), \quad \text{as stated.}
\end{aligned}$$

The inequality referred to above is as follows:

THEOREM 14. *If g is a continuous function of period ω, and*

$$\frac{1}{q} < k \leqslant 1, \qquad \int_0^\omega |g(x)-g(x-h)|^q\,dx \leqslant \mu^q h^{kq} \quad (0<h<\omega),$$

then

$$V_q(g;\ 0,\ \omega) \leqslant \frac{32k\omega^{k-1/q}}{\pi q'(k-1/q)^2}\,\mu.$$

We may suppose that g is $P_0(\omega)$. Let H denote the right-hand member of the conclusion. If the conclusion is false, there is a subdivision

$$0<x_0<x_1<\ldots<x_k<\omega$$

such that

$$\sum_{i=1}^{k} |g(x_i)-g(x_{i-1})|^q > H^q,$$

and therefore a sequence (c_i) such that

$$S_{q'}(c_i) \leqslant 1, \qquad \sum_{0<i\leqslant j\leqslant k} c_i\Big(g(x_j)-g(x_{j-1})\Big) > H/2^{1+1/q} \quad [\mathbf{6},\ 249].$$

Let $\bar{x}(t)=0$ $(t\leqslant x)$ and $\bar{x}(t)=1$ $(t>x)$, and let

$$f(t) = \sum_{0<i\leqslant j\leqslant k} c_i\{\bar{x}_{j-1}(t)-\bar{x}_j(t)-(x_j-x_{j-1})/\omega\} \quad (0\leqslant t<\omega);$$

elsewhere define f so that it has period ω. Then f is $P_0(\omega)$, and

$$\int_0^\omega |f(x)-f(x-h)|^{q'}\,dx \leqslant hV_{q'}(f;\ -\omega,\ \omega)^{q'} \quad (0<h<\omega).$$

Choose α such that $1/q<\alpha<k$. Then $g^+_{-\alpha}$ is $L^qP_0(\omega)$ and $(g^+_{-\alpha})_\alpha{}^+$ is equivalent to g (B′); and since $(g^+_{-\alpha})_\alpha{}^+$ is continuous (Lemma 6), it is identical with g. So

$$\begin{aligned}
H/2^{1+1/q} &< \sum_{j=1}^{k}\sum_{i=1}^{j} c_i\Big(g(x_j)-g(x_{j-1})\Big)\\
&= \sum_{j=1}^{k}\int_0^\omega \Big(\sum_{i=1}^{j} c_i\Big(\bar{x}_{j-1}(t)-\bar{x}_j(t)\Big)\Big)\,dg(t) \quad \text{(since } g \text{ is continuous)}\\
&= \int_0^\omega f(t)\,dg(t) \qquad \Big(\text{since } \int_0^\omega dg(t)=0\Big)\\
&= \int_0^\omega f\,d(g^+_{-\alpha})_\alpha{}^+\\
&= \int_0^\omega f^-_{\alpha-1}(t)\,g^+_{-\alpha}(t)\,dt \qquad \text{(Theorem 11)}\\
&\leqslant \Big(\int_0^\omega |f^-_{\alpha-1}(t)|^{q'}\,dt\Big)^{1/q'}\Big(\int_0^\omega |g^+_{-\alpha}(t)|^q\,dt\Big)^{1/q} \quad \text{[Hölder].}
\end{aligned}$$

We now apply Lemma 7 to each factor of this last product, since (for the first factor) $0<1-\alpha<1/q'<1$ and (for the second) $0<\alpha<k\leqslant 1$; this gives

$$H<\frac{2^{1+1/q}k\omega^{k-1/q}}{q'\,\Gamma(\alpha)\,\Gamma(1-\alpha)}\,\frac{\mu V_{q'}(f;\ -\omega,\ \omega)}{(k-\alpha)(\alpha-1/q)}.$$

It remains to evaluate $V_{q'}(f;\ -\omega,\ \omega)$, and to choose a suitable value of α. For the former we may replace f by f^* of $P(\omega)$, where

$$f^*(t)=\sum_{0<i\leqslant j\leqslant k} c_i\Big(\bar{x}_{j-1}(t)-\bar{x}_j(t)\Big)\quad(0\leqslant t<\omega);$$

then it is easily shown that $V_{q'}(f^*;\ -\omega,\ \omega)^{q'}\leqslant 4S_{q'}(c_i)^{q'}$, by dividing any subdivision of $(-\omega,\ \omega)$ into at most four parts as follows:

(i) intervals of subdivision lying wholly in $(-\omega,\ x_k-\omega)$,

(ii) the next interval to the right,

(iii) intervals lying wholly in $(-\omega,\ x_k)$ and not yet taken,

(iv) intervals remaining.

Thus

$$2^{1+1/q}V_{q'}(f;\ -\omega,\ \omega)\leqslant 2^{1+1/q+2/q'}=2^{2+1/q'}<8.$$

Take $\alpha=\frac{1}{2}(k+1/q)$ (satisfying $1/q<\alpha<k$); we then obtain a contradiction:

$$H<\frac{32k\omega^{k-1/q}\mu}{q'\pi(k-1/q)^2}.$$

11. *Fractional generalized integration by parts.* We now suppose the functions concerned to be of period 1 and mean value 0. We write (as in **10**)

$$F(t)=F(t;\ f,\ g)=\int_0^1 f(x+t)\,g(x)\,dx,\quad I(f,\ g)=\lim_{\nu\to\infty}2^{\nu}\Big(F(0)-F(2^{-\nu})\Big).$$

From the definition we have at once

LEMMA 8. *If two pairs of functions f, g, and f^*, g^*, which are $P_0(1)$, have the same Faltung $F(t)$ for each t, then $I(f,\ g)=I(f^*,\ g^*)$ provided that one side of this relation exists.*

We state explicitly the analogue of C_1 for Weyl integrals; it is derived from Theorem 12.

THEOREM C_1'. *If $\lambda > 1$, $\mu > 1$, $0 \leqslant \alpha \leqslant 1$, f is L^λ, h' is L^μ, and h is the indefinite integral (with mean value zero) of h', then*

$$\int_0^1 f_\alpha^+ h_{-\alpha}^- dx = \int_0^1 f h\, dx.$$

We now apply C_1', taking for $h(x)$ the function $g(x-t)$. Since $h_{-\alpha}^-(x) = g_{-\alpha}^-(x-t)$ for each x, we deduce that

LEMMA 9. *If $\lambda > 1$, $\mu > 1$, $0 \leqslant \alpha \leqslant 1$, f is L^λ, g' is L^μ, and g is the indefinite integral (with mean value zero) of g', then*

$$F(t;\ f_\alpha^+, g_{-\alpha}^-) = F(t;\ f, g)$$

for each t.

It follows immediately by Lemma 8 that we have

THEOREM 15. *If $\lambda > 1$, $\mu > 1$, $0 \leqslant \alpha \leqslant 1$, f is L^λ, g' is L^μ, and g is the indefinite integral (with mean value zero) of g', then*

$$I(f_\alpha^+, g_{-\alpha}^-) = I(f, g)$$

provided that one side of this relation exists.

We mention specially the case $\lambda = p$, $\mu = p'$. Writing h for g' we observe that $I(f, g) = \int_0^1 fh\, dx$ exists [*cf.* **10**]. We therefore obtain

THEOREM 16. *If the functions f, h (of period 1 and mean value 0) belong to conjugate Lebesgue classes L^p, $L^{p'}$ in a period, and $0 \leqslant \alpha \leqslant 1$, then*

$$\int_0^1 fh\, dx = I(f_\alpha^+, -h_{1-\alpha}^-).$$

As a consequence of Theorem 16 we have the following theorem on linear functionals:

THEOREM 17. *If $L(f)$ is a linear functional defined for all f which are α-th integrals of functions of L^p, where $0 \leqslant \alpha \leqslant 1$, then $L(f)$ has the form $I(f, g)$ where g is a $(1-\alpha)$-th integral of a function of $L^{p'}$.*

We have only to observe that $L(f)$ in its dependence on the α-th derivative f^* of f is a linear functional $L^*(f^*) = L\left((f^*)_\alpha^+\right)$ and has the form $\int_0^1 f^* h^*\, dx$, where h^* is $L^{p'}$ by a classical theorem of F. Riesz. There-

fore, by Theorem 16,

$$L^*(f^*) = I(f, g),$$

where $g = -(h^*)_{1-a}^-$. (We here understand by "linear functional" a "continuous" linear functional, and we suppose the distance function for f identified with the $L^{p'}$ distance-function for f^*.)

12. *Note on the linear functional theorem of* § 4†. There is a further generalization of Theorem 1 in terms of the classes W_p^* [**9**, 261]. To obtain the property of W_p^*-functions which we require, it is simpler to define W_p^* a little differently, following the order of ideas of the Moore-Pollard definition of a Stieltjes integral [**9**, 263] rather than that of the simple Riemann-Stieltjes integral. The class W_p^* defined is, however, the same in the two cases.

We consider functions defined throughout a fixed finite interval (a, b). We denote by κ a subdivision $a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$ of (a, b). If all points of subdivision κ' are points of subdivision κ'', we write $\kappa' \subset \kappa''$ or $\kappa'' \supset \kappa'$.

Given $p \geqslant 1$, a subdivision κ, and any function f, define

$$V_p^{(\kappa)}(f) = \overline{\operatorname{bnd}_{\kappa' \supset \kappa}} \left(\sum_{\kappa'} |\Delta f|^p \right)^{1/p} = \left(\sum_{i=1}^{k} V_p(f;\ x_{i-1}, x_i)^p \right)^{1/p},$$

whether this number is finite or infinite. Further define

$$V_p^*(f) = \underset{(\kappa)}{\operatorname{bnd}}\, V_p^{(\kappa)}(f).$$

We have immediately the following properties:

$V_p^*(f) \leqslant V_p(f)$, and if either is finite then so is the other.

If f is W_p, then, given $\epsilon > 0$, there is a subdivision κ_ϵ such that

$$0 \leqslant V_p^{(\kappa)}(f) - V_p^*(f) < \epsilon \quad (\textit{all } \kappa \supset \kappa_\epsilon).$$

Given $p \geqslant 1$ and a function f having ordinary discontinuities only, let the points of discontinuity be enumerated, (x_ν) say; define

$$\mathfrak{S}_p(f) = \left\{ \sum_\nu \left(|f(x_\nu + 0) - f(x_\nu)|^p + |f(x_\nu) - f(x_\nu - 0)|^p \right) \right\}^{1/p}.$$

† Section 12 is due to E. R. Love.

It is easily shown that $\mathfrak{S}_p(f) \leqslant V_p^*(f)$. Also, if f is W_p its discontinuities are necessarily ordinary. This justifies the following definition:

Given $p \geqslant 1$, let W_p^* be the class of functions f of W_p such that

$$\mathfrak{S}_p(f) = V_p^*(f).$$

We now establish an approximation theorem more complete than Lemma 2. Its completeness is apparent from its converse, Theorem 19.

THEOREM 18. *If f is W_p^*, then, given $\epsilon > 0$, there is a step-function ϕ satisfying*

$$V_p(f-\phi) < \epsilon.$$

More precisely: Given $\epsilon > 0$, there is a subdivision κ_ϵ with the property that if

$$a = x_0 \leqslant \xi_1 \leqslant x_1 \leqslant \xi_2 \leqslant x_2 \leqslant \ldots \leqslant x_{k-1} \leqslant \xi_k \leqslant x_k = b$$

contains all points of κ_ϵ among the x_i and no point of κ_ϵ among the ξ_i, then

$$V_p(f-\phi) < \epsilon,$$

where ϕ is the step-function

$$\phi(x) = f(\xi_i) \quad (x_{i-1} < x < x_i, \quad 0 < i \leqslant k), \qquad \phi(x_i) = f(x_i) \quad (0 \leqslant i \leqslant k).$$

Let the points of discontinuity of f be enumerated (t_ν) in such a way that the numbers

$$\omega_\nu^p = |f(t_\nu+0)-f(t_\nu)|^p + |f(t_\nu)-f(t_\nu-0)|^p$$

form a decreasing sequence. Also let

$$s_n^p = \sum_{\nu=1}^{n} \omega_\nu^p, \quad s^p = \sum_{\nu=1}^{\infty} \omega_\nu^p = \mathfrak{S}_p(f)^p = V_p^*(f)^p.$$

Given $\epsilon > 0$, let $\lambda = \epsilon^p/2^{2p+4}$.

There is a subdivision κ_1 such that, for all $\kappa \supset \kappa_1$,

$$V_p^{(\kappa)}(f)^p < V_p^*(f)^p + \lambda.$$

Choose n so large that both $s_n^p > s^p - \lambda$ and all the t_ν which are points of κ_1 have suffix less than n. Then, for all $\nu > 2n$,

$$\omega_\nu^p < \lambda/n$$

$\Big($for otherwise $\omega_\nu^p \geqslant \lambda/n$ $(\nu \leqslant 2n)$, so $\lambda \leqslant \sum_{n+1}^{2n} \omega_\nu^p \leqslant s^p - s_n^p < \lambda\Big)$.

Given $\nu > 2n$, f has oscillation at t_ν not greater than $2^{1-1/p}\omega_\nu$ [Hölder], and therefore less than $2^{1-1/p}(\lambda/n)^{1/p}$. By a classical generalization of a theorem on uniform continuity, there is a subdivision κ_2, containing the

points t_ν (all $\nu \leqslant 2n$), such that f has oscillation in (α, β) less than $2(\lambda/n)^{1/p}$ for any pair $\alpha < \beta$ of adjacent points of κ_2, (α, β) being open at either end-point if that end-point is a t_ν with $\nu \leqslant 2n$.

Let κ_ϵ be the subdivision consisting of all points of both κ_1 and κ_2 and also any further points required to ensure that no two of the points t_ν with $\nu \leqslant 2n$ are adjacent points of κ_ϵ.

Let ϕ be any step-function defined, with reference to κ_ϵ, as in the enunciation. Given i such that $0 < i \leqslant k$, we have (as in the proof of Lemma 2)

$$V_p(f-\phi;\ x_{i-1}, x_i)^p \leqslant 3V_p(f;\ x_{i-1}, x_i)^p.$$

If x_i is a t_ν with $\nu \leqslant 2n$, then x_{i-1} is not, and $\xi_i < x_i$; so, given any subdivision $x_{i-1} = y_0 < y_1 < \ldots < y_{h-1} < y_h = x_i$ (with $h > 1$),

$$\begin{aligned}
&\sum_{j=1}^{h} |f(y_j)-\phi(y_j)-f(y_{j-1})+\phi(y_{j-1})|^p + |f(x_i)-f(x_i-0)|^p \\
&\quad = |f(y_1)-f(\xi_i)|^p + \sum_{j=2}^{h-1} |f(y_j)-f(y_{j-1})|^p \\
&\qquad\qquad + |f(y_h)-f(y_h-0)|^p + |f(\xi_i)-f(y_{h-1})|^p \\
&\quad < 2^p\lambda/n + V_p(f;\ x_{i-1}, x_i)^p + 2^p\lambda/n,
\end{aligned}$$

whence

$$V_p(f-\phi;\ x_{i-1}, x_i)^p \leqslant V_p(f;\ x_{i-1}, x_i)^p - |f(x_i)-f(x_i-0)|^p + 2^{p+1}\lambda/n.$$

Similarly, if x_{i-1} is a t_ν with $\nu \leqslant 2n$, we find

$$V_p(f-\phi;\ x_{i-1}, x_i)^p \leqslant V_p(f;\ x_{i-1}, x_i)^p - |f(x_{i-1}+0)-f(x_{i-1})|^p + 2^{p+1}\lambda/n.$$

Let κ denote the subdivision (x_i), and let Σ'_i denote summation over those values of i for which neither x_{i-1} nor x_i is a t_ν with $\nu \leqslant 2n$; then

$$s_{2n}^p + \Sigma'_i V_p(f;\ x_{i-1}, x_i)^p \leqslant V_p^{(\kappa)}(f)^p,$$

and

$$\begin{aligned}
V_p(f-\phi)^p &\leqslant 2^p \sum_{i=1}^{k} V_p(f-\phi;\ x_{i-1}, x_i)^p && \text{(Lemma 1)} \\
&\leqslant 2^p \left\{ \sum_{i=1}^{k} V_p(f;\ x_{i-1}, x_i)^p + 2\Sigma'_i V_p(f;\ x_{i-1}, x_i)^p - s_{2n}^p + 4n2^{p+1}\lambda/n \right\} \\
&\leqslant 2^p \left\{ V_p^{(\kappa)}(f)^p + 2\left(V_p^{(\kappa)}(f)^p - s_{2n}^p\right) - s_{2n}^p + 2^{p+3}\lambda \right\} \\
&< 2^p \left\{ 3\left(V_p{}^*(f)^p + \lambda - s^p + \lambda\right) + 2^{p+3}\lambda \right\} \\
&= 2^p(6+2^{p+3})\lambda < \epsilon^p.
\end{aligned}$$

The converse theorem is as follows:

Theorem 19. *If f is a function such that, given $\epsilon > 0$, there is a step-function ϕ with $V_p(f-\phi) < \epsilon$, then f is $W_p{}^*$.*

Given $\epsilon > 0$, there is a step-function ϕ such that $V_p(f-\phi)^p < \epsilon/3$. It follows, in particular, that f is W_p, since $V_p(f) \leqslant V_p(f-\phi)+V_p(\phi) < \infty$ [**9**, 260].

Let c_ν $(0 < \nu \leqslant n)$ be the discontinuities of ϕ. Let κ be a subdivision containing these points c_ν and also at least one other point between each pair of points c_ν, and such that

$$\operatorname{osc}(f;\ \alpha_\nu,\ c_\nu)^p < |f(c_\nu)-f(c_\nu-0)|^p+\epsilon/(6n) \quad \text{if} \quad (\alpha_\nu,\ c_\nu) \text{ is an interval of } \kappa,$$

$$\operatorname{osc}(f;\ c_\nu,\ \beta_\nu)^p < |f(c_\nu+0)-f(c_\nu)|^p+\epsilon/(6n) \quad \text{if} \quad (c_\nu,\ \beta_\nu) \text{ is an interval of } \kappa.$$

There is a $\kappa' \supset \kappa$ such that

$$\sum_{\kappa'} |\Delta f|^p > V_p^{(\kappa)}(f)^p-\epsilon/3.$$

Let Σ' denote summation over those intervals of κ' neither end-point of which is a c_ν; and let $(\alpha_\nu',\ c_\nu)$ and $(c_\nu,\ \beta_\nu')$ $(0 < \nu \leqslant n)$ denote the remaining intervals of κ'. Then

$$\sum_{\kappa'} |\Delta f|^p = \Sigma' |\Delta(f-\phi)|^p + \sum_{\nu=1}^{n} \left(|f(\beta_\nu')-f(c_\nu)|^p + |f(c_\nu)-f(\alpha_\nu')|^p \right)$$

$$< V_p(f-\phi)^p + \sum_{\nu=1}^{n} \left(|f(c_\nu+0)-f(c_\nu)|^p + |f(c_\nu)-f(c_\nu-0)|^p + 2\epsilon/(6n) \right);$$

so
$$V_p{}^*(f)^p \leqslant V_p^{(\kappa)}(f)^p < \sum_{\kappa'} |\Delta f|^p + \epsilon/3 < \mathfrak{S}_p(f)^p + \epsilon.$$

Therefore
$$\mathfrak{S}_p(f) = V_p{}^*(f).$$

From Lemma 2 and Theorem 19 we deduce immediately that, if $p > 1$,

$$W_{p-0} \subset W_p{}^* \subset W_p.$$

There are examples to show that $W_p{}^*$ is identical with neither W_{p-0} nor W_p.

Similar work establishes Theorems 18 and 19 in terms of the definitions of $\mathfrak{S}_p(f)$, $V_p{}^*(f)$ and $W_p{}^*$ given in **9**, 261, but there are some extra complications. Both definitions thus lead to the deduction that $W_p{}^*$ is the class of functions which can be approximated by step-functions as in Theorem 18; so the two classes $W_p{}^*$ defined are identical.

We require definitions of lower and upper Stieltjes integrals in the Moore-Pollard sense; for these we proceed as follows:

Given functions f and g, and a subdivision κ, let $\bar{F}(\kappa) = \bar{F}(\kappa;\ f,\ g)$ denote the upper bound of all approximative sums

$$\sum_{i=1}^{k} f(\xi_i')\left(g(x_i') - g(x_{i-1}')\right),$$

such that all points of κ are points x_i' and

$$a = x_0' \leqslant \xi_1' \leqslant x_1' \leqslant \xi_2' \leqslant x_2' \leqslant \ldots \leqslant x_{k-1}' \leqslant \xi_k' \leqslant x_k' = b.$$

And let $\bar{I}$ denote the lower bound of $\bar{F}(\kappa)$ for all κ.

Similarly define $\underline{F}(\kappa)$ and $\underline{I}$.

It is simple to show that $\underline{I} \leqslant \bar{I}$, that a necessary and sufficient condition that f should be integrable with respect to g in the Moore-Pollard sense is $\underline{I} = \bar{I}$, and that if this condition is satisfied then

$$\underline{I} = \int f\,dg = \bar{I}.$$

We are therefore justified in using $\underline{I}$ and $\bar{I}$ as lower and upper integrals:

$$\underline{I} = \underline{\int} f\,dg, \qquad \bar{I} = \overline{\int} f\,dg.$$

Our generalization of Theorem 1 is as follows:

THEOREM 20. *If $q > 1$ and L is a functional defined throughout $W_q{}^*$ satisfying*

$$L\left(\sum_{i=1}^{k} c_i f_i\right) = \sum_{i=1}^{k} c_i L(f_i) \qquad [c_i \text{ constants}, f_i \text{ in } W_q{}^*],$$

$$|L(f)| \leqslant M\left(|f(a)| + V_q(f;\ a,\ b)\right) \qquad [f \text{ in } W_q{}^*],$$

$$L(f) = 0 \qquad [\text{if } f(t) = 0 \text{ for all but one value of } t],$$

then there is a g of $W_{q'}$ such that, for all f of $W_q{}^*$,

$$\underline{\int} f\,dg \leqslant L(f) \leqslant \overline{\int} f\,dg,$$

these lower and upper integrals being in the Moore-Pollard sense just defined. Further, if f of $W_q{}^*$ has no one-sided discontinuity in common with g, then the Moore-Pollard integral exists.

We define g, and prove that it is $W_{q'}$, exactly as in Theorem 1.

Given f of $W_q{}^*$ and $\epsilon > 0$, there is a subdivision κ_1 with the property: If $a = x_0 < x_1 < \ldots < x_{k-1} < x_k = b$ is a subdivision whose points include

those of κ_1, then there are points ξ_i $(x_{i-1} < \xi_i < x_i,\ 0 < i \leqslant k)$ such that the function

$$\phi = \sum_{i=1}^{k} f(\xi_i)(\bar{x}_{i-1} - \bar{x}_i - \dot{x}_i) + \sum_{i=0}^{k} f(x_i)\,\dot{x}_i$$

satisfies $V_q(f-\phi) < \epsilon/M$ (Theorem 18).

Therefore†

$$\left| L(f) - \sum_{i=1}^{k} f(\xi_i)\big(g(x_i) - g(x_{i-1})\big) \right| = |L(f-\phi)| \leqslant M V_q(f-\phi) < \epsilon,$$

whence $$L(f) - \epsilon < \sum_{i=1}^{k} f(\xi_i)\big(g(x_i) - g(x_{i-1})\big) \leqslant \bar{F}(\kappa),$$

where κ is the subdivision (x_i). This is for any $\kappa \supset \kappa_1$. It follows that

$$L(f) - \epsilon < \bar{F}(\kappa)$$

for any subdivision κ whatever, and hence that

$$L(f) - \epsilon \leqslant \bar{\int} f\,dg.$$

Thus, using a similar argument for lower integrals, we establish that

$$\underline{\int} f\,dg \leqslant L(f) \leqslant \bar{\int} f\,dg.$$

Now suppose that there is no point at which both f and g are discontinuous on the same side. Let n_1 be the number of points of κ_1, and H an upper bound to both $|f(x)|$ and $|g(x)|$. There is a subdivision $\kappa_2 \supset \kappa_1$ in each closed interval of which either f or g has oscillation less than $\epsilon/(4n_1H)$ [*cf.* **9**, 265]. Given points

$$a = x_0 \leqslant \xi_1 \leqslant x_1 \leqslant \xi_2 \leqslant x_2 \leqslant \ldots x_{k-1} \leqslant \xi_k \leqslant x_k = b$$

of which the x_i are distinct points which include all points of κ_2, let

$$\phi = \sum_{i=1}^{k} f(\xi_i)(\bar{x}_{i-1} - \bar{x}_i - \dot{x}_i) + \sum_{i=0}^{k} f(x_i)\,\dot{x}_i$$

and $$\phi' = \sum_{i=1}^{k} f(\xi_i')(\bar{x}_{i-1} - \bar{x}_i - \dot{x}_i) + \sum_{i=0}^{k} f(x_i)\,\dot{x}_i,$$

where $\xi_i' = \xi_i$, unless ξ_i is a point of κ_1 in which case ξ_i' is to be some point in (x_{i-1}, x_i) not of κ_1. There are not more than $2n_1$ values of i at which

† It is easy to deduce from Theorems 18 and 19 that the sum of two functions of $W_q{}^*$, such as f and $-\phi$, is itself of $W_q{}^*$.

ξ_i' differs from ξ_i, therefore

$$|L(\phi'-\phi)| = \left| \sum_{i=1}^{k} \Big(f(\xi_i')-f(\xi_i)\Big)\Big(g(x_i)-g(x_{i-1})\Big) \right| \leqslant 2n_1\, 2H\, \epsilon/(4n_1 H) = \epsilon.$$

Further $$|L(f-\phi')| \leqslant MV_q(f-\phi') < \epsilon \qquad \text{(Theorem 18)}$$

as before. And so

$$\left| L(f) - \sum_{i=1}^{k} f(\xi_i)\Big(g(x_i)-g(x_{i-1})\Big) \right| = |L(f-\phi)|$$

$$\leqslant |L(f-\phi')| + |L(\phi'-\phi)| < 2\epsilon.$$

Therefore the Moore-Pollard integral exists, and

$$L(f) = \int f\,dg.$$

References.

1. Hardy, G. H., "Weierstrass's non-differentiable function", *Trans. American Math. Soc.*, 17 (1916), 301–325.
2. Hardy, G. H., and Littlewood, J. E., "Some properties of fractional integrals, I" *Math. Zeitschrift*, 27 (1928), 565–606.
3. Hardy, G. H., and Littlewood, J. E., "A convergence criterion for Fourier series", *Math. Zeitschrift*, 28 (1928), 612–634.
4. Hardy, G. H., and Littlewood, J. E., "Some new cases of Parseval's theorem", *Math. Zeitschrift*, 34 (1932), 620–636.
5. Hardy, G. H., Littlewood, J. E., and Polya, G., *Inequalities* (Cambridge, 1934).
6. Love, E. R., and Young, L. C., "Sur une classe de fonctionelles linéaires", *Fund. Math.*, 28 (1937), 243–257.
7. Titchmarsh, E. C., *Theory of functions* (Oxford, 1932).
8. Weyl, H., "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung", *Vierteljahrsschrift d. naturf. Ges. in Zürich*, 62 (1917), 296–302.
9. Young, L. C., "An inequality of the Hölder type, connected with Stieltjes integration", *Acta Math.*, 67 (1936), 251–282.
10. Young, L. C., "Inequalities connected with binary derivation, integrated Lipschitz conditions and Fourier series", *Math. Zeitschrift*, 43 (1937), 255–270.

Trinity College,
Cambridge.

SOME THREEFOLDS OF GENERA ZERO

By LEONARD ROTH.

[Received 28 August, 1936.—Read 12 November, 1936.—Received in revised form 1 January, 1937.]

One of the outstanding problems in algebraic geometry is that of determining necessary and sufficient conditions for the rationality of a threefold; it is complicated by the existence of threefolds whose genera and plurigenera are all zero, but which are nevertheless known to be irrational. It has been suggested by Castelnuovo (in conversation with the author) that this fact makes it probable that the rationality of a threefold depends not only upon its genera but also upon the values of its moduli; if this is the case, the search for conditions of rationality may prove an arduous one.

A notable class of threefolds of genera zero, introduced by Fano, consists of the $V_3^{2\pi-2}$ of $S_{\pi+1}$ whose prime sections are regular surfaces of genera unity. The type $\pi = 3$ is the general quartic primal of S_4, and the type $\pi = 4$ is the complete intersection of a quadric and cubic primal of S_5. These varieties have been shown by Fano* to be irrational and birationally distinct, thereby proving the assertion made above; and from the latter variety Enriques† has deduced the existence of irrational involutions of S_3, although the involutions so obtained prove to be of a high order.

The present paper continues the investigation of these threefolds, taking for point of departure the interesting review of the subject given by Fano‡. First we establish a number of general results, and show that the threefolds $V_3^{2\pi-2}$ of general type contain only surface intersections with

* Fano (4), (5).
† Enriques (3).
‡ Fano (6).

primals. We obtain all the general types for $\pi \leqslant 9$, and some interesting special types for $\pi = 8, 9, 13$. We show that all these threefolds can be transformed into particular quartic primals of S_4 (save the type $\pi = 4$) which, for increasing values of π, have increasingly complex singularities. It is found, also, that all the types, with the exception of $\pi = 3$, can be represented on a double space S_3 having a sextic branch-surface. In § 5 we obtain for purposes of comparison other examples of quartic primals which occupy a position intermediate between the general types of $V_3^{2\pi-2}$; we find also that the general cubic primal of S_4 enters into the present series, as a specialised type.

Finally, in § 7, we give a brief summary of the principal results; these tend to confirm Fano's conjectures, in particular, that the varieties $V_3^{2\pi-2}$ form a limited number of birationally distinct types and that they are in general irrational, although they appear to approach rationality for increasing values of π.

1. Let V be a non-singular threefold of order $2\pi-2$ and sectional genus π, situated in $S_{\pi+1}$, such that its prime sections $|F|$ are regular surfaces of genera unity. These surfaces are characterised by the fact that they possess an i-canonical curve of order zero ($i = 1, 2, 3, \ldots$); hence the system $|F'|$ of first adjoints, and all its multiples, are of order zero. It follows that the geometrical genus P_g and all the plurigenera of V are zero. Moreover, the superficial irregularity of V is zero, since it is zero for the system $|F|$; in order to show that V is completely regular, it remains to prove that the tridimensional irregularity $P_g - P_a$ is also zero.

For this purpose we determine the characters of the system $|kF|$, where k is any integer; by the ordinary formulae* the grade, curvilinear, and arithmetic genera of the system are respectively

$$\left.\begin{aligned} n_0^{(k)} &= 2k^3(\pi-1), \\ n_1^{(k)} &= 2k^2(k-1)(\pi-1)+1, \\ n^{(k)} &= \binom{k}{3}(2\pi-3)+\binom{k}{2}\pi+k+\binom{k-1}{3}. \end{aligned}\right\} \tag{1}$$

Now the virtual canonical system of V is represented by $|F'-F|$, *i.e.* the system $|-F|$. Hence, from (1), its characters are

$$\Omega_0 = -2(\pi-1), \quad \Omega_1 = 1-3(\pi-1), \quad \Omega_2 = -(\pi+2). \tag{2}$$

* Severi (**14**).

Substituting these in the formula

$$2P_a = \Omega_0 - \Omega_1 + \Omega_2 + 4, \tag{3}$$

we find that $P_a = 0$.

Thus, finally, V is a completely regular threefold of genera zero.

2. *Some general properties.*

An arbitrary prime section C of F is a canonical curve of genus π. If we assume that this is of general type, it is known to lie on $\binom{\pi-2}{2}$ linearly independent quadrics $V^2_{\pi-2}$ of $S_{\pi-1}$ ($\pi > 3$). Since F is normal in S_π, the system $|C|$ of its prime sections is complete and of dimension π. Now the quadrics of S_π passing through one of the quadrics $V^2_{\pi-2}$ which contain C form a system of dimension $\pi+1$, and cut out on F a residual system contained in $|C|$. Hence it follows that one (at least) of these quadrics must contain F; so we may obtain $\binom{\pi-2}{2}$ linearly independent quadrics of S_π which contain F. Reasoning in the same manner with regard to these latter, we obtain the result:

If C is a general canonical curve, V lies on $\binom{\pi-2}{2}$ linearly independent quadrics of $S_{\pi+1}$.

2.1. We now make the further assumption that $|C|$ is the system of minimum genus, without base-points, contained in a generic section F. Then it is known that:

(i) Every other system contained in F is a multiple of $|C|$; and every system, without base-points, is regular.

(ii) The postulation formula for primals $V^l_{\pi-1}$ applies to F for all values of l.

We now show that*

The primals of order k determine, for all values of k, a complete system of surfaces on V.

* Here we may assume that $\pi > 3$, since the following results are known to hold for $\pi = 3$.

Let $\nu(h, \pi)$ denote the number of linearly independent primals $V^h_{\pi-1}$ of S_π which contain F, so that, by (ii),

$$\nu(h, \pi) = \binom{h+\pi}{\pi} - \left\{2(\pi-1)\binom{h+1}{2} - h(\pi-1)+2\right\}, \tag{4}$$

for all values of h.

If $\nu(h, \pi+1)$ is the number of linearly independent primals V^h_π containing V, we have

$$\nu(h, \pi+1) - \nu(h-1, \pi+1) = \nu(h, \pi) \quad (\text{all } h). \tag{5}$$

Writing $h = 2, 3, \ldots, k$ in (5) and adding the equations so obtained, we derive

$$\nu(k, \pi+1) = \binom{k+\pi+1}{\pi+1} - d(k) - 1, \tag{6}$$

where $d(k)$ is the virtual dimension of $|kF|$, namely

$$d(k) = n_0^{(k)} - n_1^{(k)} + n_2^{(k)} + 2, \tag{7}$$

and where the characters are given by (1).

It follows from (6) that, in order to establish the required result, we have only to show that the system $|kF|$ is regular for all values of k. This we show by induction. In the first place, $|F|$ is known to be regular. Consider the system $|2F| = |F+F|$; this is such that $|2F|$ is the adjoint of some system, namely $|3F|$, and cuts on F a system of curves of zero redundance. Hence, according to the criterion given by B. Segre*, $|2F|$ is regular. Similarly, the system $|3F| = |2F+F|$ is found to be regular; and so on, until the system $|kF|$ is obtained. Thus, finally, the theorem enunciated is established.

2.2. Hence follows the result:

Every surface on V is the complete intersection of V with a primal.

First, suppose that Φ is any non-singular surface on V; its section (ϖ, Φ) by an arbitrary prime ϖ lies on a surface F and, by (i), must be the complete intersection of F with a primal in ϖ, of some order k. Moreover, since any system of curves on F is self-adjoint, it follows that, on the surface Φ, canonical curves are cut out by primals of order $k-1$.

* B. Segre (13), 73.

On the other hand, since the virtual canonical system on V is given by $|-F|$, it follows that canonical curves are also cut out on Φ by the surfaces $|\Phi-F|$. Hence, equivalent systems are cut out on Φ by the surfaces $|(k-1)F|$ and $|\Phi-F|$; and hence, provided that Φ can vary in a system at least ∞^1, it follows from Severi's criterion* that the two systems of surfaces are themselves equivalent, so that $|\Phi| \equiv |kF|$; whence, by § 2.1, Φ is the complete intersection of V with a primal of order k.

In any case, we have the result that, on any prime section (ϖ', Φ) of Φ, the surfaces $|(k-1)F|$ and $|\Phi-F|$ cut equivalent sets. Now the curve (ϖ', Φ) is also a complete intersection, and is one of the complete system cut, on the section F_1 of V by ϖ', by primals of order k in ϖ'. Since, for varying ϖ', the systems of surfaces $|(k-1)F|$ and $|\Phi-F|$ cut equivalent sets on the curve (ϖ', Φ), it follows that these surfaces cut equivalent systems of curves on F_1. Hence $|(k-1)F| \equiv |\Phi-F|$, and $|\Phi| \equiv |kF|$, whence the desired result follows from § 2.1.

Suppose now that Φ possesses singularities; then primals of arbitrarily high order l which are drawn through Φ meet V again in a system of surfaces $|\Psi|$ which are irreducible and non-singular; to these we apply the preceding result, that they must be intersections of V with primals of some order m. Thus $|\Psi| \equiv |mF|$. But we have $|\Phi+\Psi| \equiv |lF|$, whence

$$|\Phi| \equiv |(l-m)F| \equiv |kF|,$$

say $(k > 0)$. Hence the theorem is completely established.

We note, in conclusion, that if a surface Φ is cut out on V by a primal of order k, then its adjoints are cut out by primals of order $k-1$.

2.3. *Projection of V from a curve.*

Suppose that V is projected from a normal non-special curve c, of order ν and genus ρ, lying in S_r $(r \leqslant \pi-3, \pi > 4)$. We have, then,

$$r = \nu - \rho. \tag{8}$$

A space $S_{\pi-1}$ drawn through c meets V again in a curve C' of genus π', say, intersecting c in i points, where

$$\pi = \pi' + \rho + i - 1. \tag{9}$$

* Severi (15).

Then the projection V' of V on $S_{\pi-r+1}$ has order $2\pi-2-\nu-i$ and sectional genus π'. Its prime sections are regular surfaces of genus unity and order $2\pi'-2$, whence

$$2\pi'-2 = 2\pi-2-\nu-i.$$

Hence, from (9), $$\nu = 2\rho-2+i. \tag{10}$$

Thus, from (8), (9), (10) it follows that $1+r = \pi-\pi'$, so that V' lies in $S_{\pi'+1}$.

The neighbourhood of c projects into a scroll of genus ρ and order i. The projection V' of V therefore forms one of the same series of threefolds, but it contains a scroll which is not present in the more general type having sectional genus π'.

Hence, in conclusion,

The projection of V from a normal non-special curve of order ν and genus ρ, lying in S_r, is of order $2(\pi-r-2)$ and sectional genus $\pi-r-1$; it contains a scroll of order $\nu+2-2\rho$ and genus ρ.

The types $\pi = 3, 4$.

3. We turn now to consider some special cases. For $\pi = 3$ we have the general quartic primal V^4 of S_4. Fano has proved that this is irrational, in two different ways*: first, by showing that it cannot contain a homaloidal system of surfaces and secondly, by showing that the only system of regular surfaces of genera unity and dimension greater than unity which V^4 contains, is that of its prime sections—a property which suffices to distinguish V^4 from the space S_3. The results obtained may be summarized as follows:

The general quartic primal of S_4 is irrational; it contains no system of surfaces of genera unity and dimension exceeding unity, other than that of its prime sections. Hence the only birational transformations which it can possess are projective.

3.1. For $\pi = 4$ we have the intersection V_3^6 of a quadric and cubic primal of S_5. This has been shown by Fano* to be irrational and, by virtue of the known properties of V^4, birationally distinct from the latter, since V^4 contains no system of surfaces of genera unity and dimension five. Evi-

* Fano (**4**), (5).

dently V_3^6 contains ∞^1 lines; the spaces S_3 passing through a line l meet V_3^6 again in quintics of genus 2, trisecant to l. Hence the ∞^3 planes through l meet V_3^6 again in pairs of points of an involution I_2; it follows, therefore, that V_3^6 admits of at least ∞^1 birational self-transformations other than possible projective transformations. We have, in conclusion:

The V_3^6 of S_5 which is the intersection of a quadric and cubic primal of general type, situated in general position, is irrational. It contains no system of surfaces of genera unity and dimension at least two, other than that of its prime sections and the systems derived from it by projection from the lines of V_3^6. Hence, any birational self-transformation of V_3^6 is the product of a finite number of such projections and, possibly, of projective transformations.

3.2. From a line V_3^6 projects into a double S_3 having for branch-surface a special type of sextic; such a double space is, therefore, irrational. The double space with a general sextic branch-surface (which corresponds to the type $\pi = 2$ of the threefolds under discussion) must include this as a special case, whence it follows that

The double S_3 with a general sextic branch-surface is irrational.

Now a quartic primal V^4 of S_4 which possesses a node projects from it into a double S_3 having for branch-surface a sextic which, as regards particularity, occupies a position intermediate between the above-mentioned surfaces. Hence

The quartic primal of S_4 whose only singularity is a node is irrational.

3.3. It was by means of V_3^6 that Enriques* first established the existence of an irrational involution in S_3, without, however, obtaining the order of the involution so constructed. Later, Aprile† showed that Enriques' method leads to an involution of order 216, but that, by a modification of the procedure, V_3^6 can be represented on S_3 by an involution of order 36.

It is natural to compare the above results with those obtained for the general cubic primal V^3 of S_4. It is known that the latter can be represented on S_3 by an involution of order 2; thus, apart from § 3.1, it is suspected,

* Enriques (3).

† Aprile (1).

and, as we shall see, there is strong reason for supposing that V_3^6 is far more "general" than V^3. In fact, we may show* that:

If V_3^6 acquires three planes of the same system of the containing quadric, it becomes birationally identical with V^3.

For, consider the threefold represented on V^3 by quadric sections which pass through three skew lines. The system of surfaces so obtained is of dimension 5, curvilinear genus 4, and grade 6; its image is a V_3^6 containing three planes corresponding to prime sections of V^3 through pairs of the base-lines.

Thus it is necessary to specialize V_3^6 very considerably in order to identify it with V^3.

3.4. We examine now some representations of V_3^6 upon primals of S_4. From an external point V_3^6 projects into a V^6 having a double surface ${}^4F^6$, intersection of a quadric and cubic primal of S_4. From § 2.2 and the known properties of V_3^6, it follows that

The sextic primal of S_4 having a double ${}^4F^6$ is irrational, and contains only complete intersections with primals.

From a point of itself, V_3^6 projects into a V^5 having a double quadric and an exceptional plane cospatial with it. Thus *the quintic primal of S_4 having a double quadric is irrational; on it a base is provided by the quadric and the exceptional plane.* These results may be obtained by representing a quadric of S_5 by means of quadrics of S_4 passing through a quadric surface. But the quadric may also be represented on S_4 by quadrics passing through a rational quartic curve ${}^0C^4$. Hence it follows that

The sextic primal of S_4 having a triple rational quartic is irrational, and contains only complete intersections with primals.

If the quadric containing V_3^6 is a line-cone, it possesses two systems of generating spaces S_3, and may be represented on S_4 by quadrics passing through two cospatial planes. In this case V_3^6 has three double points and contains two pencils of cubic surfaces; it may be transformed into a primal V^6 of S_4 having two cospatial triple planes, or, if projected from a point of itself, into a primal V^5 with two cospatial double planes. Whether this

* Fano (6).

type of V_3^6 is irrational or not is not known; the proof given in the general case cannot be applied here.

4. For $\pi = 5$, we have the intersection V_3^8 of three quadrics of S_6. Using Fano's method, we can show that if this contains a homaloidal system of surfaces cut out by primals of order n, the surfaces must have a base-line of multiplicity greater than n with, possibly, other singular points and curves. All such singular points must be of multiplicity less than $2n$ and all such singular curves of multiplicity less than n. Thus we succeed in narrowing the question of rationality very considerably, although the conclusion remains doubtful.

By § 2 . 3, if V_3^8 is projected from one of its ∞^1 lines, we obtain a primal V^4 which contains a cubic scroll R and, consequently, seventeen nodes; and conversely, a general quartic primal of this type can be transformed into V_3^8. Thus,

A quartic primal of S_4 which contains a cubic scroll (and, consequently, seventeen nodes) is birationally identical with V_3^8. On it a base is provided by a prime section and the scroll.

5. *Some quartic primals of S_4.*

In relation to those quartic primals which we have considered and others which appear later, it is worth while examining some other types. It is well known that a primal V^4 is rational if it contains a surface, reducible or irreducible, having one apparent triple point; the correspondence between V^4 and S_3 is then obtained by projection. For example, if V^4 contains a Bordiga surface ${}^3F^6$ it is rational*; so also, if it contains three skew planes†.

Among the quartic primals having double curves we can specify two types which are certainly rational and which will serve as criteria in the sequel. In the first place,

If V^4 possesses a double rational quartic, it is rational.

For, as in § 3 . 4, we can represent a quadric of S_5 by means of quadrics of S_4 passing through a curve ${}^0c^4$. It follows that a Del Pezzo V_3^4 can be represented on a V^4 for which ${}^0c^4$ is double, so that the latter must be

* Roth (11).
† Roth (12).

rational. In the second place,

If V^4 possesses a tacnodal line, it is rational.

An arbitrary prime section of V^4 is a rational quartic surface and hence, by Fano's theorem*, V^4 must itself be rational.

5.1. We turn now to the case where V^4 possesses a double elliptic quartic. Consider first the system of quadrics of S_4 which pass through a curve ${}^1c^4$. The characteristic curves of the system are elliptic quartics meeting ${}^1c^4$ in four points. Hence the system represents a Del Pezzo $V_4{}^4$ of S_6. It is seen, however, that the ∞^1 quadric surfaces containing ${}^1c^4$ correspond to the points of a double line on $V_4{}^4$; it follows that

A V^4 with a double elliptic quartic is birationally equivalent to a $V_3{}^8$ of S_6 having two double points.

Whether or not this type of V^4 is rational, there certainly exist subspecies which are rational and for which the double curve consists of four cospatial lines. For example, it is easily shown that the V^4 whose equation in homogeneous coordinates is

$$x_1{}^2x_2{}^2-x_3{}^2x_4{}^2=x_5{}^3(x_5+a),$$

where a is a linear function of the coordinates, is rational.

6. With the examples given in the previous section for comparison, we continue the examination of the threefolds $V_3^{2\pi-2}$ $(\pi>5)$. The types $\pi=6$, 7 are obtained as the intersections of quadrics with Del Pezzo fourfolds, since a generic prime section is then a general surface of the requisite type.

The Del Pezzo $V_4{}^5$, section of the Grassmannian of lines of S_4, is rational, and can be represented on S_4 by means of quadrics passing through a rational cubic curve†. Since V_3^{10} is the intersection of $V_4{}^5$ with a quadric, it follows that

V_3^{10} may be transformed into a quartic primal V^4 of S_4 having a double rational cubic. V^4 contains only complete intersections with other primals.

* Fano (7).

† Todd (16).

For, by §2.2, V_3^{10} contains only complete intersections, and the given representation on V^4 introduces no exception to this rule.

The prime containing the double curve meets V^4 in a rational quartic scroll R^4; this contains an infinity of conics bisecant to the double curve, which therefore correspond to conics on V_3^{10}. From one of these conics V_3^{10} projects into a primal V^4 of S_4 containing a rational quartic scroll (§2.3) and, therefore, twenty-one nodes at points of this scroll. Hence,

V_3^{10} may be projected into a primal V^4 of S_4 containing a rational quartic scroll R^4. On V^4, a base is provided by a prime section and R^4.

6.1. The type $\pi = 7$ is obtained* as the intersection of a quadric with a Segre $V_4{}^6$ of S_8. Since the latter can be represented on S_4 by means of quadrics passing through two skew lines, it follows that

V_3^{12} may be transformed into a primal V^4 of S_4 with two skew double lines.

From this representation on V^4 it is clear that V_3^{12} contains an infinity of lines, conics, and rational cubics. By projection from one or more of these curves we obtain other representations on quartic primals of S_4. Thus, by §2.3,

V_3^{12} projects from a rational cubic into a primal V^4 containing a rational quintic scroll and, consequently, twenty-four nodes.

From a pair of skew lines V_3^{12} projects into a primal V^4, the neighbourhood of each line giving rise to a cubic scroll. Each of these scrolls imposes seventeen nodes on V^4, but nine nodes of each set coincide with the common points of the scrolls. Hence,

V_3^{12} projects from two skew lines into a primal V^4 containing two cubic scrolls and, therefore, twenty-five nodes.

6.2. The preceding constructions for $V_3^{2\pi-2}$ are, in effect, based on the fact that, for $\pi < 8$, a surface $F^{2\pi-2}$ of genera unity and general type (that is, containing no linear system of curves, without base-points, of genus less than π) can be obtained as the intersection of a quadric and a Del Pezzo threefold. It does not seem to have been pointed out before that, while similar constructions exist for $\pi \geqslant 8$, they do not lead to general surfaces

* Fano has since shown that there exists a second type of V_3^{12} which is rational and which contains only complete intersections.

of the required type. For instance, in the case $\pi = 9$, we may obtain a surface F^{16} as the intersection of a quadric and a Del Pezzo ${V_3}^8$. Since the latter is the Veronesean of quadrics of S_3, it follows that F^{16} *is birationally identical with the general quartic surface of* S_3. Hence, for $\pi \geqslant 8$, we cannot obtain general types of threefolds $V_3^{2\pi-2}$ (supposing that they exist) by imitating this procedure.

Before considering the question of existence in the general case, we may note some interesting special types for which $\pi > 7$. One species of V_3^{14} has been obtained by Fano* as the Grassmannian of lines common to a linear system ∞^4 of linear complexes of S_5; this threefold contains only complete intersections and is birationally identical with the general cubic primal V^3 of S_4, prime sections of V_3^{14} corresponding to sections of V^3 by primals of order 7 having for quadruple base-curve an elliptic quintic.

However, a V_3^{14} birationally identical with this can be obtained far more simply. The quadrics of S_4 which contain a given conic c represent the prime sections of a ${V_4}^8$ of S_9 having sectional genus 2. To primes containing c correspond quadrics ${V_3}^2$. Then, since a primal V^4 for which c is double may break up into a prime and a V^3, both containing c, it follows that

The intersection of ${V_4}^8$ *with a quadric, residual to a quadric threefold, is a* V_3^{14} *which is birationally identical with the cubic primal of* S_4.

By taking a S_7-section of the whole figure and representing the rational surface ${V_2}^8$ on a plane, it is verified that the curve sections of V_3^{14} are general canonical curves; but the surface sections, which are represented by quadric sections of V^3 through c, are evidently not general.

The most general type of Del Pezzo ${V_3}^7$ is the projection of the Veronesean ${V_3}^8$ mentioned above; hence, the V_3^{14} obtained as the intersection of a quadric and the cone standing on ${V_3}^7$ is only a special case of a type which we now consider.

6.3. The cone standing on ${V_3}^8$ may be represented on S_4 by quadrics which touch a given S_3 at a given point; hence the intersection of a quadric with the cone will be represented by a primal V^4 with a tacnode at the given point. Thus

A special type of V_3^{16} *is birationally identical with a primal* V^4 *having a tacnode.*

* Fano (8).

Now V^4 projects from the tacnode into a double space S_3 having a quartic branch-surface; whence

V_3^{16} may be represented on a double S_3 with a quartic branch-surface.

This result gives some idea of the degree to which V_3^{16} is specialized, since all the general threefolds $V_3^{2\pi-2}$ save V^4 project into double spaces whose branch-surfaces are sextics with various singularities.

If the quartic branch-surface acquires a trope, it becomes identical with the general cubic primal and hence with the V_3^{14} discussed in § 6.2. It is therefore probable that V_3^{16} is birationally distinct from V_3^{14} and hence also from the cubic primal.

6.4. The question arises whether there exist types more general than those we have given for $\pi = 8, 9$. If we make the assumption that the general V_3^{14} contains a line (and all the types hitherto considered contain ∞^1 lines) and project from it on to S_7, we obtain a V_3^{10} which contains a cubic scroll. Conversely, such a threefold is easily constructed; for a Del Pezzo V_4^5 contains an infinity of cubic scrolls, and a quadric of S_7 not passing through V_4^5 can be made to contain one of these scrolls, which imposes twelve conditions on the quadric.

Similarly, if we assume that the general V_3^{16} contains a line, the projection from it is a V_3^{12} of S_8 containing a cubic scroll; such threefolds certainly exist and can be constructed in the manner described. It follows, then, that, for $\pi \leqslant 9$, there exist threefolds $V_3^{2\pi-2}$ whose surface sections are of general type.

6.5. The various types which we have found to be equivalent birationally to the cubic primal V^3 of S_4 are all transforms of the V_3^{24} of S_{14} represented by the quadric sections of V^3. In view of the assumptions made in § 6.4, we note that this threefold contains ∞^2 conics but no lines; through each point of V_3^{24} there pass six conics, corresponding to the lines of V^3, so that, if a projection is made from a curve, the projected threefold contains one scroll arising from the neighbourhood of this curve, and another from the conics meeting it.

We have seen in § 3.3 that V_3^{24} can be projected into a special type of V_3^6. It can also be projected into a primal V^4 of S_4. For the quadrics containing an elliptic quintic of V^3 cut out a system of surfaces which represent a V^4; the base-curve corresponds to an elliptic scroll R of order 10, which has twenty-five improper nodes but imposes no other nodes on

V^4. Hence

V^3 is birationally identical with a primal V^4 containing an elliptic decimic scroll and, in consequence, twenty-five nodes.

7. In this section we examine the conclusions which may be drawn from the previous results, taken as a whole.

The threefolds V^4 and ${V_3}^6$ of genera zero, have been termed "semi-rational" by Fano, since they occupy a position between the rational threefolds and those having the genus or some plurigenus greater than zero. and the representations that we have given for the types $5 \leqslant \pi \leqslant 9$, when compared with the results of § 5, indicate that these also are probably irrational. We shall, for convenience, use the name "semi-rational" for all threefolds of the series.

The threefolds $V_3^{2\pi-2}$ whose prime sections possess pluricanonical curves of order zero have an obvious analogy with the Del Pezzo surfaces; and since the latter form a finite series, the question arises whether the semi-rational threefolds which are not cones also consist of a finite number of types. In support of this conjecture we have shown that, with the exception of ${V_3}^6$, all the threefolds considered are birationally identical with quartic primals of S_4, and that all, with the exception of V^4, can be represented on a double S_3 with a sextic branch-surface. Moreover, as π increases, the number of singularities imposed on the primal or the branch-surface, as the case may be, increases as well; so that, if all semi-rational threefolds can be so represented, the result follows.

7.1. It may be pointed out that it is possible to construct another series of $V_3^{2\pi-2}$, which we may call semi-rational three-folds of the second species. It is known* that there exist regular surfaces $F^{2\pi-2}$ of genera unity whose sections are canonical curves containing a ${g_3}^1$; such surfaces are obtained as the intersection of a cubic primal of S_π with a rational $V_3^{\pi-2}$ generated by ∞^1 planes, residual to $\pi-4$ planes ($\pi > 4$). In analogy with these surfaces we may consider the $V_3^{2\pi-2}$ obtained as the intersection of a cubic primal of $S_{\pi+1}$ with a rational $V_4^{\pi-2}$ generated by ∞^1 spaces S_3, residual to $\pi-4$ such spaces ($\pi > 4$). These threefolds contain a rational pencil of cubic surfaces. We may represent the generating spaces of $V_4^{\pi-2}$ by a pencil of spaces containing a plane ϖ of S_4; its prime sections correspond to plane-forms $V_3^{\pi-2}$, for which ϖ is $(\pi-3)$-ple, passing through $\pi-3$ planes ϖ_i ($i = 1, 2, \ldots$) cospatial with ϖ; this representation

* Du Val (2).

is, of course, analogous to that of the rational planar threefolds. It follows, then, that $V_3^{2\pi-2}$ is represented on S_4 by a primal of order $2\pi-2$ for which ϖ is $(2\pi-5)$-ple and the planes ϖ_i are triple. Hence, in conclusion,

The semi-rational threefold of the second species, whose curve sections contain a g_3^1, is birationally identical with a primal $V^{2\pi-2}$ of S_4 containing a $(2\pi-5)$-ple plane and $\pi-3$ triple planes cospatial with it $(\pi > 4)$.

This type of threefold, which does not seem to have been studied before, is apparently birationally distinct from the semi-rational threefolds of the first species; and it appears likely that the various types $\pi = 5, 6, 7, \ldots$ are birationally distinct from one another*.

There is yet another type having $\pi = 7$ which seems to be distinct from the threefolds of both the first and second species. Consider the fourfold represented on S_4 by means of quadrics with a common plane; evidently this is a rational V_4^5, normal in S_8, generated by ∞^1 spaces S_3. It follows that *the intersection of V_4^5 with a cubic primal, residual to three generating spaces, is a V_3^{12} of sectional genus* 7, *representable on the general cubic primal of S_4.*

For comparison we note that the V_3^{12} of the second species is represented by a primal V^{12} having a multiple plane of order 9 and four triple planes cospatial with it.

Incidentally, we observe that the intersection of V_4^5 with a cubic primal, residual to two generating spaces, is birationally identical with a primal V^4 containing a plane.

7.2. Assuming that the threefolds of the first species are all irrational, one nevertheless observes in them an apparent approach to rationality with increasing values of π. Thus, we have obtained representations on primals V^4 or double spaces S_3 whose specialisation increases with π; there are the representations on involutions of S_3 of decreasing order; and there are the increasingly wider groups of birational self-transformations of which $V_3^{2\pi-2}$ is capable. Further, there is the increasing dimension of the system of surfaces having unit genera, which $V_3^{2\pi-2}$ contains: for $\pi = 3$, the system is that of the prime sections; for $\pi = 4$, the maximum dimension is 5; and for $\pi = 5$ it seems that the dimension is 6, while it is known that for a rational threefold the dimension of such a system is at least 34 (that of all quartic surfaces of S_3).

* Fano has now shown that the series of such threefolds is a finite one.

7.3. In conclusion, we note that there exist other types of threefolds of genera zero which are, presumably, distinct from the preceding. One such type, mentioned by Fano*, consists of the primals V^n of S_4 having an $(n-2)$-ple line l. To calculate the invariants of V^n, we first find the value of Ω_0, using James's formulae† for the intersection of four primals with a common multiple curve. If the primals are of order m_i $(i = 1, 2, 3, 4)$ and contain an r_i-ple line, they meet elsewhere in I points, where

$$I = m_1 m_2 m_3 m_4 - \Sigma m_1 r_2 r_3 r_4 + 3 r_1 r_2 r_3 r_4.$$

On V^n the virtual canonical system is cut out by primals of order $n-5$, passing $n-4$ times through l; hence the virtual grade Ω_0 of the system is obtained by putting $m_1 = n$, $m_2 = m_3 = m_4 = n-5$, $r_1 = n-2$, $r_2 = r_3 = r_4 = n-4$ in the equation above; this gives

$$\Omega_0 = n(n-5)^3 - (n-4)^2 (n-6)(n-1) = -3n^2 + 35n - 96.$$

Substituting in (3), where $P_a = 0$ and $2\Omega_1 - 2 = 3\Omega_0$, we have

$$\Omega_2 = -\tfrac{1}{2}(n-6)(3n-17).$$

For $n = 4$, we obtain the primal V^4 with a double line; but for $n > 4$, it would seem that V^n does not contain any surfaces of genera unity, in which case it is not only irrational but birationally distinct from $V_3^{2\pi-2}$. Another class of threefolds of genera zero which may be worth investigation is that of the primals V^n of S_4 having an $(n-3)$-ple plane; if the threefolds considered in § 7.1 are irrational, then it may be supposed that the former are even further removed from rationality.

[*Added* 15 *November*, 1937. Since the above paper was written there have appeared two works by Fano‡ which make important contributions to the subject. The main conclusions reached are that threefolds whose curve sections are canonical curves of genus π exist only for values of $\pi \leqslant 37$, while, for $\pi > 10$, they are rational. The rationality or irrationality of the types considered above thus remains unsettled; so also does the question of the existence of types, for values of $\pi > 8$, containing only complete intersections with primals.]

---

* Fano (6).

† James (9), 458.

‡ Fano, *Scritti offerti a L. Berzolari* (1936), 329; *Mem. R. Accad. d'Italia*, 8 (1937), 23.

References.

1. Aprile, *Rassegna di Mat. e Fis.*, 1 (1921), 133.
2. Du Val, *Rend. Lincei* (6), 15 (1932), 276.
3. Enriques, *Rend. Lincei* (5), 21 (1912), 81.
4. Fano, *Atti Acc. Torino*, 43 (1908), 973.
5. ————, *ibid.*, 50 (1914), 1067.
6. ————, *Atti Congresso Internaz. Bologna*, IV (1928), (1931), 115.
7. ————, *Annali di Mat.* (3), 24 (1915), 49.
8. ————, *Rend. Lincei* (6), 11 (1930), 329.
9. James, *Proc. Camb. Phil. Soc.*, 21 (1923), 435.
10. Pannelli, *Rend. Lincei* (5), 15 (1906), 620.
11. Roth, *Proc. London Math. Soc.* (2), 30 (1930), 118.
12. ————, *ibid.* (2), 35 (1933), 249.
13. B. Segre, *Quelques résultats nouveaux dans la géométrie sur une* V_3 *algébrique* (Brussels, 1936).
14. Severi, *Rend. Palermo*, 28 (1909), 33.
15. ————, *Atti Ist. Veneto*, 70 (1910), 373.
16. Todd, *Proc. London Math. Soc.* (2), 30 (1930), 513.

Imperial College of Science,
London, S.W.7.

ON INFINITE SOLUBLE GROUPS—I

By K. A. Hirsch.

[Received and read 20 May, 1937.]

Introduction.

An infinite group $\mathfrak{G}$ is called "soluble"*, if its derived series terminates in the 1-element after a finite number of steps or, what is equivalent, if it possesses at least one finite series of subgroups

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_l = 1,$$

each $\mathfrak{G}_i$ self-conjugate in the preceding one, leading from $\mathfrak{G}$ to the 1-element, the factor-groups $\mathfrak{G}_i/\mathfrak{G}_{i+1}$ of which are all Abelian groups.

Very little is known about infinite soluble groups. Indeed, this definition covers so wide a field that no attempt has been, nor is likely to be, made to establish a systematic theory of these groups, since each of the Abelian factor-groups $\mathfrak{G}_i/\mathfrak{G}_{i+1}$ can be of an arbitrarily complicated structure.

We therefore propose to introduce an additional finiteness condition which restricts the class of groups with which we are dealing so that it becomes narrow enough to be treated on the same lines as in the theory of finite soluble groups, yet leaves enough generality to ensure that the results remain interesting intrinsically and as applied to some actual examples of infinite soluble groups.

There are, of course, many different ways of selecting such a finiteness condition. Some of the more obvious choices lead, as we shall see, to one and the same class of groups, which we have called S-groups and which form the subject of this paper.

* O. Schreier, "Ueber den Jordan-Hölderschen Satz", *Abh. Math. Sem. Hamburg*, 6 (1929), 300–303.

The following three definitions of S-groups are equivalent:

(i) S-groups are soluble groups in which the Abelian factor-groups of the derived or any other series of the above type all have finite numbers of generators.

(ii) S-groups are soluble groups satisfying the "maximal condition" or Teiler-kettensatz for subgroups, *i.e.* in which every increasing chain of subgroups has only a finite number of different terms.

(iii) S-groups are groups with a series of subgroups, each self-conjugate in the preceding one, of finite length and with cyclical factor-groups.

In the present introductory paper we give a proof of the equivalence of these definitions and consider some elementary properties of S-groups corresponding to the Jordan-Hölder and related theorems in the theory of finite groups. Subsequent papers will contain a detailed examination of special classes and examples of S-groups. My cordial thanks are due to Mr. P. Hall for his constant helpful interest and for many valuable suggestions.

1.1. *Definition.* A group $\mathfrak{G}$ is called an "S-group" if it possesses at least one finite series of subgroups,

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_l = 1, \tag{1.11}$$

each $\mathfrak{G}_i$ self-conjugate in the preceding one, leading from $\mathfrak{G}$ to the 1-element, the factor-groups $\mathfrak{G}_i/\mathfrak{G}_{i+1}$ of which are all Abelian groups with finite numbers of generators.

1.2. THEOREM. *The system of S-groups is closed with respect to subgroups and factor-groups. I.e. if $\mathfrak{G}$ is an S-group, and if $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$, then $\mathfrak{H}$ is an S-group; and if $\mathfrak{H}$ is self-conjugate in $\mathfrak{G}$, then $\mathfrak{G}/\mathfrak{H}$ is an S-group.*

For the proof of (1.2) we use the obvious

1.21. LEMMA. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{H}$ be any three subgroups of a group $\mathfrak{G}$, $\mathfrak{A}$ self-conjugate in $\mathfrak{B}$. Then $\mathfrak{A} \cap \mathfrak{H}$ is self-conjugate in* $\mathfrak{B} \cap \mathfrak{H}$; and if furthermore $\mathfrak{H}$ is self-conjugate in $\mathfrak{G}$, then $\{\mathfrak{A}, \mathfrak{H}\}$ is self-conjugate in* $\{\mathfrak{B}, \mathfrak{H}\}$.*

---

* If $\mathfrak{M}$ and $\mathfrak{N}$ are any two subgroups of a group $\mathfrak{G}$, we denote by $\mathfrak{M} \cap \mathfrak{N}$ the meet or common part of $\mathfrak{M}$ and $\mathfrak{N}$; and by $\{\mathfrak{M}, \mathfrak{N}\}$ the join of, or the group generated by, $\mathfrak{M}$ and $\mathfrak{N}$.

Proof of (1.2). If we put

$$\mathfrak{H}_i = \mathfrak{G}_i \cap \mathfrak{H},$$

$$\mathfrak{K}_i = \{\mathfrak{G}_i, \mathfrak{H}\} \quad (i = 0, 1, \ldots, l),$$

then, by the preceding lemma, $\mathfrak{H}_{i+1}$ is self-conjugate in $\mathfrak{H}_i$; and if $\mathfrak{H}$ is self-conjugate in $\mathfrak{G}$, $\mathfrak{K}_{i+1}$ is self-conjugate in $\mathfrak{K}_i$. The "isomorphism theorem"* gives

$$\mathfrak{H}_i/\mathfrak{H}_{i+1} \cong \{\mathfrak{H}_i, \mathfrak{G}_{i+1}\}/\mathfrak{G}_{i+1},$$

which, as a subgroup of $\mathfrak{G}_i/\mathfrak{G}_{i+1}$, is Abelian;

$$\mathfrak{K}_i/\mathfrak{K}_{i+1} \cong \mathfrak{G}_i/\mathfrak{G}_i \cap \mathfrak{K}_{i+1},$$

which, as a factor-group of a subgroup of $\mathfrak{G}_i/\mathfrak{G}_{i+1}$, is Abelian: Of the terms in the two series

$$\mathfrak{H} = \mathfrak{H}_0 \supseteq \mathfrak{H}_1 \supseteq \ldots \supseteq \mathfrak{H}_l = 1,$$

$$\mathfrak{G}/\mathfrak{H} = \mathfrak{K}_0/\mathfrak{H} \supseteq \mathfrak{K}_1/\mathfrak{H} \supseteq \ldots \supseteq \mathfrak{K}_l/\mathfrak{H} = 1,$$

the different ones, therefore, satisfy the conditions of the definition (1.1).

1.3. THEOREM. *S-groups satisfy the "maximal condition" or Teilerkettensatz for subgroups. I.e. an arbitrary increasing chain of subgroups of an S-group $\mathfrak{G}$ has only a finite number of different terms. Conversely, if an infinite soluble group $\mathfrak{G}$ satisfies the maximal condition for subgroups, it is an S-group.*

We shall apply the

1.31. LEMMA. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be any two subgroups of a group $\mathfrak{G}$, $\mathfrak{A}$ contained in $\mathfrak{B}$, and $\mathfrak{K}$ a self-conjugate subgroup of $\mathfrak{G}$. If $\mathfrak{K}$ has the same meet with $\mathfrak{A}$ as with $\mathfrak{B}$, and the same join with $\mathfrak{A}$ as with $\mathfrak{B}$, then $\mathfrak{A} = \mathfrak{B}$.*

Proof. We show that under the conditions of the lemma every element of $\mathfrak{B}$ belongs to $\mathfrak{A}$. An arbitrary element B of $\mathfrak{B}$ is contained in the join $\{\mathfrak{B}, \mathfrak{K}\} = \{\mathfrak{A}, \mathfrak{K}\}$, and consequently, $\mathfrak{K}$ being self-conjugate in $\mathfrak{G}$, is of the form $A.K$, where $A \in \mathfrak{A}$ and $K \in \mathfrak{K}$. This $K(= A^{-1}B)$ is also contained in $\mathfrak{B}$, therefore in the meet $\mathfrak{B} \cap \mathfrak{K} = \mathfrak{A} \cap \mathfrak{K}$, therefore in $\mathfrak{A}$. So we see finally that B, as the product of two elements of $\mathfrak{A}$, lies in $\mathfrak{A}$.

* *Cf. e.g.* van der Waerden, *Moderne Algebra*, I (1st. ed., 1930), 136.

Proof of 1.3. Let $\mathfrak{G}$ be an S-group and let

$$\mathfrak{H}_1 \leqq \mathfrak{H}_2 \leqq \ldots \tag{1.32}$$

be an arbitrary increasing chain of subgroups of $\mathfrak{G}$.

(i) We show that Theorem (1.3) holds for $\mathfrak{G}$, if it can be proved for a self-conjugate subgroup $\mathfrak{K}$ of $\mathfrak{G}$ and for its factor-group.

We put

$$\mathfrak{K}_i = \mathfrak{H}_i \cap \mathfrak{K},$$

$$\mathfrak{L}_i = \{\mathfrak{H}_i, \mathfrak{K}\} \quad (i = 1, 2, \ldots).$$

Then we have

$$\mathfrak{K}_1 \leqq \mathfrak{K}_2 \leqq \ldots,$$

$$\mathfrak{L}_1 \leqq \mathfrak{L}_2 \leqq \ldots,$$

therefore, by assumption, from some index h_1 onwards we have

$$\mathfrak{K}_{h_1} = \mathfrak{K}_{h_1+1} = \ldots,$$

and from some index h_2 onwards we have

$$\mathfrak{L}_{h_2} = \mathfrak{L}_{h_2+1} = \ldots.$$

Applying Lemma (1.31) for the series (1.32), we see that from $h = \max(h_1, h_2)$ onwards we have

$$\mathfrak{H}_h = \mathfrak{H}_{h+1} = \ldots.$$

(ii) Now consider the last term $\mathfrak{N} \supset 1$ of the derived series of $\mathfrak{G}$. This is an Abelian self-conjugate subgroup of $\mathfrak{G}$, and the maximal condition obviously holds for it. Therefore we may use induction over the length of the derived series of $\mathfrak{G}$ and assume that the maximal condition holds for the factor-group $\mathfrak{G}/\mathfrak{N}$. Then the above argument shows that it also holds for $\mathfrak{G}$, and the first part of Theorem (1.3) is proved.

(iii) Conversely, if an infinite soluble group $\mathfrak{G}$ satisfies the maximal condition for subgroups, then in its derived series the Abelian factor-groups must all have finite numbers of generators; *i.e.* $\mathfrak{G}$ is an S-group.

Exactly as in Abstract Algebra the maximal condition for subgroups is equivalent to the "basis theorem": that the group and all its subgroups have finite numbers of generators. Therefore, from Theorem (1.3) we obtain immediately

1.33. THEOREM. *S-groups and all their subgroups have finite numbers of generators. Conversely, if an infinite soluble group $\mathfrak{G}$ and all its subgroups have finite numbers of generators, then it is an S-group.*

1.4. Abelian groups with finite numbers of generators are direct products of cyclical groups. Therefore, a series of type (1.11) for an S-group $\mathfrak{G}$ may be refined by inserting further terms such that the factor-groups become cyclical groups and, if finite, of prime order.

Definition. Such a series will be called an "A-series" and the number of factor-groups in it the "length" of the A-series or, shortly*, "A-length" of $\mathfrak{G}$.

The A-series of an S-group correspond in some respect to the composition series of a finite soluble group. The chief series and characteristic series of such a group also have their analogues in S-groups.

THEOREM. *Any S-group $\mathfrak{G}$ possesses finite series of self-conjugate subgroups*

$$\mathfrak{G} = \mathfrak{N}_0 \supset \mathfrak{N}_1 \supset \ldots \supset \mathfrak{N}_k = 1 \tag{1.41}$$

leading from $\mathfrak{G}$ to the 1-element, the factor-groups $\mathfrak{N}_i/\mathfrak{N}_{i+1}$ of which are, if infinite, free Abelian groups, and if finite, elementary Abelian groups, i.e. of order p^n (p a prime) and type $(p, p, \ldots, p)$.

Proof. We proceed to form a series of type (1.41) in which the terms are not only self-conjugate, but characteristic subgroups of $\mathfrak{G}$. Let

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_l = 1$$

be any A-series for $\mathfrak{G}$. Consider the meet $\mathfrak{M}_1$ of the set of all groups

$$(\mathfrak{G}_1, \ \mathfrak{G}_1^{(1)}, \ \mathfrak{G}_1^{(2)}, \ \ldots)$$

upon which $\mathfrak{G}_1$ is mapped by the automorphisms of $\mathfrak{G}$. Since any automorphism must permute among themselves the groups of this set, their meet is left unchanged by it, and $\mathfrak{M}_1$ is a characteristic subgroup of $\mathfrak{G}$. Moreover, since all groups of the set have Abelian factor-groups under $\mathfrak{G}$, they all contain the derived group of $\mathfrak{G}$, and so does their meet $\mathfrak{M}_1$; $\mathfrak{G}/\mathfrak{M}_1$ is, therefore, an Abelian group.

If now $\mathfrak{G}/\mathfrak{G}_1$ is infinite, then $\mathfrak{G}/\mathfrak{M}_1$ has no elements of finite order, except the 1-element. For, if G^n belongs to $\mathfrak{M}_1$, it belongs to $\mathfrak{G}_1$; but, since $\mathfrak{G}/\mathfrak{G}_1$ has no elements of finite order except 1, this implies that G itself belongs

* This A-length is not an invariant feature of the group. It merely states that, among other A-series, $\mathfrak{G}$ has one with just so many factor-groups. ($\mathfrak{G}$ has possibly shorter and, if infinite, certainly longer A-series). The same convention applies to similar terms introduced later for other series.

to $\mathfrak{G}_1$; for the same reason G belongs to $\mathfrak{G}^{(1)}$, $\mathfrak{G}^{(2)}$, ... and finally to $\mathfrak{M}_1$. Thus $\mathfrak{G}/\mathfrak{M}_1$ is a free Abelian group.

If $\mathfrak{G}/\mathfrak{G}_1$ is finite and of order p, and if G is any element of $\mathfrak{G}$, then G^p belongs to $\mathfrak{G}_1$, and also to $\mathfrak{G}_1^{(1)}$, $\mathfrak{G}_1^{(2)}$, ..., therefore to $\mathfrak{M}_1$. Thus $\mathfrak{G}/\mathfrak{M}_1$ has, apart from the 1-element, only elements of order p, *i.e.* it is an elementary Abelian group.

We may assume now, by induction over the A-length of $\mathfrak{G}$, that we have already found for $\mathfrak{M}_1$ (which is, as a subgroup of $\mathfrak{G}$, of A-length not exceeding $l-1$) the required series

$$\mathfrak{M}_1 \supset \mathfrak{M}_2 \supset \ldots \supset \mathfrak{M}_k = 1,$$

where the $\mathfrak{M}_i$ are characteristic subgroups of $\mathfrak{M}_1$. Then

$$\mathfrak{G} = \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \ldots \supset \mathfrak{M}_k = 1$$

is a series of type (1.41) for $\mathfrak{G}$, since a characteristic subgroup of a characteristic subgroup is characteristic.

Of special importance in the following are those series of type (1.41) in which the free Abelian factor-groups are completely split up; that is, in which the dimensions of the free Abelian factor-groups are as small, or the number of infinite factor-groups is as large, as possible. In such a series, if $\mathfrak{N}_i/\mathfrak{N}_{i+1}$ is infinite, there exists no self-conjugate subgroup $\mathfrak{N}$ of $\mathfrak{G}$ between $\mathfrak{N}_i$ and $\mathfrak{N}_{i+1}$ with an infinite factor-group $\mathfrak{N}_i/\mathfrak{N}$.

Definition. Such a series will be called a "B-series" and the number of factor-groups in it the "length" of the B-series or "B-length of $\mathfrak{G}$".

1.42. THEOREM. *Any two A-series of an S-group $\mathfrak{G}$ have the same number of infinite factor-groups.*

In any two B-series of an S-group $\mathfrak{G}$ the infinite factor-groups are isomorphic in pairs.

Proof. These theorems are inferences from O. Schreier's "refinement theorem"*. Let $\mathfrak{G}$ be an S-group and let

$$\text{(i)} \quad \mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_l = 1,$$

$$\text{(ii)} \quad \mathfrak{G} = \mathfrak{G}_0' \supset \mathfrak{G}_1' \supset \ldots \supset \mathfrak{G}_{l'}' = 1,$$

be any two A-series of $\mathfrak{G}$. By the refinement theorem (i) and (ii) have refinements the factor-groups of which are isomorphic in pairs. If now

* *Cf. loc. cit.*

$\mathfrak{G}_i/\mathfrak{G}_{i+1}$ is finite, there is nothing to prove. But if $\mathfrak{G}_i/\mathfrak{G}_{i+1}$ is infinite, any refinement between $\mathfrak{G}_i$ and $\mathfrak{G}_{i+1}$

$$\mathfrak{G}_i \supset \mathfrak{H} \supset \ldots \supset \mathfrak{K} \supset \mathfrak{G}_{i+1}$$

gives rise to some finite factor-groups $\mathfrak{G}_i/\mathfrak{H}$, ..., whereas $\mathfrak{K}/\mathfrak{G}_{i+1}$ again is an infinite cyclical group. This follows from the fact that all subgroups of an infinite cyclical group are infinite cyclical groups of finite index.

The total number of infinite factor-groups is, therefore, unaffected by the process of refinement and must be the same in any two A-series.

The theorem for B-series follows in exactly the same way. This time the refinement theorem is applied to the case where the set of operators of $\mathfrak{G}$ consists of all inner automorphisms of $\mathfrak{G}$, in other words, where the self-conjugate subgroups of $\mathfrak{G}$, and only they, are "permissible".

We have thus dealt satisfactorily with the infinite factor-groups in the A- and B-series of an S-group $\mathfrak{G}$. The invariants which we have obtained, viz. the number of infinite factor-groups in an A-series and the set of dimension numbers of the infinite factor-groups in a B-series, correspond to the invariants which are given for finite soluble groups by the Jordan-Hölder theorem, viz. the cyclical factor-groups of prime order in any composition series and the elementary Abelian factor-groups of prime-power order in any chief-series.

The finite factor-groups of the A- and B-series present, however, considerable difficulties. They are, of course, highly arbitrary; one may, for example, always insert before an infinite factor-group some finite ones. Consequently only a further restricted and specialised type of A- and B-series can be expected to yield a uniqueness theorem. The most obvious restriction would be that no such "superfluous" terms occur in the A- or B-series, in other words, that the series which arise from dropping any term are no longer A- or B-series. That this is not sufficient is shown by the following example.

Let $\mathfrak{G}$ be the group generated by two elements P and Q of infinite order with the relation $Q^{-1}PQ = P^{-1}$. It is an S-group, because P generates a self-conjugate subgroup and

$$\mathfrak{G} \supset \{P\} \supset 1$$

is an A-series with two infinite cyclical factor-groups. But Q^2 is permutable with P and generates, in fact, the central of $\mathfrak{G}$. Therefore

$$\mathfrak{G} \supset \{Q^2, P\} \supset \{Q^2\} \supset 1$$

is another A-series for $\mathfrak{G}$ with two infinite cyclical and one finite factor-group of order 2. In this series no term can be dropped.

Narrower types of A- and B-series are those in which the number of finite factor-groups is as small as possible. If we call such series "minimal" A- or B-series, the theorem to be expected, the complete analogue to the Jordan-Hölder theorem, would be:

In a minimal A- or B-series of an S-group $\mathfrak{G}$ the infinite and the finite factor-groups are, apart from isomorphisms, uniquely determined.

This theorem holds for a wide subclass of S-groups, as will be shown in a subsequent paper. We have, however, not been able to prove it in general.

King's College,
Cambridge.

A THEOREM ON GENERAL MEASURE FUNCTIONS

By R. RADO.

[Received 3 May, 1937.—Read 20 May, 1937.]

P. Hall† has proved the following theorem:

Let $a_1, a_2, \ldots, a_n$ be a finite number of aggregates. Assume the following condition to be satisfied: Any k of the aggregates a_ν $(1 \leqslant k \leqslant n)$ contain between them at least k different elements. Then it is possible to select one element out of every a_ν in such a way that the selected elements are different from each other.

Evidently the condition stated is also necessary in order that the conclusion of the theorem may be true.

The object of this note is to generalize the theorem in two directions.

(*a*) We replace "number of elements of an aggregate" by a very general measure function Γ which is defined for every aggregate of a given system of aggregates and whose functional values are the elements of a given ordered aggregate. Γ is assumed to be monotonic, additive, and distributive; these terms will be defined later.

(*b*) Instead of selecting single elements we select subsets a_ν^* of the given sets a_ν; instead of postulating that the selected elements are different from each other we prescribe the "structure" of the system of sets a_ν^*, or, in other words, the way in which these sets a_ν^* overlap, in terms of the measure function Γ. This is done by postulating that the ordered system of sets $a_1^*, a_2^*, \ldots, a_n^*$ is "equimeasurable" with a given system $b_1, b_2, \ldots, b_n$.

Theorem I asserts that, in order that this selection of subsets a_ν^* may be possible, certain conditions, whose necessity is immediately obvious, are also sufficient.

An additional aspect is introduced by the fact that we admit sets of "infinite" measure.

† *Journal London Math. Soc.*, 10 (1934), 26–30.

The application of the theorem to the most natural measure, the measure given by the cardinal number of a set, leads to the notion of "cardinal number of a system of sets" which may be of some interest.

The argument which Mr. Hall used in order to prove his special case of the theorem is not capable of extension along the lines mentioned.

I should like to thank the Referee for his helpful criticisms and suggestions.

1. *Definitions.*

1. Let S be an abstract aggregate, fixed throughout. The letters a, b, c, d, e denote subsets of S. Sets whose elements are subsets of S are denoted by Gothic capitals $\mathfrak{F}, \mathfrak{M}, \ldots$. 0 denotes the empty set. We use the symbol "$\prec$" to denote the relation of elements to the class to which they belong, the symbol "$\subset$" to denote the relation of inclusion between two classes.

We start by defining a *measure* in S. Let $\mathfrak{M}$ be a system of subsets of S which has the property that

$$a,\ b \prec \mathfrak{M}$$

implies

$$a+b,\ ab,\ a-ab \prec \mathfrak{M}.$$

$\mathfrak{M}$ is the system of all "measurable" sets. We do not postulate that the complement of a measurable set is measurable.

From now onwards any one of the letters a, b, c, d, e denotes an arbitrary measurable set, *i.e.*, an element of $\mathfrak{M}$.

We divide the elements of $\mathfrak{M}$ into non-overlapping classes, the class to which a belongs being denoted by $|a|$. $|a|$ is "the measure of a". Instead of $|0|$, the measure of the empty set, we write 0.

The system of classes $|a|$ is ordered, by means of a transitive relation "$<$". Thus, given sets a, b, exactly one of the three relations

$$|a| < |b|,$$

$$|a| = |b|,$$

$$|a| > |b|$$

holds. We also use the relation "$\leqslant$" to denote the logical sum of the statements expressed by the first two relations.

We say, the relation "$<$" defines a measure Γ in $\mathfrak{M}$. Γ is called *monotonic*, if $a \subset b$ implies $|a| \leqslant |b|$. Γ is called *continuous*, if $|a| \geqslant |b|$ implies the existence of a set a' satisfying

$$a' \subset a;\quad |a'| = |b|.$$

Γ is called *additive* if

$$|a_1 a_2| = |b_1 b_2| = 0; \quad |a_1| = |b_1|; \quad |a_2| = |b_2|$$

implies $$|a_1 + a_2| = |b_1 + b_2|.$$

Γ is called *distributive*, if

$$|b_1 b_2| = 0; \quad |a| \geqslant |b_1 + b_2|$$

implies the existence of a_1, a_2 satisfying

$$|a_1 a_2| = 0; \quad a_1 + a_2 \subset a;$$
$$|a_1| = |b_1|; \quad |a_2| = |b_2|.$$

We denote these four properties of a measure by M, C, A, D. There are three alternative definitions of D, obtained from the one mentioned by replacing one or both of the relations

$$|a_1 a_2| = 0,$$
$$|b_1 b_2| = 0$$

by a corresponding relation

$$a_1 a_2 = 0,$$
$$b_1 b_2 = 0.$$

If A holds, these four definitions of D are equivalent. We shall sometimes use the one, sometimes the other form of the definition of D. A similar remark* applies to the definition of A.

---

* To be more explicit, we assert that

(i) if, for a given Γ,

$$a_1 a_2 = b_1 b_2 = 0; \quad |a_1| = |b_1|; \quad |a_2| = |b_2|$$

implies $$|a_1 + a_2| = |b_1 + b_2|,$$

then $$|c_1 c_2| = |d_1 d_2| = 0; \quad |c_1| = |d_1|; \quad |c_2| = |d_2|$$

implies $$|c_1 + c_2| = |d_1 + d_2|;$$

(ii) if the first implication of (i) holds and, furthermore, if

$$|a| \geqslant |b_1 + b_2|; \quad b_1 b_2 = 0$$

implies the existence of a_1, a_2, such that

$$a_1 + a_2 \subset a; \quad |a_1 a_2| = 0; \quad |a_1| = |b_1|; \quad |a_2| = |b_2|,$$

then $$|c| \geqslant |d_1 + d_2|; \quad |d_1 d_2| = 0$$

implies the existence of c_1, c_2, such that

$$c_1 + c_2 \subset c; \quad c_1 c_2 = 0; \quad |c_1| = |d_1|; \quad |c_2| = |d_2|.$$

The truth of these assertions follows easily from the fact, that, in either case,

$$a' a'' = 0; \quad |a''| = 0$$

implies $$|a'| = |a' + a''|.$$

Obviously D implies C. But the implication $C \rightarrow D$ does not hold. For, define Γ by

$$\Gamma S = \{1, 2\}; \quad 0 < |\{1\}| = |\{2\}| = |\{1, 2\}|,$$

and put $$a = \{1\}, \quad b_1 = \{1\}, \quad b_2 = \{2\}.$$

Then $$|a| = |b_1 + b_2|,$$

but sets a_1, a_2, in accordance with the definition of D, do not exist.

We say that a set a is of *finite measure*, if

$$a' \subset a; \quad |a'| = |a|$$

implies $$|a - a'| = 0.$$

Later on (Lemma 4) we shall justify this expression by proving that the property of a set a of being of finite measure is, in fact, a property of its measure $|a|$. We denote by $\mathfrak{F}$ the aggregate of all sets of finite measure.

A statement like: "Γ is monotonic in $\mathfrak{F}$" means that Γ is monotonic provided that the sets concerned are in $\mathfrak{F}$.

Examples of a monotonic, distributive, additive measure are:

(1) S is a given abstract aggregate, $\mathfrak{M}$ the system of all subsets of S, $|a|$ the cardinal of a. (More precisely: $|a|$ is the class of all b which have the same cardinal as a.)

(2) S is a Euclidean space, $\mathfrak{M}$ the system of all point sets in S which are Lebesgue measurable, $|a|$ the Lebesgue measure of a.

2. Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables whose coefficients are real non-negative numbers. Then, given sets $a_1, a_2, \ldots, a_n$, we define the set $f(a_1, a_2, \ldots, a_n)$ as follows.

Let $\phi_\nu(\tau)$ be a function, defined for all elements τ of S, which assumes real positive values in a_ν and the value 0 in $S - a_\nu$. Then $f(a_1, a_2, \ldots, a_n)$ is the set of all τ for which

$$f\big(\phi_1(\tau), \phi_2(\tau), \ldots, \phi_n(\tau)\big) > 0.$$

This set is independent of the particular choice of functions $\phi_\nu(\tau)$. Its definition is consistent with the definition of $a_1 + a_2$, $a_1 a_2$. Some obvious rules hold for this correspondence between polynomials and operations with sets. They may be summarized as follows:

Let

$$f(y_1, \ldots, y_m), \; f_1(x_1, \ldots, x_n), \; f_2(x_1, \ldots, x_n), \; \ldots, \; f_m(x_1, \ldots, x_n)$$

be $m+1$ polynomials with real non-negative coefficients,

$$F(x_1, \ldots, x_n) = f\Big(f_1(x_1, \ldots, x_n), \ \ldots, \ f_m(x_1, \ldots, x_n)\Big).$$

Then, for any sets a_ν,

$$F(a_1, \ldots, a_n) = f\Big(f_1(a_1, \ldots, a_n), \ \ldots, \ f_m(a_1, \ldots, a_n)\Big).$$

If $f(x_1, \ldots, x_n)$ has absolute term zero, then $f(a_1, \ldots, a_n)$ is measurable. We call a polynomial f *normal* if its absolute term vanishes.

Let $a_1, a_2, \ldots, a_n$; $b_1, b_2, \ldots, b_n$ be $2n$ given sets. If, for *every* normal polynomial f,

$$|f(a_1, a_2, \ldots, a_n)| \geqq |f(b_1, b_2, \ldots, b_n)|,$$

we express this fact by

$$|a_1, a_2, \ldots, a_n| \geqq |b_1, b_2, \ldots, b_n|.$$

It is obvious that, in this definition, it suffices to consider only polynomials f whose coefficients are either zero or one and which do not contain any variable to a higher power than the first. We call a polynomial of this special type *unitary*.

The relation

$$|a_1, a_2, \ldots, a_n| = |b_1, b_2, \ldots, b_n| \tag{1}$$

is, by definition, equivalent to the system of relations

$$|a_1, \ldots, a_n| \geqq |b_1, \ldots, b_n|,$$

$$|b_1, \ldots, b_n| \geqq |a_1, \ldots, a_n|.$$

For $n = 1$ these definitions are consistent with previous definitions. By

$$|a_1, \ldots, a_n| \equiv |b_1, \ldots, b_n| \tag{2}$$

we denote the following relation between the given sets. Let a be any set obtained from $a_1, \ldots, a_n$ by a finite number of applications of the processes indicated by

$$a'+a'', \quad a'a'', \quad a'-a'a''.$$

Suppose that b is the set which is obtained by exactly the same processes in the same order and applied to the corresponding sets, but this time

starting from $b_1, b_2, \ldots, b_n$. Then (2) asserts that always

$$|a| = |b|.$$

Thus (2) expresses the fact that, so far as our particular measure is concerned, the two systems of sets

$$a_1, \ldots, a_n,$$

$$b_1, \ldots, b_n$$

are indistinguishable by purely inner properties (are equimeasurable).

For every n, (2) implies (1). If $n = 1$, (2) is equivalent to (1). But if $n \geqslant 2$, (2) is actually stronger than (1) as is shown by the example

$$a_1 = a_2 = b_1 = \{1, 2, 3, \ldots\},$$

$$b_2 = \{2, 4, 6, \ldots\},$$

$$|a| = \text{cardinal of } a.$$

It will follow from later results (Lemma 7) that for sets of finite measure (1) and (2) are equivalent.

2. *The main theorem, preliminary lemmas.*

1. In stating our theorems we shall make use of notations and definitions explained in § 1.

THEOREM I. *Let Γ be a monotonic, distributive, and additive measure. Suppose that $b_\nu \prec \mathfrak{F}$ $(1 \leqslant \nu \leqslant n)$. Then*

$$|a_1, a_2, \ldots, a_n| \geqslant |b_1, b_2, \ldots, b_n| \tag{3}$$

is a necessary and sufficient condition in order that it may be possible to find subsets a_ν^* of a_ν satisfying

$$|a_1^*, a_2^*, \ldots, a_n^*| = |b_1, b_2, \ldots, b_n|. \tag{4}$$

The special case considered by Hall is obtained by considering the measure of example (1) on p. 64 and putting $b_\nu = \{\nu\}$.

The necessity of the condition (3) is an immediate consequence of M. For if sets a_ν^* exist, then, for every normal polynomial f,

$$f(a_1, \ldots, a_n) \supset f(a_1^*, \ldots, a_n^*),$$

$$|f(a_1, \ldots, a_n)| \geqslant |f(a_1^*, \ldots, a_n^*)| = |f(b_1, \ldots, b_n)|,$$

i.e. (3) holds.

If, for a certain measure Γ, the assertion of Theorem I is true, then Γ *must* be monotonic and distributive in $\mathfrak{F}$. This is easily established. Suppose first, that a_1 and b_1 are two sets $(b_1 \prec \mathfrak{F})$ satisfying $a_1 \supset b_1$. Then, putting $a_1^* = b_1$, we have

$$a_1^* \subset a_1,$$

$$|a_1^*| = |b_1|.$$

Hence, by the necessity part of Theorem I, applied for $n = 1$, we have

$$|a_1| \geqslant |b_1|,$$

i.e. M.

Next, let us suppose that

$$b_1, b_2 \prec \mathfrak{F},$$

$$|b_1 b_2| = 0, \quad |a| \geqslant |b_1 + b_2|. \tag{5}$$

Then

$$|a, a| \geqslant |b_1, b_2|. \tag{6}$$

For (6) means that

$$|a| \geqslant |b_1|; \quad |a| \geqslant |b_2|;$$

$$|a+a| \geqslant |b_1+b_2|; \quad |aa| \geqslant |b_1 b_2|,$$

and these relations are true, as consequences of (5) and M. By the sufficiency part of Theorem I we conclude from (6) the existence of sets a_ν^* satisfying

$$a_\nu^* \subset a,$$

$$|a_1^*, a_2^*| \equiv |b_1, b_2|.$$

In particular, $$|a_1^* a_2^*| = |b_1 b_2| = 0.$$

Hence Γ is distributive in $\mathfrak{F}$.

Additivity, the third property of Γ required in Theorem I, does not follow from the validity of the criterion contained in Theorem I. For consider the following measure Γ defined for all subsets of $\{1, 2\}$.

$$0 = |\{1\}| = |\{2\}| < |\{1, 2\}|.$$

The assertion of Theorem I holds for this special case, but Γ is not additive, as is evident from

$$|\{1\}+0| < |\{1\}+\{2\}|.$$

2. In order to prove Theorem I, we establish a number of lemmas. These lemmas are trivial in the two special cases (1) and (2) on p. 64.

Lemmas 1 and 2 state a certain generalized kind of continuity of Γ.

LEMMA 1. *Suppose that* $|a| \leqslant |b|$; $|a+c| > |b|$. *Then there is a set* c' *satisfying* $c' \subset c$; $|a+c'| = |b|$.

Proof. By C we can find a set d such that $d \subset b$; $|a| = |d|$.

Case α. Suppose that $|c-ac| \leqslant |b-d|$.

Then, by C, there is a set e such that

$$e \subset b-d; \quad |c-ac| = |e|.$$

Then, by A and M,

$$|a+c| = |a+(c-ac)| = |d+e| \leqslant |b| < |a+c|,$$

which is a contradiction.

Case β. Suppose that $|c-ac| > |b-d|$.

Then, by C, there is a set c' such that

$$c' \subset c-ac; \quad |c'| = |b-d|.$$

Then, by A,

$$|a+c'| = |d+(b-d)| = |b|.$$

Hence c' has the properties required.

LEMMA 2. *Suppose that*

$$|a_1| \leqslant |b_1|,$$

$$|a_\nu+c| > |b_\nu| \quad (1 \leqslant \nu \leqslant n).$$

Then there is a set c' *satisfying*

$$c' \subset c; \quad |a_\nu+c'| \geqslant |b_\nu| \quad (1 \leqslant \nu \leqslant n);$$

$$|a_{\nu_0}+c'| = |b_{\nu_0}|$$

for at least one value of ν_0 $(1 \leqslant \nu_0 \leqslant n)$.

Proof. The case $n=1$ is proved by Lemma 1. Let us suppose that Lemma 2 is proved if n is replaced by $n-1$. Then there is a d such that

$$d \subset c,$$

$$|a_\nu+d| \geqslant |b_\nu| \quad (1 \leqslant \nu < n),$$

$$|a_{\nu_1}+d| = |b_{\nu_1}|,$$

where $1 \leqslant \nu_1 < n$.

Case α. Suppose that $|a_n+d| > |b_n|$.

Then we may put $c' = d$.

Case β. Suppose that $|a_n+d| \leqslant |b_n|$.

Since $$|(a_n+d)+c| = |a_n+c| > |b_n|,$$

an application of Lemma 1 shows the existence of an e such that

$$e \subset c; \quad |(a_n+d)+e| = |b_n|.$$

Then $c' = d+e$ has the properties required.

The remaining six lemmas show that the system $\mathfrak{F}$ of all sets of finite measure has a number of properties generally associated with sets of finite measure.

LEMMA 3. *If* $a \subset b$; $b \prec \mathfrak{F}$, *then* $a \prec \mathfrak{F}$.

Proof. Let $a' \subset a$; $|a'| = |a|$.

Then $$a'+(b-a) \subset b;$$

$$|a'+(b-a)| = |a+(b-a)| \quad \text{(by } A\text{)}$$

$$= |b|.$$

Hence, as a consequence of $b \prec \mathfrak{F}$,

$$\big|b-\big(a'+(b-a)\big)\big| = 0,$$

i.e. $$|a-a'| = 0.$$

Since this holds for every such a', we infer from the definition of $\mathfrak{F}$ that

$$a \prec \mathfrak{F}.$$

LEMMA 4. *If* $|a| = |b|$; $b \prec \mathfrak{F}$, *then* $a \prec \mathfrak{F}$.

Proof. Let $a' \subset a$; $|a'| = |a|$.

Then $$|a'+(a-a')| = |a| = |b|.$$

Hence, by D, there are sets b_1, b_2 such that

$$b_1 b_2 = 0; \quad b_1 + b_2 \subset b;$$

$$|a'| = |b_1|; \quad |a - a'| = |b_2|.$$

Then

$$|b_1| = |a'| = |a| = |b|,$$

$$|b - b_1| = 0 \qquad (\text{since } b \prec \mathfrak{F}),$$

$$b_2 \subset b - b_1; \quad |b_2| = 0 \quad (\text{by } M),$$

$$|a - a'| = |b_2| = 0.$$

Hence the conclusion.

LEMMA 5. *If* $|a| \leqslant |b|$; $b \prec \mathfrak{F}$, *then* $a \prec \mathfrak{F}$

Proof. By C there is b' such that

$$b' \subset b; \quad |a| = |b'|.$$

Then, by Lemmas 3 and 4,

$$b' \subset \mathfrak{F}; \quad a \prec \mathfrak{F}.$$

LEMMA 6. *If*

$$a' \subset a; \quad b' \subset b; \quad |a'| = |b'|; \quad |a| = |b|; \quad a \prec \mathfrak{F},$$

then

$$|a - a'| = |b - b'|.$$

Proof. We have $b \prec \mathfrak{F}$. Hence there is no loss of generality in assuming that

$$|a - a'| \leqslant |b - b'|.$$

By C, there is a b'' such that

$$b'' \subset b - b'; \quad |a - a'| = |b''|.$$

Then

$$b' + b'' \subset b,$$

and, by A,

$$|b' + b''| = |a' + (a - a')| = |a| = |b|.$$

Hence, since $b \prec \mathfrak{F}$,

$$|b - (b' + b'')| = 0; \quad |(b - b') - b''| = 0;$$

$$|b - b'| = |\big((b - b') - b''\big) + b''| = |0 + b''| = |a - a'|.$$

LEMMA 7. *If* $a,\ b \prec \mathfrak{F}$, *then* $a+b \prec \mathfrak{F}$.

Proof. We shall first assume that $ab = 0$.

Suppose that

$$a_1 \subset a;\quad b_1 \subset b;\quad |a_1+b_1| = |a+b|.$$

We have to prove that these relations imply that

$$|(a+b)-(a_1+b_1)| = 0.$$

By D, we can find sets a_2, b_2, satisfying

$$a_2 b_2 = 0;\quad a_2+b_2 \subset a_1+b_1;$$
$$|a_2| = |a|;\quad |b_2| = |b|.$$

Case α. Suppose that

$$|a_1 b_2| > |b_1 - b_1 b_2|.$$

Then C implies the existence of a c such that

$$c \subset a_1 b_2;\quad |c| = |b_1 - b_1 b_2|.$$

Then, making use of M and A, we have

$$\begin{aligned} |a_1| &\geqslant |a_1-(a_1 b_2-c)| = |(a_1-a_1 b_2)+c| \\ &= |(a_1-a_1 b_2)+(b_1-b_1 b_2)| = |(a_1+b_1)-b_2| \\ &\geqslant |a_2| = |a| \geqslant |a_1|. \end{aligned}$$

Hence, since $a_1 \prec \mathfrak{F}$,

$$|a_1 b_2 - c| = 0,$$

$$|a_1 b_2| = |(a_1 b_2-c)+c| = |0+c| = |b_1-b_1 b_2| < |a_1 b_2|,$$

i.e. a contradiction.

Case β. Suppose that

$$|a_1 b_2| \leqslant |b_1 - b_1 b_2|.$$

Then C implies the existence of a d such that

$$d \subset b_1 - b_1 b_2;\quad |a_1 b_2| = |d|.$$

Then

$$|b_2| = |a_1 b_2 + b_1 b_2| - |d + b_1 b_2| \leqslant |b_1| \leqslant |b| = |b_2|.$$

Hence $|b_1| = |b|$ and, since $b \prec \mathfrak{F}$,

$$|b - b_1| = 0.$$

Similarly it follows that

$$|a - a_1| = 0.$$

Therefore

$$|(a+b)-(a_1+b_1)| = |(a-a_1)+(b-b_1)| = |0+0| = 0.$$

The lemma is therefore proved in the case $ab = 0$.

If a and b are arbitrary sets of finite measure, then

$$a - ab \prec \mathfrak{F},$$

and the above result, if applied to the sets $a-ab$ and b, shows that

$$a+b = (a-ab)+b \prec \mathfrak{F}.$$

COROLLARY. *If the sets b_ν are of finite measure,* (1) *and* (2) *are equivalent.*

Before we prove this result we make a general remark. Let $c_1, c_2, \ldots, c_n$ be n sets of $\mathfrak{M}$,

$$1 \leqslant k \leqslant n; \quad 1 \leqslant \nu_1 < \nu_2 < \ldots < \nu_k \leqslant n.$$

Let $$\mu_1, \mu_2, \ldots, \mu_l \quad (l = n-k,\ 1 \leqslant \mu_1 < \mu_2 < \ldots < \mu_l \leqslant n)$$

be those indices ν $(1 \leqslant \nu \leqslant n)$ which are not ν_κ's. Then†

$$c_{(\nu_\kappa)} = c_{(\nu_1, \nu_2, \ldots, \nu_k)} = c_{\nu_1} c_{\nu_2} \ldots c_{\nu_k} - c_{\nu_1} c_{\nu_2} \ldots c_{\nu_k} (c_{\mu_1} + \ldots + c_{\mu_l})$$

is the set of all elements τ of $c_1+c_2+\ldots+c_n$ which belong to each of the sets c_{ν_κ} but to none of the remaining $n-k$ sets c_ν. Therefore two sets $c_{(\nu_\kappa)}$ corresponding to different systems (ν_κ) have no elements in common. Moreover, any set c obtained from $c_1, c_2, \ldots, c_n$ by a finite number of applications of the operations of addition, subtraction and multiplication is the sum of certain sets $c_{(\nu_\kappa)}$.

Hence, by A, (2) is equivalent to

$$|a_{(\nu_\kappa)}| = |b_{(\nu_\kappa)}| \quad \text{for every } (\nu_\kappa).$$

Now suppose that (1) holds for certain sets a_ν, b_ν satisfying $b_\nu \prec \mathfrak{F}$. We have to show that (2) follows.

† In particular, $c_{(1,2,\ldots,n)} = c_1 c_2 \ldots c_n$.

Using our previous notation, we have, by (1) and Lemmas 5, 7, 6,

$$b' = b_{\nu_1} b_{\nu_2} \dots b_{\nu_k} \prec \mathfrak{F},$$

$$a' = a_{\nu_1} \dots a_{\nu_k} \prec \mathfrak{F} \quad (\text{since } |a'| = |b'|),$$

$$a_{\mu_1} + \dots + a_{\mu_l}, \quad b_{\mu_1} + \dots + b_{\mu_l} \prec \mathfrak{F},$$

$$a'' = a'(a_{\mu_1} + \dots + a_{\mu_l}) \prec \mathfrak{F},$$

$$b'' = b'(b_{\mu_1} + \dots + b_{\mu_l}) \prec \mathfrak{F},$$

$$|a'| = |b'|; \quad |a''| = |b''|;$$

$$|a_{(\nu_\kappa)}| = |a' - a''| = |b' - b''| = |b_{(\nu_\kappa)}|.$$

Therefore, by the foregoing remark, (2) is satisfied.

LEMMA 8. *If*

$$|a_1, a_2| \geqslant |b_1, b_2|; \tag{7}$$

$$|a_1| = |b_1|; \quad |a_2| = |b_2|; \quad a_1, a_2 \prec \mathfrak{F}$$

then

$$|a_1 a_2| = |b_1 b_2|.$$

Proof. In the case where a_ν, b_ν are sets of a finite number of elements and $|a|$ denotes the number of elements of a, the assertion of Lemma 8 is an immediate consequence of

$$|a_1| + |a_2| = |a_1 + a_2| + |a_1 a_2| \geqslant |b_1 + b_2| + |b_1 b_2|$$

$$= |b_1| + |b_2| = |a_1| + |a_2|.$$

In the general case we cannot argue in this way since, for instance, symbols like $|a_1| + |a_2|$ have no meaning.

By (7),

$$|a_1 a_2| \geqslant |b_1 b_2|.$$

Hence there is a c satisfying

$$c \subset a_1 a_2; \quad |c| = |b_1 b_2|.$$

Then, by M and Lemma 6,

$$|a_1 - a_1 a_2| \leqslant |a_1 - c| = |b_1 - b_1 b_2|.$$

Hence there exists a d such that

$$d \subset b_1 - b_1 b_2; \quad |a_1 - a_1 a_2| = |d|.$$

Then, by A, M and (7),

$$|a_1+a_2| = |(a_1-a_1a_2)+a_2| = |d+b_2| \leqslant |(b_1-b_1b_2)+b_2|$$
$$= |b_1+b_2| \leqslant |a_1+a_2|.$$

Since, by Lemma 7, $b_1+b_2 \prec \mathfrak{F}$, this implies that

$$|(b_1+b_2)-(d+b_2)| = 0,$$

i.e. $$|(b_1-b_1b_2)-d| = 0; \quad |b_1-b_1b_2| = |d| = |a_1-a_1a_2|.$$

Then, by Lemma 6,

$$|b_1b_2| = |b_1-(b_1-b_1b_2)| = |a_1-(a_1-a_1a_2)|$$
$$= |a_1a_2|.$$

3. *Proof of the main theorem.*

1. In this section we prove the sufficiency part of Theorem I, under the additional assumption that

$$b_\kappa b_\lambda = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n). \tag{8}$$

In this special case (4) is equivalent to

$$|a_\nu^*| = |b_\nu| \quad (1 \leqslant \nu \leqslant n), \tag{9}$$

$$|a_\kappa^* a_\lambda^*| = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n). \tag{10}$$

For, obviously, (9) and (10) follow from (4) and (8). On the other hand, suppose that (9) and (10) are satisfied. Then, as an immediate consequence of (8), (9), (10), and A,

$$|a_1^*, \ldots, a_n^*| = |b_1, \ldots, b_n|.$$

Hence, for any normal polynomials

$$f(x) = f(x_1, \ldots, x_n),$$
$$g(x) = g(x_1, \ldots, x_n),$$

we have

$$|f(a^*)-f(a^*)g(a^*)| = |f(a^*)| = |f(b)| = |f(b)-f(b)g(b)|.$$

This makes it obvious that (4) holds.

(9) and (10) can be replaced by

$$|a^*_{\nu_1} a^*_{\nu_2} \dots a^*_{\nu_k}| = |b_{\nu_1} b_{\nu_2} \dots b_{\nu_k}| \tag{11}$$

$$(1 \leqslant k \leqslant n;\ 1 \leqslant \nu_1 < \nu_2 < \dots < \nu_k \leqslant n).$$

Let N be the number of *unitary* normal polynomials (the definition of unitary polynomials was given on p. 65) satisfying

$$|f(a_1, \dots, a_n)| > |f(b_1, \dots, b_n)|.$$

If $N = 0$, then, in particular, (11) holds with $a_\nu^* = a_\nu$. Then (4) is satisfied with $a_\nu^* = a_\nu$. Hence we may assume that in our particular case

$$N = N_0 > 0,$$

and, furthermore, that the theorem is true in all cases in which $N < N_0$.

Then (11) is not satisfied with $a_\nu^* = a_\nu$ for all values of k, $\nu_1, \dots, \nu_k$. There is a number m $(1 \leqslant m \leqslant n)$ which has the property that, after renumbering the sets a_ν and b_ν suitably,

$$|a_1 a_2 \dots a_m| > |b_1 b_2 \dots b_m|, \tag{12}$$

$$|a_1 a_2 \dots a_m a_\nu| = |b_1 b_2 \dots b_m b_\nu| \quad (m < \nu \leqslant n). \tag{13}$$

If $m = n$, there is no equation (13).

Without loss of generality, we may assume that

$$a_1 a_2 \dots a_m a_\nu = 0 \quad (m < \nu \leqslant n). \tag{14}$$

For (13) implies $$|a_1 a_2 \dots a_m a_\nu| = 0 \quad (m < \nu \leqslant n).$$

Put, for the moment,

$$a_1' = a_1 - \sum_{\nu=m+1}^{n} a_1 a_2 \dots a_m a_\nu.$$

Then $$|a_1', a_2, \dots, a_n| \equiv |a_1, a_2, \dots, a_n|,$$

$$a_1' a_2 \dots a_m a_\nu = 0 \quad (m < \nu \leqslant n).$$

Therefore the sets $a_1', a_2, \dots, a_n$ may be used instead of $a_1, a_2, \dots, a_n$, and if the conclusion holds for the former then it holds, *a fortiori*, for the latter. Instead of a_1', we write a_1 again.

Any unitary normal polynomial f can be written uniquely in the form

$$f(x_1, \dots, x_n) = \bar{f}(x_1, \dots, x_m) + \sum_{\nu=m+1}^{n} x_\nu g_\nu(x_1, \dots, x_{\nu-1}).$$

Then $\bar{f}$ and $x_\nu g_\nu$ are unitary and normal. Instead of $g_\nu(x_1, \dots, x_{\nu-1})$ we write $g_\nu(x_1, \dots, x_n)$.

We distinguish three types of f's:

(α) $\bar{f} \neq 0$; $\bar{f}$ is divisible by x_1. We denote polynomials f of this type by f_1.

$$f = f_1 = x_1 g(x_2, \ldots, x_m) + \sum_{m+1}^{n} x_\nu g_\nu(x_1, \ldots, x_n); \quad g \neq 0.$$

(β) $\bar{f} = 0$. We denote polynomials f of this type by f'.

$$f = f' = \sum_{m+1}^{n} x_\nu g_\nu(x_1, \ldots, x_n).$$

(γ) $\bar{f}$ is not divisible by x_1. We denote f's of this type by f_1''.

$$f = f_1'' = x_{\nu_1} x_{\nu_2} \ldots x_{\nu_l} + g(x_1, \ldots, x_n),$$

where g has non-negative coefficients and

$$1 \leqq l < m; \quad 1 < \nu_1 < \nu_2 < \ldots < \nu_l \leqq m.$$

In particular,

$$f_1^{(0)}(x_1, \ldots, x_n) = x_1 x_2 \ldots x_m$$

is an f_1.

We have to consider $m+1$ cases.

Case 1. Suppose that, for every f_1,

$$|f_1(a_1, \ldots, a_n)| > |f_1(b_1, \ldots, b_n)|. \tag{15}$$

Put

$$d_0 = a_1 a_2 \ldots a_m.$$

Then, for any $d \subset d_0$,

$$f_1(a_1 - d, a_2, \ldots, a_n) = (a_1 - d) g(a_2, \ldots, a_m) + \sum_{\nu=m+1}^{n} a_\nu g_\nu(a_1 - d, a_2, \ldots, a_n).$$

Now

$$d \subset d_0 \subset g(a_2, \ldots, a_m),$$

$$(a_1 - d) g(a_2, \ldots, a_m) = a_1 g(a_2, \ldots, a_m) - d,$$

$$\sum_{\nu=m+1}^{n} a_\nu g_\nu(a_1, \ldots, a_n) \subset \sum_{m+1}^{n} a_\nu [g_\nu(a_1 - d, a_2, \ldots, a_n) + d]$$

$$= \sum_{m+1}^{n} a_\nu g_\nu(a_1 - d, \ldots, a_n) + \sum_{m+1}^{n} a_\nu d$$

$$= \sum_{m+1}^{n} a_\nu g_\nu(a_1 - d, \ldots, a_n) + 0 \quad \text{[by (14)]}.$$

Hence $$\sum_{m+1}^{n} a_\nu g_\nu(a_1-d, \ldots, a_n) = \sum_{m+1}^{n} a_\nu g_\nu(a_1, \ldots, a_n),$$

$$(16)\quad f_1(a_1-d, a_2, \ldots, a_n) = \Big(a_1 g(a_2, \ldots, a_m)-d\Big) + \sum_{m+1}^{n} a_\nu g_\nu(a_1, \ldots, a_n)$$
$$= f_1(a_1, \ldots, a_n)-d.$$

On the other hand,

$$(17)\quad f'(a_1-d, a_2, \ldots, a_n) = f'(a_1, a_2, \ldots, a_n),$$

$$(18)\quad f_1''(a_1-d, a_2, \ldots, a_n) = f_1''(a_1, a_2, \ldots, a_n).$$

We have

$$|f_1^{(0)}(a_1, \ldots, a_n)-d_0| = |a_1 \ldots a_m - d_0| = 0 \leqslant |f_1^{(0)}(b_1, \ldots, b_n)|,$$

and, for every f_1,

$$|(f_1(a_1, \ldots, a_n)-d_0)+d_0| = |f_1(a_1, \ldots, a_n)| > |f_1(b_1, \ldots, b_n)|.$$

Therefore an application of Lemma 2, with $c = d_0$, shows the existence of a $d'(d' \subset d_0)$ such that, for *every* f_1,

$$(19)\quad \Big|\Big(f_1(a_1, \ldots, a_n)-d_0\Big)+d'\Big| \geqslant |f_1(b_1, \ldots, b_n)|,$$

and, for *at least one* f_1, the inequality in (19) is replaced by equality. Then, by (19), (17), (18), (3),

$$\Big|f_1\Big(a_1-(d_0-d'), a_2, \ldots, a_n\Big)\Big| = |f_1(a_1, \ldots, a_n)-(d_0-d')|$$
$$= \Big|\Big(f_1(a_1, \ldots, a_n)-d_0\Big)+d'\Big| \geqslant |f_1(b_1, \ldots, b_n)|$$

(equality for at least one f_1),

$$\Big|f'\Big(a_1-(d_0-d'), a_2, \ldots, a_n\Big)\Big| = |f'(a_1, \ldots, a_n)| \geqslant |f'(b_1, \ldots, b_n)|,$$

$$\Big|f_1''\Big(a_1-(d_0-d'), a_2, \ldots, a_n\Big)\Big| = |f_1''(a_1, \ldots, a_n)| \geqslant |f_1''(b_1, \ldots, b_n)|.$$

Hence

$$(20)\quad |a_1-(d_0-d'), a_2, \ldots, a_n| \geqslant |b_1, b_2, \ldots, b_n|.$$

Let N' be the value of the number N associated with (20) in the same way as N_0 is associated with (3). Then

$$(21)\quad N' < N_0.$$

For, obviously, $N' \leqslant N_0$. But the fact that for some special polynomial f_1 there is equality in (19), while in (15) there is no sign of equality, shows that (21) holds. Hence, by our induction hypothesis, the theorem is true for the sets occurring in (20). There are sets a_ν^* satisfying (4) and

$$a_1^* \subset a_1 - (d_0 - d') \subset a_1,$$

$$a_\nu^* \subset a_\nu \quad (1 < \nu \leqslant n).$$

Theorem I is therefore proved in case 1.

More generally, let $1 \leqslant \mu \leqslant m$. A unitary normal polynomial $f(x_1, \ldots, x_n)$ is said to be of type f_μ if

$$f = x_\mu g(x_1, \ldots, x_{\mu-1}, x_{\mu+1}, \ldots, x_m) + \sum_{\nu=m+1}^{n} x_\nu g_\nu(x_1, \ldots, x_n),$$

$$g \not\equiv 0,$$

where $x_\mu g$ and $x_\nu g_\nu$ $(m < \nu \leqslant n)$ are unitary and normal.

Case μ. Suppose that, for every unitary normal f of the type f_μ, we have

$$|f_\mu(a_1, \ldots, a_n)| > |f_\mu(b_1, \ldots, b_n)|.$$

Then the theorem is proved in the same way as in case 1.

Case $m+1$. Suppose that, for every μ $(1 \leqslant \mu \leqslant m)$, there is a polynomial $f = f^{(\mu)}$ of the type f_μ satisfying

$$|f^{(\mu)}(a_1, \ldots, a_n)| = |f^{(\mu)}(b_1, \ldots, b_n)|. \tag{22}$$

We shall show that this case cannot occur. Put

$$f^{(\mu)}(x_1, \ldots, x_n) = x_\mu g^{(\mu)}(x_1, \ldots, x_{\mu-1}, x_{\mu+1}, \ldots, x_m) + \sum_{\nu=m+1}^{n} x_\nu g_\nu^{(\mu)}(x_1, \ldots, x_n), \tag{23}$$

where $x_\mu g^{(\mu)}$ and $x_\nu g_\nu^{(\mu)}$ are unitary, and $g^{(\mu)} \not\equiv 0$. Write $f(a)$ instead of $f(a_1, \ldots, a_n)$.

If $m \geqslant 2$ then, as a consequence of (3),

$$|f^{(1)}(a), f^{(2)}(a)| \geqslant |f^{(1)}(b), f^{(2)}(b)|.$$

Moreover,

$$f^{(\mu)}(b) \prec \mathfrak{F},$$

$$|f^{(\mu)}(a)| = |f^{(\mu)}(b)|.$$

Hence, by an application of Lemma 8,

$$|f^{(1)}(a) f^{(2)}(a)| = |f^{(1)}(b) f^{(2)}(b)|.$$

After $m-1$ applications of Lemma 8 we find that

$$\left| \prod_{\mu=1}^{m} f^{(\mu)}(a) \right| = \left| \prod_{\mu=1}^{m} f^{(\mu)}(b) \right|.$$

Obviously, by (23),

$$\prod_{\mu=1}^{m} f^{(\mu)}(a) = a_1 a_2 \dots a_m + \sum_{\nu=m+1}^{n} a_\nu h_\nu(a),$$

$$\prod_{\mu=1}^{m} f^{(\mu)}(b) = b_1 b_2 \dots b_m + \sum_{\nu=m+1}^{n} b_\nu h_\nu(b),$$

where $x_\nu h_\nu(x)$ are unitary polynomials. Therefore

$$|a_1 a_2 \dots a_m + a'| = |b_1 b_2 \dots b_m + b'|,$$

where $$a' = \Sigma a_\nu h_\nu(a), \quad b' = \Sigma b_\nu h_\nu(b).$$

By (3), $|a'| \geqslant |b'|$. Hence, there is a c satisfying

$$c \subset a', \quad |c| = |b'|.$$

In view of $b_\nu \prec \mathfrak{F}$ and Lemmas 3 and 7,

$$b_1 b_2 \dots b_m + b' \prec \mathfrak{F}.$$

Consequently, by Lemma 6,

$$|b_1 \dots b_m| = |(b_1 \dots b_m + b') - b'| \quad \text{(true because of } b_1 \dots b_m b' = 0)$$

$$= |(a_1 \dots a_m + a') - c| \geqslant |a_1 \dots a_m|.$$

This contradicts (12). Theorem I is therefore proved in the special case (8).

2. In this section we show that, if Theorem I holds for every n and all sets b_ν of $\mathfrak{F}$ satisfying (8), then it also holds for arbitrary sets b_ν of $\mathfrak{F}$.

We adopt the notation used on p. 72. As mentioned on p. 74, (4) is equivalent to

$$|a^{*}_{(\nu_\kappa)}| = |b_{(\nu_\kappa)}| \quad \text{for every } (\nu_\kappa).$$

We denote by $$\dots, c_{(\nu_\kappa)}, \dots$$

the system of all the $2^n - 1$ sets $c_{(\nu_\kappa)}$ which correspond to the possible choices $\nu_1, \dots, \nu_k$ of k. We arrange the systems (ν_κ) in some fixed order. Similarly

$$\dots, c_{\nu_1} c_{\nu_2} \dots c_{\nu_k}, \dots$$

denotes the system of all products of the sets c_ν, arranged in the corresponding order.

(3) obviously implies that

$$|\ldots,\ a_{\nu_1}\ldots a_{\nu_k},\ \ldots| \geqq |\ldots,\ b_{\nu_1}\ldots b_{\nu_k},\ \ldots|.$$

Furthermore, since $$b_{\nu_1}\ldots b_{\nu_k} \supset b_{(\nu_\kappa)},$$

we have $$|\ldots,\ b_{\nu_1}\ldots b_{\nu_k},\ \ldots| \geqq |\ldots,\ b_{(\nu_\kappa)},\ \ldots|.$$

Hence $$|\ldots,\ a_{\nu_1}\ldots a_{\nu_k},\ \ldots| \geqq |\ldots,\ b_{(\nu_\kappa)},\ \ldots|.$$

Since the sets occurring on the right-hand side of this inequality belong to $\mathfrak{F}$ and do not overlap, we can apply the result proved in the previous section. We find certain sets $d^{(\nu_\kappa)}$ satisfying

$$d^{(\nu_\kappa)} \subset a_{\nu_1}\ldots a_{\nu_k},$$

$$|\ldots,\ d^{(\nu_\kappa)},\ \ldots| \equiv |\ldots,\ b_{(\nu_\kappa)},\ \ldots|.$$

In particular, for any two different systems (ν_κ), $(\nu'_{\kappa'})$, we have

$$|d^{(\nu_\kappa)} d^{(\nu'_{\kappa'})}| = |b_{(\nu_\kappa)} b_{(\nu'_{\kappa'})}| = 0.$$

Therefore, if we put

$$e^{(\nu_\kappa)} = d^{(\nu_\kappa)} - d^{(\nu_\kappa)} \sum_{(\nu'_{\kappa'}) \neq (\nu_\kappa)} d^{(\nu'_{\kappa'})},$$

we have

$$|\ldots,\ e^{(\nu_\kappa)},\ \ldots| \equiv |\ldots,\ d^{(\nu_\kappa)},\ \ldots| \equiv |\ldots,\ b_{(\nu_\kappa)},\ \ldots|,$$

$$e^{(\nu_\kappa)} e^{(\nu'_{\kappa'})} = 0 \quad \text{for} \quad (\nu_\kappa) \neq (\nu'_{\kappa'}).$$

We put $$c_\lambda = \sum_{\lambda \prec \{\nu_1,\ \ldots,\ \nu_k\}} e^{(\nu_\kappa)},$$

where the sum is extended over all 2^{n-1} systems (ν_κ) which contain the index λ. It is readily verified that

$$c_\lambda \subset a_\lambda,$$

$$c_{(\nu_\kappa)} = e^{(\nu_\kappa)}.$$

Hence $$|c_{(\nu_\kappa)}| = |b_{(\nu_\kappa)}|,$$

$$|c_1,\ c_2,\ \ldots,\ c_n| \equiv |b_1,\ b_2,\ \ldots,\ b_n|.$$

(4) is satisfied, with $a_\nu^* = c_\nu$. This completes the proof of Theorem I.

We add the remark that the foregoing reduction of the general case to the case of non-overlapping b_ν's does not make use of any property of the measure Γ except the fact that Γ is additive and monotonic.

4. *Sets of infinite measure.*

Our next object is to extend the assertion of Theorem I to sets b_ν of infinite measure. So far, however, we have been able only to handle a special type of sets of infinite measure which, nevertheless, seems to include all cases of general interest.

For convenience we introduce the following definition. A measure Γ is called *regular* in $\mathfrak{M}$ if Γ is defined in $\mathfrak{M}$ and

$$|a_1, \ldots, a_n| \geqslant |b_1, \ldots, b_n|,$$

$$a_1, \ldots, a_n, b_1, \ldots, b_n \prec \mathfrak{M}$$

implies the existence of sets a_ν^* of $\mathfrak{M}$ satisfying

$$a_\nu^* \subset a_\nu; \quad |a_1^*, \ldots, a_\nu^*| = |b_1, \ldots, b_n|.$$

By Lemmas 3 and 7, the system $\mathfrak{F}$ of all sets of finite measure is closed with respect to the operations of addition, multiplication, and subtraction. Theorem I asserts that every monotonic, additive, and distributive measure is regular in $\mathfrak{F}$. On p. 67 it was shown that regularity implies distributivity.

THEOREM II. *If Γ is a distributive measure satisfying*

$$|a+b| = \max(|a|, |b|) \tag{24}$$

for every a, b of $\mathfrak{M}$, then Γ is regular in $\mathfrak{M}$.

Proof. Evidently (24) implies A and M. The argument of paragraph 3, section 2 proves that it is sufficient to show that the relations

$$|a_1, \ldots, a_n| \geqslant |b_1, \ldots, b_n|, \tag{25}$$

$$b_\kappa b_\lambda = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n), \tag{26}$$

$$|b_1| \leqslant |b_2| \leqslant \ldots \leqslant |b_n|$$

imply the existence of sets a_ν^* satisfying

$$a_\nu^* \subset a_\nu; \quad |a_\nu^*| = |b_\nu| \quad (1 \leqslant \nu \leqslant n),$$

$$|a_\kappa^* a_\lambda^*| = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n).$$

Instead of (25) we shall use only [what, in fact, is equivalent to (25)]

$$|a_\nu| \geqslant |b_\nu| \quad (1 \leqslant \nu \leqslant n).$$

We choose any pair of indices ρ, σ $(1 \leqslant \rho < \sigma \leqslant n)$. We define sets $a_\rho^{(1)}$, $a_\sigma^{(1)}$ as follows.

Case α. Suppose that $|b_\rho| < |b_\sigma|$.

Then we choose $a_\rho^{(1)}$ satisfying

$$a_\rho^{(1)} \subset a_\rho; \quad |a_\rho^{(1)}| = |b_\rho|.$$

The possibility of this choice follows from C and $|a_\rho| \geqslant |b_\rho|$. Put

$$a_\sigma^{(1)} = a_\sigma - a_\rho^{(1)} a_\sigma.$$

Case β. Suppose that $|b_\rho| = |b_\sigma|$.

Case β_1. If $|a_\rho a_\sigma| < |b_\rho|$, put

$$a_\rho^{(1)} = a_\rho - a_\rho a_\sigma,$$

$$a_\sigma^{(1)} = a_\sigma.$$

Case β_2. If $|a_\rho a_\sigma| \geqslant |b_\rho|$, choose $a_\rho^{(1)}$, $a_\sigma^{(1)}$ satisfying

$$a_\rho^{(1)} a_\sigma^{(1)} = 0; \quad a_\rho^{(1)} + a_\sigma^{(1)} \subset a_\rho a_\sigma;$$

$$|a_\rho^{(1)}| = |a_\sigma^{(1)}| = |b_\rho|.$$

The possibility of this choice follows from D and

$$|a_\rho a_\sigma| \geqslant |b_\rho| = \max(|b_\rho|, |b_\sigma|) = |b_\rho + b_\sigma|.$$

We always have

$$a_\rho^{(1)} a_\sigma^{(1)} = 0, \tag{27}$$

$$|a_\rho^{(1)}| \geqslant |b_\rho|; \quad |a_\sigma^{(1)}| \geqslant |b_\sigma|. \tag{28}$$

(27) is obvious. To prove (28) we observe that

$$c \subset d; \quad |c| < |d|,$$

by (24), implies that

$$|c| < |d| = |c + (d - c)| = \max(|c|, |d - c|) = |d - c|,$$

i.e.

$$|d - c| = |d|.$$

Using this remark and the fact that, in case α,

$$|a_\rho^{(1)} a_\sigma| \leqslant |a_\rho^{(1)}| = |b_\rho| < |b_\sigma| \leqslant |a_\sigma|,$$

while, in case β_1,

$$|a_\rho a_\sigma| < |b_\rho| \leqslant |a_\rho|,$$

we find that

in case α: $\quad |a_\sigma^{(1)}| = |a_\sigma - a_\rho^{(1)} a_\sigma| = |a_\sigma| \geqslant |b_\sigma|,$

in case β_1: $\quad |a_\rho^{(1)}| = |a_\rho - a_\rho a_\sigma| = |a_\rho| \geqslant |b_\rho|.$

This proves (28).

We put $\quad a_\nu^{(1)} = a_\nu \quad \text{for} \quad \nu \neq \rho, \sigma.$

Then

$$a_\nu^{(1)} \subset a_\nu \quad (1 \leqslant \nu \leqslant n),$$

$$a_\rho^{(1)} a_\sigma^{(1)} = 0,$$

$$|a_\nu^{(1)}| \geqslant |b_\nu| \quad (1 \leqslant \nu \leqslant n).$$

Now we apply to the sets $a_\nu^{(1)}$ the same process which we applied to the sets a_ν, this time with a different pair ρ, σ. We continue in this way until, after $N = \frac{1}{2}n(n-1)$ steps, we arrive at sets $a_\nu^{(N)}$ satisfying

$$a_\nu^{(N)} \subset a_\nu \quad (1 \leqslant \nu \leqslant n),$$

$$a_\kappa^{(N)} a_\lambda^{(N)} = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n),$$

$$|a_\nu^{(N)}| \geqslant |b_\nu| \quad (1 \leqslant \nu \leqslant n).$$

An application of C now shows the existence of $a_\nu{}^*$ satisfying

$$a_\nu{}^* \subset a_\nu^{(N)} \subset a_\nu; \quad |a_\nu{}^*| = |b_\nu|.$$

Then $\qquad a_\kappa{}^* a_\lambda{}^* = 0 \quad (\kappa < \lambda)$

and $\qquad |a_1{}^*, \ldots, a_n{}^*| \equiv |b_1, \ldots, b_n|.$

Theorem II is therefore proved.

5. *Composition of measure functions.*

Suppose that Γ_1 and Γ_2 are two measure functions, defined for all sets of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ respectively. If $a \prec \mathfrak{M}_1$, we denote by $|a|_1$ the measure of a as measured by Γ_1. Similarly, $|b|_2$ is the "value" of Γ_2, defined for every $b \prec \mathfrak{M}_2$.

Now let us assume that $\mathfrak{M}_1 \subset \mathfrak{M}_2$. Then we define in $\mathfrak{M}_2$ a measure $\Gamma_3 = \Gamma_1 + \Gamma_2$, by means of the following relations:

if $a, b \prec \mathfrak{M}_1$, $|a|_1 \leqslant |b|_1$, then $|a|_3 \leqslant |b|_3$,

if $a \prec \mathfrak{M}_1$, $b \prec \mathfrak{M}_2 - \mathfrak{M}_1$, then $|a|_3 < |b|_3$,

if $a, b \prec \mathfrak{M}_2$, $|a|_2 \leqslant |b|_2$, then $|a|_3 \leqslant |b|_3$.

These postulates define a transitive relation $|c|_3 < |d|_3$ in $\mathfrak{M}_2$.

THEOREM III. *Suppose that* Γ_1 *and* Γ_2 *are regular*† *additive monotonic measures in* $\mathfrak{M}_1$, $\mathfrak{M}_2$ *respectively* ($\mathfrak{M}_1 \subset \mathfrak{M}_2$). *Furthermore, suppose that*

(i) $a \subset b \prec \mathfrak{M}_1$, $a \prec \mathfrak{M}_2$ *implies* $a \prec \mathfrak{M}_1$,

(ii) $a_1 \prec \mathfrak{M}_1$ *implies* $|a_1|_2 = 0$,

(iii) $a_1 \prec \mathfrak{M}_1$, $a_2 \prec \mathfrak{M}_2 - \mathfrak{M}_1$ *implies the existence of* a_2' *such that*

$$a_2' \subset a_2; \quad a_2' \prec \mathfrak{M}_1; \quad |a_1|_1 = |a_2'|_1.$$

Then $\Gamma_3 = \Gamma_1 + \Gamma_2$ *is a regular measure in* $\mathfrak{M}_2$.

Proof. First of all, we show that Γ_3 is monotonic and additive. Suppose first that

$$a' \subset a,$$

$$a', \ a \prec \mathfrak{M}_2.$$

Case α. $a \prec \mathfrak{M}_1$.

Then, by hypothesis (i), $a' \prec \mathfrak{M}_1$, and, since Γ_1 is monotonic

$$|a'|_1 \leqslant |a|_1,$$

i.e. $|a'|_3 \leqslant |a|_3$.

Case β'. $a \prec \mathfrak{M}_2 - \mathfrak{M}_1$, $a' \prec \mathfrak{M}_1$.

Then, by definition of Γ_3,

$$|a'|_3 < |a|_3.$$

Case β''. $a, \ a' \prec \mathfrak{M}_2 - \mathfrak{M}_1$.

Then, as in case α,

$$|a'|_3 \leqslant |a|_3.$$

Hence Γ_3 is monotonic.

† The definition of regularity is given on p. 81.

Now suppose that four sets $a,\ a',\ b,\ b'$ satisfy

$$a,\ a',\ b,\ b' \prec \mathfrak{M}_2,$$

$$aa' = bb' = 0,$$

$$|a|_3 = |b|_3; \quad |a'|_3 = |b'|_3;$$

$$|a'|_3 \prec |a|_3.$$

We have to prove that

$$|a+a'|_3 = |b+b'|_3.$$

From the definition of Γ_3 it follows that $a \prec \mathfrak{M}_1$ implies $b \prec \mathfrak{M}_1$, and *vice versa*.

Case α. $\qquad a \prec \mathfrak{M}_1.$

Then $a' \prec \mathfrak{M}_1$. For otherwise we should have

$$a' \prec \mathfrak{M}_2 - \mathfrak{M}_1, \quad |a|_3 < |a'|_3,$$

which is a contradiction. Hence

$$a,\ a',\ b,\ b' \prec \mathfrak{M}_1,$$

and the conclusion follows from the additivity of Γ_1.

Case β'. $\qquad a \prec \mathfrak{M}_2 - \mathfrak{M}_1; \quad a' \prec \mathfrak{M}_1.$

Then $a+a' \prec \mathfrak{M}_2 - \mathfrak{M}_1$. For otherwise

$$a+a' \prec \mathfrak{M}_1,$$

$$a = (a+a') - a' \prec \mathfrak{M}_1,$$

which is not true. Hence

$$b \prec \mathfrak{M}_2 - \mathfrak{M}_1; \quad b' \prec \mathfrak{M}_1;$$

$$b+b' \prec \mathfrak{M}_2 - \mathfrak{M}_1;$$

$$|a'|_2 = |b'|_2 = 0,$$

and the conclusion follows from

$$|a+a'|_2 = |a|_2 = |b|_2 = |b+b'|_2.$$

Case β''. $\qquad a,\ a' \prec \mathfrak{M}_2 - \mathfrak{M}_1.$

In this case $\qquad a,\ a',\ b,\ b' \prec \mathfrak{M}_2 - \mathfrak{M}_1,$

and the result follows from the definition of Γ_3 and the additivity of Γ_2.

Now we proceed to prove Theorem III. Let us assume that

$$|a_1, \ldots, a_n|_3 \geqslant |b_1, \ldots, b_n|_3.$$

We have to show the existence of sets a_ν^* satisfying

$$a_\nu^* \prec \mathfrak{M}_2; \quad a_\nu^* \subset a_\nu;$$

$$|a_1^*, \ldots, a_n^*|_3 \equiv |b_1, \ldots, b_n|_3. \tag{29}$$

There is no loss of generality† in assuming that

$$b_\kappa b_\lambda = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n).$$

Under this assumption, (29) is equivalent to

$$|a_\nu^*|_3 = |b_\nu|_3 \quad (1 \leqslant \nu \leqslant n), \tag{30}$$

$$|a_\kappa^* a_\lambda^*|_3 = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n). \tag{31}$$

We may assume that

$$b_1, b_2, \ldots, b_m \prec \mathfrak{M}_1,$$

$$b_{m+1}, b_{m+2}, \ldots, b_n \prec \mathfrak{M}_2 - \mathfrak{M}_1,$$

where $0 \leqslant m \leqslant n$.

Let $1 \leqslant \mu \leqslant m$. We define a set a_μ' as follows.

Case α. Suppose that $a_\mu \prec \mathfrak{M}_1$.

Then we put $a_\mu' = a_\mu$.

Case β. Suppose that $a_\mu \prec \mathfrak{M}_2 - \mathfrak{M}_1$.

Since $b_1 + b_2 + \ldots + b_m \prec \mathfrak{M}_1$, hypothesis (iii) shows the existence of a_μ' satisfying

$$a_\mu' \subset a_\mu; \quad a_\mu' \prec \mathfrak{M}_1; \quad |a_\mu'|_1 = |b_1 + \ldots + b_m|_1.$$

Obviously $$|a_1', \ldots, a_m'|_1 \geqslant |b_1, \ldots, b_m|_1.$$

Hence, in view of the regularity of Γ_1, we can find a_μ^* $(1 \leqslant \mu \leqslant m)$ such that

$$a_\mu^* \prec \mathfrak{M}_1; \quad a_\mu^* \subset a_\mu'; \quad |a_1^*, \ldots, a_m^*|_1 \equiv |b_1, \ldots, b_m|_1.$$

Put, for $m < \nu \leqslant n$,

$$a_\nu' = a_\nu - a_\nu(a_1^* + \ldots + a_m^*).$$

† § 3, section 2.

Then, by hypothesis (ii),

$$|a_\nu(a_1^*+\ldots+a_m^*)|_2 \leqslant |a_1^*+\ldots+a_m^*|_2 = 0,$$

$$|a'_{m+1}, \ldots, a_n'|_2 = |a_{m+1}, \ldots, a_n|_2 = |b_{m+1}, \ldots, b_n|_2.$$

Now we use the regularity of Γ_2 and find sets a_ν'' $(m < \nu \leqslant n)$ satisfying

$$a_\nu'' \prec \mathfrak{M}_2; \quad a_\nu'' \subset a_\nu' \quad (m < \nu \leqslant n),$$

$$|a''_{m+1}, \ldots, a_n''|_2 = |b_{m+1}, \ldots, b_n|_2.$$

Then
$$|a_\kappa'' a_\lambda''|_2 = 0 \quad (m < \kappa < \lambda \leqslant n).$$

Hence, if

$$a_\nu^* = a_\nu'' - a_\nu''(a''_{m+1}+\ldots+a''_{\nu-1}+a''_{\nu+1}+\ldots+a_n'') \quad (m < \nu \leqslant n),$$

we have
$$a_\kappa^* a_\lambda^* = 0 \quad (1 \leqslant \kappa < \lambda \leqslant n),$$

$$|a_\mu^*|_1 = |b_\mu|_1 \quad (1 \leqslant \mu \leqslant m),$$

$$|a_\nu^*|_2 = |b_\nu|_2 \quad (m < \nu \leqslant n),$$

i.e. (30) and (31). Thus the theorem is proved.

6. *Application to cardinal numbers.*

In this last paragraph we apply Theorems I, II, III to the most natural measure Γ, *i.e.* the measure defined by the cardinal of a set.

We start from a fixed abstract aggregate S. We take as $\mathfrak{M}$ the system of all subsets of S. A relation

$$|a| \leqslant |b|$$

is now, by definition, equivalent to the fact that the cardinal of b is not less than the cardinal of a. The symbol $|a|$ may be considered as a notation for the cardinal of a.

Let $\mathfrak{M}_1$ be the system of all finite subsets of S, and put $\mathfrak{M}_2 = \mathfrak{M}$. Then, in the notation used in previous paragraphs, $\mathfrak{F} = \mathfrak{M}_1$.

We shall consider three measures, Γ, Γ_1, Γ_2. Their values are denoted by $|a|$, $|a|_1$, $|a|_2$. Γ has been defined above. Γ_1 is identical with Γ, but is defined only in $\mathfrak{M}_1$. Γ_2 is defined in $\mathfrak{M}_2$, by taking

$$|a|_2 = 0, \quad \text{if} \quad a \prec \mathfrak{M}_1,$$

$$|a|_2 = |a|, \quad \text{if} \quad a \prec \mathfrak{M}_2 - \mathfrak{M}_1.$$

Γ_1 satisfies the hypotheses of Theorem I. Γ_2 satisfies the hypotheses of Theorem II. Hence both Γ_1 and Γ_2 are regular in $\mathfrak{M}_1$, $\mathfrak{M}_2$, respectively.

Γ_1 and Γ_2 satisfy the hypotheses of Theorem III. Hence $\Gamma_1+\Gamma_2$ is regular in $\mathfrak{M}_2$. But evidently $\Gamma_1+\Gamma_2=\Gamma$. Therefore Γ is regular in $\mathfrak{M}$.

A relation

$$|a_1, \ldots, a_n| \equiv |b_1, \ldots, b_n| \tag{32}$$

is, in our special case, equivalent to the following statement. It is possible to define a one-one correspondence between all elements of $a_1+\ldots+a_n$ and all elements of $b_1+\ldots+b_n$ which has the property that, for $1 \leqslant \nu \leqslant n$, a_ν is mapped on b_ν.

This can be seen in the following way. First, if a correspondence of the described kind exists, then, by means of this correspondence, every set a obtained from $a_1, \ldots, a_n$ by the three processes of addition, multiplication, and subtraction, is mapped on the set b obtained from $b_1, \ldots, b_n$ in the same way. Hence $|a|=|b|$, *i.e.* (32) holds.

If, on the other hand, (32) is true, then, using the notation of § 3, section 2,

$$|a_{(\nu_\kappa)}| = |b_{(\nu_\kappa)}|.$$

Hence there is a one-one correspondence between the elements of $a_{(\nu_\kappa)}$ and $b_{(\nu_\kappa)}$. For different systems (ν_κ) these mappings do not interfere since sets $a_{(\nu_\kappa)}$ belonging to different systems (ν_κ) do not overlap. Hence there is a one-one mapping of $a_1+\ldots+a_n$ on $b_1+\ldots+b_n$, and this correspondence has the properties required.

Therefore the fact that Γ is regular may be expressed in the following way.

THEOREM IV. *Let $a_1, \ldots, a_n$; $b_1, \ldots, b_n$ be two ordered systems of n abstract sets. Suppose that the cardinal of any set a obtained from $a_1, \ldots, a_n$ by a finite number of applications of the processes of addition and multiplication is at least equal to the cardinal of the set b obtained from $b_1, \ldots, b_n$ by exactly the same processes. Then there is a one-one correspondence between all elements of $b_1+\ldots+b_n$ and some elements of $a_1+\ldots+a_n$ which maps every b_ν on a subset of the corresponding set a_ν.*

Theorem IV suggests the introduction of the notion of the "cardinal of a system of n sets", a direct generalization of the notion of the cardinal of one set. We say that two ordered systems

$$a_1, a_2, \ldots, a_n,$$

$$b_1, b_2, \ldots, b_n,$$

of sets have the same *generalized cardinal* if, and only if, there is a one-one correspondence between all elements of $a_1+\ldots+a_n$ and all elements of $b_1+\ldots+b_n$ which maps every a_ν on the corresponding b_ν.

For the sake of clearness we introduce a special symbol for the generalized cardinal, say

$$[a_1,\ a_2,\ \ldots,\ a_n].$$

Then
$$[a_1,\ \ldots,\ a_n] = [b_1,\ \ldots,\ b_n]$$

means the same thing as

$$|a_1,\ \ldots,\ a_n| \equiv |b_1,\ \ldots,\ b_n|.$$

There is a natural extension of the definition of inequalities

$$[a] \geqslant [b]$$

between ordinary cardinals to generalized cardinals. We say that an inequality

(33)
$$[a_1,\ \ldots,\ a_n] \geqslant [b_1,\ \ldots,\ b_n]$$

expresses the fact that there is a one-one correspondence between some elements of $a_1+\ldots+a_n$ and all elements of $b_1+\ldots+b_n$ which maps every b_ν on a subset of the corresponding a_ν.

For $n = 1$, this agrees with the ordinary definition.

An equivalent definition is: (33) expresses the fact that, for suitable sets $a_\nu' \subset a_\nu$,

$$[a_1',\ a_2',\ \ldots,\ a_n'] = [b_1,\ b_2,\ \ldots,\ b_n].$$

Then Theorem IV asserts that (33) is equivalent to

$$|a_1,\ \ldots,\ a_n| \geqslant |b_1,\ \ldots,\ b_n|.$$

There are, however, two fundamental differences between the cases $n = 1$ and $n > 1$.

(1) For $n = 1$, any two cardinals are comparable if we allow the use of Zermelo's axiom. For $n > 1$, this is not so, as is shown by the following example.

$$n = 2, \quad a_1 = \{1,\ 2\}, \quad b_1 = \{1,\ 2\},$$

$$a_2 = \{3,\ 4\}, \quad b_2\ \{1,\ 3\}.$$

Here none of the relations

$$[a_1, a_2] \geqq [b_1, b_2],$$

$$[a_1, a_2] \leqq [b_1, b_2]$$

is true.

In fact, Theorem IV gives a criterion for comparability of generalized cardinals.

(2) For $n = 1$, Bernstein's theorem holds; this asserts that

$$[a_1] \geqq [b_1],$$

$$[a_1] \leqq [b_1],$$

implies $$[a_1] = [b_1].$$

For $n > 1$, this is no longer the case, as is shown by the fact that the two relations

$$[\{1, 2, 3, \ldots\}, \{1, 2, 3, \ldots\}] \geqq [\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}],$$

$$[\{1, 2, 3, \ldots\}, \{1, 2, 3, \ldots\}] \leqq [\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}]$$

are true, while, on the other hand,

$$[\{1, 2, 3, \ldots\}, \{1, 2, 3, \ldots\}] = [\{1, 2, 3, \ldots\}, \{2, 4, 6, \ldots\}]$$

is not satisfied.

As an illustration of Theorem IV, consider the example

$$a_1 = \{1, 4\}, \quad b_1 = \{2, 3\},$$

$$a_2 = \{2, 4\}, \quad b_2 = \{1, 3\},$$

$$a_3 = \{3, 4\}, \quad b_3 = \{1, 2\}.$$

Here $$|a_\nu| \geqq |b_\nu| \qquad (1 \leqq \nu \leqq 3),$$

$$|a_\kappa a_\lambda| \geqq |b_\kappa b_\lambda|, \quad |a_\kappa + a_\lambda| \geqq |b_\kappa + b_\lambda| \quad (1 \leqq \kappa < \lambda \leqq 3),$$

$$|a_1 a_2 a_3| \geqq |b_1 b_2 b_3|, \quad |a_1 + a_2 + a_3| \geqq |b_1 + b_2 + b_3|,$$

but, nevertheless,

$$[a_1, a_2, a_3] \geqq [b_1, b_2, b_3]$$

is not true. In fact, the two generalized cardinals are not comparable, as is seen from

$$|a_1 a_2 a_3| > |b_1 b_2 b_3|.$$

$$|a_1 a_2 + a_3| < |b_1 b_2 + b_3|.$$

The obvious extension of Theorem IV to the case $n = \infty$ is not true. Not even the extension of Hall's theorem to $n = \infty$ is true. Put, for instance,

$$a_1 = \{1, 2, 3, \ldots\},$$

$$a_\nu = \{\nu - 1\} \quad (\nu = 2, 3, 4, \ldots).$$

Any sum of k sets a_ν (k finite or infinite) contains at least k different elements, but the conclusion of Hall's theorem does not hold.

However, if all sets a_ν are finite, Theorem IV is true for $n = \infty$.

The University,
Sheffield.

THE RELATION BETWEEN DE LA VALLÉE POUSSIN AND ABEL SUMMABILITY

By B. KUTTNER.

[Received 10 June, 1937.—Read 17 June, 1937.]

1. In a recent paper*, Hyslop has shown that summability $(V\text{-}P)$ and summability $(A, 2)$ are equivalent for all those series Σa_n which satisfy the condition that $a_n = O(n^k)$, where k is any constant. In that paper, an example was given to show that, when no restriction is imposed on the a_n's, summability $(A, 2)$ does not imply summability $(V\text{-}P)$. The question whether or not summability $(V\text{-}P)$ implies summability $(A, 2)$ was, however, left unsettled. The object of the present paper is to settle this question by proving the following

THEOREM. *If a series is summable* $(V\text{-}P)$, *then it is summable* $(A, 2)$ *to the same sum.*

2. We first prove a number of lemmas.

LEMMA 1. *Let* V_k *be the* k*-th* $(V\text{-}P)$ *sum of*

$$\sum_{n=0}^{\infty} a_n,$$

i.e., let

$$V_k = \sum_{n=0}^{k} \frac{(k!)^2}{(k-n)!\,(k+n)!}\, a_n.$$

Then

$$a_n = 2 \sum_{k=0}^{n} (-1)^{n-k} \frac{n(n+k-1)!}{(k!)^2\,(n-k)!}\, V_k,$$

the term $n(n+k-1)!$ *being replaced*† *by* $\frac{1}{2}$ *when* $n = k = 0$.

* J. M. Hyslop, "Some relations between the de la Vallée Poussin and Abel methods of summability", *Proc. London Math. Soc.* (2), 40 (1936), 449–467. See that paper for definitions of summability $(V\text{-}P)$ and (A, μ).

† We suppose this done throughout this paper. 0! is taken to mean 1.

Proofs of this result have been given by Douglass* and by Rutledge†. The following proof was suggested by a Referee.

The result is trivial when $n = 0$, and we therefore suppose that $n > 0$. Let

$$f_n(s) = \sum_{m=0}^{n} a_m \cos 2ms.$$

Then
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f_n(s) \cos 2ns \, ds.$$

Now, if $n > 0$,

$$\cos 2ns = \sum_{k=0}^{n} (-1)^{n-k} 2^{2k} \frac{n(n+k-1)!}{(n-k)!\,(2k)!} (\cos s)^{2k}, \tag{1}$$

and hence

$$a_n = \frac{1}{\pi} \sum_{k=0}^{n} (-1)^{n-k} \frac{2^{2k} n(n+k-1)!}{(n-k)!\,(2k)!} \int_{-\pi}^{\pi} f_n(s)(\cos s)^{2k} ds. \tag{2}$$

We have, for $k \leqslant n$,

$$\begin{aligned} \int_{-\pi}^{\pi} f_n(s)(\cos s)^{2k} ds &= \sum_{m=0}^{n} a_m \int_{-\pi}^{\pi} \cos 2ms \, (\cos s)^{2k} ds \\ &= \frac{\pi}{2^{2k-1}} \sum_{m=0}^{k} \frac{(2k)!}{(k-m)!\,(k+m)!} a_m \\ &= \frac{\pi}{2^{2k-1}} \frac{(2k)!}{(k!)^2} V_k, \end{aligned}$$

and, on substituting this in (2), we obtain the lemma at once.

Lemma 2. *Let*

$$F_k(y) = 2 \sum_{n=k}^{\infty} (-1)^{n-k} \frac{n(n+k-1)!}{(k!)^2 (n-k)!} e^{-n^2 y}.$$

Then, if V_k is bounded, the series

$$\sum_{n=0}^{\infty} a_n e^{-n^2 y}$$

* R. D. Douglass, "Stirling expansions derived by finite de la Vallée Poussin summation", *Journal Math. and Physics Mass. Inst. Tech.*, 10 (1931), 131–156.

† G. Rutledge, "The inverse matrix for de la Vallée Poussin summation", *ibid.*, 11 (1932), 73–82.

converges for all positive values of y to

$$\sum_{k=0}^{\infty} V_k F_k(y).$$

It follows from Lemma 1 that

$$\sum_{n=0}^{\infty} a_n e^{-n^2 y} = 2 \sum_{n=0}^{\infty} e^{-n^2 y} \sum_{k=0}^{n} (-1)^{n-k} \frac{n(n+k-1)!}{(k!)^2 (n-k)!} V_k, \tag{3}$$

provided that the series converge. Now, if $n > 0$,

$$2 \sum_{k=0}^{n} \frac{n(n+k-1)!}{(k!)^2 (n-k)!} \leqslant \sum_{k=0}^{n} \frac{(2n)!}{n!\, k!\, (n-k)!} = \frac{(2n)!}{(n!)^2} 2^n = O(2^{3n}),$$

by Stirling's theorem. It follows that the double series on the right of (3) is absolutely convergent. Hence the series on the left of (3) converges; further, we may invert the order of summation on the right of (3), and the lemma follows at once.

LEMMA 3.

$$F_k(y) = \frac{(2k)!}{2^{2k+1} (k!)^2} \frac{i}{\pi} \int_C \frac{\sin s}{(\cos s)^{2k+1}} \vartheta_3(s, e^{-y})\, ds,$$

where the contour C encircles the point $s = \frac{1}{2}\pi$ once in a counterclockwise direction, but does not enclose any other poles of the integrand.

It may be proved, by taking as the contour C the rectangle whose vertices are $\pm Ri$, $\pi \pm Ri$, and then making $R \to \infty$, that, if $m \geqslant 1$,

$$\frac{1}{2\pi i} \int_C \frac{\sin s}{(\cos s)^{2m+1}}\, ds = 0,$$

and it is evident that

$$\frac{1}{2\pi i} \int_C \frac{\sin s}{\cos s}\, ds = -1.$$

We deduce with the aid of equation (1) that, if $n > 0$,

$$\frac{1}{2\pi i} \int_C \frac{\sin s \cos 2ns}{(\cos s)^{2k+1}}\, ds = 0 \quad \text{if} \quad n < k$$

$$= (-1)^{n-k+1} \frac{2^{2k} n(n+k-1)!}{(n-k)!\,(2k)!} \quad \text{if} \quad n \geqslant k.$$

It follows that

$$\frac{1}{2\pi i} \int_C \frac{\sin s}{(\cos s)^{2k+1}} \vartheta_3(s, e^{-y})\, ds = -2 \sum_{n=k}^{\infty} (-1)^{n-k} \frac{2^{2k} n(n+k-1)!}{(n-k)!\,(2k)!} e^{-n^2 y},$$

which proves the lemma.

Lemma 4.

$$F_k(y) = \frac{(2k)!}{2^{2k}(k!)^2}\,\frac{i}{(\pi y)^{\frac{1}{2}}}\int_\Gamma \frac{\sin s}{(\cos s)^{2k+1}}\exp\left(-\frac{s^2}{y}\right)ds,$$

where Γ is a contour which starts from $+\infty$, goes above the real axis to the origin, and returns to $+\infty$ below the real axis.

By Jacobi's transformation,

$$\vartheta_3(s, e^{-y}) = (\pi/y)^{\frac{1}{2}}\exp(-s^2/y)\,\vartheta_3(\pi i s/y, e^{-\pi^2/y}).$$

Hence, by Lemma 3,

$$F_k(y) = \frac{(2k)!}{2^{2k+1}(k!)^2}\,\frac{i}{(\pi y)^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty}\int_C \frac{\sin s}{(\cos s)^{2k+1}}\exp\left(-\frac{(s-n\pi)^2}{y}\right)ds.$$

Writing s for $s-n\pi$ in each integral, we obtain

$$F_k(y) = \frac{(2k)!}{2^{2k+1}(k!)^2}\,\frac{i}{(\pi y)^{\frac{1}{2}}}\int \frac{\sin s}{(\cos s)^{2k+1}}\exp\left(-\frac{s^2}{y}\right)ds,$$

the integral being taken along a path which passes from $+\infty$ to $-\infty$ above the real axis, and from $-\infty$ to $+\infty$ below the real axis. Both halves of the path may be taken to pass through the origin. Since the integrand is an odd function, the contribution to the integral arising from the portions of the path in the first and fourth quadrants is equal to that arising from the portions in the second and third quadrants, and the lemma follows.

Lemma 5. *Let*

$$G_m(y) = \int_\Gamma \frac{f(s)}{(\cos s)^{2m}}\exp\left(-\frac{s^2}{y}\right)ds,$$

the contour Γ being the same as in Lemma 4. Suppose that at all points of Γ, $|\cos s| \geqslant 1$, and that, except at the origin, $\sigma^2 > t^2$, where $s = \sigma + it$. Suppose that, in some neighbourhood of the origin, $f(s)$ is regular and $O(|s|^\lambda)$, where $\lambda \geqslant 0$, and that, for large $|s|$ and s on the contour Γ, $f(s) = O(|s|^k)$, where k is any constant. Then, for y small,*

$$G_m(y) = O(y^{\frac{1}{2}(\lambda+1)})$$

uniformly in m.

* It may easily be verified that, on the curve $|\cos s| = 1$, $\sigma^2 > t^2$ except at the origin. Hence it is possible to draw Γ so as to satisfy these conditions.

Let Γ_1 denote the portion of Γ in some neighbourhood of the origin, and let Γ_2 denote the rest of Γ. Since $f(s)$ is regular, we may take the portion of Γ_1 above the real axis to be part of the curve $t = \sigma(1-\frac{1}{6}\sigma^2)$, and the rest of Γ_1 to be the image of this in the real axis. We then have, on Γ_1,

$$|\cos s|^2 = \tfrac{1}{2}(\cos 2\sigma + \cosh 2t) = 1 + (t^2-\sigma^2) + \tfrac{1}{3}(t^4+\sigma^4) + O(\sigma^6)$$
$$= 1 + \tfrac{1}{3}\sigma^4 + O(\sigma^6) \geqslant 1,$$

provided that the neighbourhood of the origin considered is sufficiently small. Further, b and c being positive constants, we have

$$\left|\exp\left(-\frac{s^2}{y}\right)\right| = \exp\left(\frac{-\sigma^2+t^2}{y}\right) \leqslant \exp\left(-\frac{b\sigma^4}{y}\right);$$

and, on Γ_2, we have

$$\left|\exp\left(-\frac{s^2}{y}\right)\right| \leqslant \exp\left(-\frac{c\sigma^2}{y}\right),$$

Hence

$$G_m(y) = O\left\{\int_0^\infty \sigma^\lambda \exp\left(-\frac{b\sigma^4}{y}\right) d\sigma + \int_A^\infty \sigma^k \exp\left(-\frac{c\sigma^2}{y}\right) d\sigma\right\},$$

where $A > 0$, and the result follows.

LEMMA 6.

$$\text{(i)} \quad F_k(y) = O(k^{-\frac{1}{2}});$$

$$\text{(ii)} \quad F_k(y) = O\left\{\frac{y}{(ky-1)^2(k+1)^{\frac{1}{2}}}\right\},$$

where the O's are uniform in k and y.

We write

$$I_k(y) = \int_\Gamma \frac{\sin s}{(\cos s)^{2k+1}} \exp\left(-\frac{s^2}{y}\right) ds,$$

so that, by Lemma 4,

$$F_k(y) = \frac{(2k)!}{2^{2k}(k!)^2} \frac{i}{(\pi y)^{\frac{1}{2}}} I_k(y).$$

By Lemma 5, $I_k(y) = O(y^{\frac{1}{2}})$; and, by Stirling's theorem,

$$\frac{(2k)!}{2^{2k}(k!)^2} = O(k^{-\frac{1}{2}}) \quad \text{as} \quad k \to \infty.$$

The first part of the lemma follows at once. To prove the second part, we have, integrating by parts,

$$I_k(y) = \frac{1}{ky}\int_\Gamma \frac{s}{(\cos s)^{2k}} \exp\left(-\frac{s^2}{y}\right) ds.$$

Hence

$$(ky-1)\, I_k(y) = \int_\Gamma \frac{\sin s}{(\cos s)^{2k+1}} (s\cot s - 1) \exp\left(-\frac{s^2}{y}\right) ds \tag{4}$$

$$= J_k(y),$$

say. Integrating by parts again, we have

$$J_k(y) = \frac{1}{ky}\int_\Gamma \frac{s}{(\cos s)^{2k}} (s\cot s - 1) \exp\left(-\frac{s^2}{y}\right) ds$$

$$-\frac{1}{2k}\int_\Gamma \frac{1}{(\cos s)^{2k}} (\cot s - s\,\mathrm{cosec}^2 s) \exp\left(-\frac{s^2}{y}\right) ds,$$

whence

$$(ky-1)\, J_k(y) = \int_\Gamma \frac{\sin s}{(\cos s)^{2k+1}} (s\cot s - 1)^2 \exp\left(-\frac{s^2}{y}\right) ds$$

$$-\tfrac{1}{2}y\int_\Gamma \frac{1}{(\cos s)^{2k}} (\cot s - s\,\mathrm{cosec}^2 s) \exp\left(-\frac{s^2}{y}\right) ds$$

$$= O(y^{\frac{3}{2}}),$$

by Lemma 5. Hence, by (4),

$$I_k(y) = O\left\{\frac{y^{\frac{3}{2}}}{(ky-1)^2}\right\},$$

and the lemma follows.

3. We are now in a position to prove the theorem. The result will evidently follow from Lemma 2 if we can show that

(*a*) for fixed k, $F_k(y) \to 0$ as $y \to 0$,

(*b*) $\sum_{k=0}^{\infty} |F_k(y)|$ is bounded for y sufficiently small, and

(*c*) $\sum_{k=0}^{\infty} F_k(y) = 1$.

Now (a) follows at once from Lemma 6 (ii). To prove (b), we have, by Lemma 6,

$$(5)\quad \sum_{k=0}^{\infty} |F_k(y)| = O\left\{y \sum_{0\leqslant k<A(y)} \frac{1}{(ky-1)^2(k+1)^{\frac{1}{2}}}\right\} + O\left\{\sum_{A(y)\leqslant k\leqslant B(y)} \frac{1}{k^{\frac{1}{2}}}\right\} + O\left\{y \sum_{k>B(y)} \frac{1}{(ky-1)^2(k+1)^{\frac{1}{2}}}\right\},$$

where $A(y) = y^{-1} - y^{-\frac{1}{2}}$ and $B(y) = y^{-1} + y^{-\frac{1}{2}}$. The expression on the right of (5) is

$$(6)\quad O\left\{y \sum_{0\leqslant k\leqslant \frac{1}{2}y^{-1}} \frac{1}{(k+1)^{\frac{1}{2}}}\right\} + O\left\{y^{\frac{3}{2}} \sum_{0\leqslant k<A(y)} \frac{1}{(ky-1)^2}\right\} + O(1) + O\left\{y^{\frac{3}{2}} \sum_{k>B(y)} \frac{1}{(ky-1)^2}\right\}.$$

Now
$$\sum_{k>B(y)} \frac{1}{(ky-1)^2} \leqslant \int_{B(y)-1}^{\infty} \frac{dk}{(ky-1)^2} = \frac{1}{y(y^{\frac{1}{2}}-y)} = O(y^{-\frac{3}{2}});$$

the second sum in (6) can be dealt with in the same way, and (b) now follows at once. Finally, (c) can be deduced at once from Lemma 2 by taking the special case in which $a_0 = 1$, and $a_n = 0$ for $n > 0$.

4. In conclusion, it may be remarked that the weaker result that, if a series is summable $(V\text{-}P)$, then it is summable (A, μ) for $1 < \mu < 2$, could have been deduced at once from trivial extensions of known theorems. It follows from a result due to Gronwall* that if,

$$(7)\quad \sum_{n=0}^{\infty} a_n$$

is summable $(V\text{-}P)$ to l, and if $\phi(s)$ denotes the function defined by the series

$$(8)\quad \sum_{n=0}^{\infty} a_n e^{-ns}$$

(extended, if necessary, by analytic continuation), then $\phi(s) \to l$ when $s \to 0$ so that $-\theta \leqslant \arg s \leqslant \theta$, where $\theta < \frac{1}{4}\pi$. Again, it is known, as a

* T. H. Gronwall, "Summation of series and conformal mapping", *Annals of Math.* (2), 33 (1932), 101–117. (Theorem 1.)

special case of a result due to Miss Cartwright*, that, if the series (8) converges for $\Re s > 0$, and tends to l when $s \to 0$ so that $-\alpha \leqslant \arg s \leqslant \alpha$, where $\alpha > \frac{1}{2}\pi(1-1/\mu)$, then the series (7) is summable (A, μ) to l, provided that

$$\sum_{n=0}^{\infty} a_n e^{-n^\mu y} \tag{9}$$

converges for $y > 0$. Further, the convergence of (9) can be established by the argument of Lemma 2. These theorems, as they stand, do not quite yield the required result, since Miss Cartwright's theorem requires the convergence of (8) for $\Re s > 0$. But it is easily seen that Miss Cartwright's result will continue to hold when this condition is not satisfied, provided that $\phi(s)$ is regular in $|\arg s| \leqslant \alpha$, $s \neq 0$. (For it follows by analytic continuation that equation (3.11) of her paper still holds.) Again, the $(V\text{-}P)$ summability of (7) implies† the regularity of $\phi(s)$ in $|w| < 1$, where $w = \operatorname{sech}^2 \frac{1}{2}s$, and this domain includes $|\arg s| \leqslant \frac{1}{4}\pi$, $s \neq 0$. The result now follows at once.

The University,
Birmingham.

* M. L. Cartwright, "On the relation between the different types of Abel summation", *Proc. London Math. Soc.* (2), 31 (1930), 81–96. (Theorem II.)

† Gronwall, *loc. cit.* [Equation (3)].

APPROXIMATION BY TRIGONOMETRICAL POLYNOMIALS AND ALMOST PERIODICITY*

By FRANTIŠEK WOLF.

[Received 6 May, 1937.—Read 20 May, 1937.]

A. S. Besicovitch proved in a paper on almost periodic functions† that the approximation by trigonometrical polynomials, uniformly in any finite interval, does not necessarily imply the almost periodicity of the approximated function. He actually showed that any arbitrary continuous function can be approximated to in this way. The aim of this paper is to show that under suitable conditions on the function (boundedness, uniform continuity) and on the set of exponents λ_n, which occur in the approximating polynomials, we can infer the almost periodicity from the uniform approximation in any finite interval.

By a reducible set of points I understand a set with enumerable derivative.

The first sections are concerned with sequences of trigonometrical polynomials which converge uniformly in any finite interval. The main theorems are enunciated in §§ 17, 18, 22, and 23.

The last sections are devoted to showing the necessity of the condition for the set of exponents λ_n.

Let us first state a series of well-known relations‡. If

$$C_{1+k}(t) = \frac{t^{k+1}}{\Gamma(k+1)} \int_0^1 (1-u)^k \cos tu \, du \quad (k > -1), \tag{1.1}$$

* I am much indebted to Mr. A. S. Besicovitch and Mr. A. C. Offord for reading the manuscript and helping me in improving its form. The paper has been written during my stay in Cambridge, which was made possible by a grant received from the Denis fund of the Czechoslovak Government in Prague.

† "On generalized almost periodic functions", *Proc. London Math. Soc.*, 25 (1926), 506.

‡ Hobson, *Theory of functions*, 2 (1926), 567.

then

$$(1.2)\quad \frac{\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)}\,C_{1+k}(\omega t)\,e^{i\lambda t}\,dt = \begin{cases} (\omega-|\lambda|)^k & \text{for} \quad \omega \geqslant |\lambda| \\ 0 & \text{for} \quad \omega \leqslant |\lambda|. \end{cases}$$

And for $n = [\frac{1}{2}(k+1)]$ we have

$$(1.3)\qquad C_{1+k}(t) = \frac{t^{k-1}}{\Gamma(k+1)} - \frac{t^{k-3}}{\Gamma(k-1)} + \ldots + (-1)^n\, C_{k+1-2n}(t),$$

and

$$(1.4)\qquad C_1(t) = \sin t, \quad C_0(t) = \cos t.$$

Further, it is evident that

$$(1.5)\qquad \lim_{t\to\infty} \frac{C_{1+k}(t)}{t^{k-1}} = \frac{1}{\Gamma(k+1)}.$$

LEMMA. *If $P(x)$ denotes a finite trigonometrical polynomial*

$$P(x) = \sum_n a_{\lambda_n} e^{i\lambda_n x},$$

then

$$\frac{2\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)}\,C_{1+k}(\omega t)\,P(x+t)\cos t\,dt$$
$$= \sum a_{\lambda_n} e^{i\lambda_n x}[(\omega-|\lambda_n+1|)^k + (\omega-|\lambda_n-1|)^k].$$

If one of the brackets becomes negative we replace it by zero.

The verification by means of (1.2) is obvious.

2. LEMMA. *If $|\lambda_n| \leqslant 1$, then, for k even, we have the formula*

$$(2.1)\qquad \frac{2\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)}\,C_{1+k}(t)\,P(x+t)\cos t\,dt = P^{(k)}(x),$$

and, for k odd, we have

$$(2.2)\qquad \frac{2\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)}\,C_{1+k}(t)\,P(x+t)\sin t\,dt = P^{(k)}(x).$$

These formulae are readily deduced from Lemma 1.

3. LEMMA. *If $A(t) = O(t^{k-1})$ for large t, then*

$$(3.1)\qquad I = \left(\int_{-\infty}^{-1} + \int_1^{\infty}\right) t^{-(1+k)}\,A(t+x)\,dt = O(x^{k-1})$$

for $k \geqslant 1$ and large x.

Suppose that

$$|A(t)| \leqslant M(1+|t|^{k-1}) \tag{3.2}$$

for all t. Then

$$\begin{aligned} |I| &\leqslant M\left(\int_{-\infty}^{-|x|}+\int_{|x|}^{\infty}\right)\frac{1+|x+t|^{k-1}}{|t|^{k+1}}\,dt+M\left(\int_{-|x|}^{-1}+\int_{1}^{|x|}\right)\frac{1+|x+t|^{k-1}}{|t|^{k+1}}\,dt \\ &\leqslant M\left(\int_{-\infty}^{-|x|}+\int_{|x|}^{\infty}\right)\left[\frac{1}{|t|^{k+1}}+\frac{2}{t^2}\right]dt+2M\int_{1}^{|x|}\left(\frac{1}{t^{k+1}}+\frac{2x^{k-1}}{t^{k+1}}\right)dt \\ &= O(x^{k-1}), \end{aligned}$$

and the lemma is proved.

4. *Definition. Let*

$$P_m(x) \underset{k}{\mathscr{C}\to} f(x) \tag{4.1}$$

signify that, as $m\to\infty$, $P_m(x)$ *converges to* $f(x)$ *uniformly in any finite interval, and that there is a constant* M *independent of* m *such that*

$$|P_m(x)| \leqslant M(1+|x|^{k-1}). \tag{4.2}$$

LEMMA. *If*

(1) $D(t) = O(t^{-k-1})$ *for large* t, *and is continuous,*

(2) $P_m(x) \underset{k}{\mathscr{C}\to} f(x)$,

then

$$\int_{-\infty}^{\infty} D(t)\,P_m(x+t)\,dt \underset{k}{\mathscr{C}\to} \int_{-\infty}^{\infty} D(t)f(x+t)\,dt. \tag{4.3}$$

This lemma follows easily from Lemma 3.

5. LEMMA. *If*

$$\begin{aligned} \gamma(\lambda,\omega) &= (\omega-\lambda)^l \quad \textit{for} \quad |\lambda| \leqslant \omega, \\ &= 0 \quad \textit{for} \quad |\lambda| > \omega, \end{aligned} \tag{5.1}$$

then

$$\delta_l(\lambda,\phi,\psi) = \frac{1}{l!}\sum_{p=0}^{l}(-1)^p\binom{l}{p}\gamma(\lambda,\phi+2p\psi) \tag{5.2}$$

is an even continuous function with the following properties:

(i) *It is non-increasing for* $\lambda \geqslant 0$.

(ii) *It has* $l-1$ *continuous differential coefficients.*

(iii)

$$\begin{cases} \delta_l(\lambda, \phi, \psi) = 1 & \text{for} \quad |\lambda| \leqslant \phi, \\ \delta_l(\lambda, \phi, \psi) = 0 & \text{for} \quad |\lambda| \geqslant \phi + 2l\psi. \end{cases} \tag{5.3}$$

(iv)

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\psi t}{\psi t}\right)^l \frac{\sin(\phi+l\psi)t}{t}\,P(x+t)\,dt = \Sigma a_{\lambda_n} e^{i\lambda_n x}\,\delta_l(\lambda_n, \phi, \psi). \tag{5.4}$$

From (1.3) and (1.4),

$$\frac{1}{(2\psi)^k}\sum_{p=0}^{k}(-1)^p\binom{k}{p}C_{1+k}\{(\phi+2p\psi)t\} = \sin(\phi+k\psi)t\left(\frac{\sin\psi t}{\psi}\right)^k. \tag{5.5}$$

Combining (1.2), (5.2) and (5.5), we get (5.4). The rest of the proof is elementary.

6. LEMMA. *If*

$$(1)\quad |\lambda_n| \leqslant \Lambda, \qquad (2)\quad P_m(x) \underset{k}{\mathcal{C}\to} f(x),$$

then

$$P_m^{(n)}(x) \underset{k}{\mathcal{C}\to} f^{(n)}(x). \tag{6.1}$$

Further, if $f(x)$ *is bounded, then it is uniformly continuous and the same is true of all its derivatives.*

In (5.4) take $\phi = \Lambda$, $l = k$, so that $\delta(\lambda_n, \phi, \psi) = 1$ for all λ_n. Then (5.4) can be written

$$P_m(x) = \int_{-\infty}^{\infty} K(t)\,P_m(x+t)\,dt = \int_{-\infty}^{\infty} K(t-x)\,P_m(t)\,dt, \tag{6.2}$$

and hence

$$P_m^{(n)}(x) = (-1)^n\int_{-\infty}^{\infty} K^{(n)}(t-x)\,P_m(t)\,dt. \tag{6.3}$$

It is evident that $k^{(n)}(t) = O(t^{-k-1})$ for large t.

By Lemma 4 we find

$$P_m^{(n)}(x) \underset{k}{\mathscr{C}\to} (-1)^n \int_{-\infty}^{\infty} k^{(n)}(t-x) f(t)\, dt. \tag{6.4}$$

Similarly we deduce from (6.2)

$$f(x) = \int_{-\infty}^{\infty} k(t-x) f(t)\, dt. \tag{6.5}$$

Since

$$|f(t)| \leqslant M(1+|t|^{k-1}), \tag{6.6}$$

we may differentiate this integral under the sign of integration and obtain

$$f^{(n)}(x) = (-1)^n \int_{-\infty}^{\infty} k^{(n)}(t-x) f(t)\, dt. \tag{6.7}$$

Now (6.7) and (6.4) yield (6.1).

If now $f(t)$ is bounded, then by (6.7) all its derivatives are bounded and uniformly continuous.

7. LEMMA. *If*

$$(1)\ |\lambda_n| \geqslant \delta > 0, \qquad (2)\ P_m(x) \underset{k}{\mathscr{C}\to} f(x),$$

then

$$\int_0^x P_m(x)\, dx \underset{k}{\mathscr{C}\to} \int_0^x f(x)\, dx. \tag{7.1}$$

Further, there is a sequence of numbers $c_m \to c$, such that

$$\int_0^x P_m(x)\, dx - c_m$$

has no absolute term and

$$\int_0^x dy \left[\int_0^y P_m(z)\, dz - c_m \right] \underset{k}{\mathscr{C}\to} \int_0^x dy \left[\int_0^y f(z)\, dz - c \right]. \tag{7.2}$$

We may, in the same manner, repeat this process of integration.

In Lemma 5 put $l = k+1$, and $\phi = \psi = \delta/(2l+1)$, so that $\phi + 2l\psi = \delta$. Then (5.4) can be written

$$\int_{-\infty}^{\infty} k(t)\, P_m(x+t)\, dt = 0, \tag{7.3}$$

since $\delta_l(\lambda_n, \phi, \psi) = 0$ for all λ_n.

Therefore, using the relation

$$\int_{-\infty}^{\infty} k(t)\,dt = 1, \tag{7.4}$$

we get

$$P_m(x) = -\int_{-\infty}^{\infty} k(t)[P_m(x+t)-P_m(x)]\,dt. \tag{7.5}$$

Since $k(t)$ is even, this can be written

$$P_m(x) = -\int_0^{\infty} k(t)\,[P_m(x+t)+P_m(x-t)-2P_m(x)]\,dt. \tag{7.6}$$

Now $$k_1(t) = \int_t^{\infty} k(\tau)\,d\tau = O(t^{-k-1}).$$

Therefore, integrating (7.6) by parts, we get

$$P_m(x) = -\int_0^{\infty} k_1(t)[P_m'(x+t)-P_m'(x-t)]\,dt,$$

and so

$$Q_m(y) = \int_0^y P_m(x)\,dx = -\int_0^{\infty} k_1(t)[P_m(y+t)+P_m(y-t)-2P_m(y)]\,dt.$$

From Lemma 4 we easily obtain (7.1).

The constant term of $Q_m(y)$ is

$$c_m = \int_{-\infty}^{\infty} k(t)\,Q_m(t+x)\,dt \underset{k}{\mathscr{C}\to} \int_{-\infty}^{\infty} k(t)\,dt \int_0^{x+t} f(y)\,dy = c,$$

and hence, integrating once more, we get

$$\int_0^x \big(Q_m(x)-c_m\big)\,dx \underset{k}{\mathscr{C}\to} \int_0^x dx\Big[\int_0^y f(z)\,dz - c\Big].$$

8. LEMMA. *If*

$$(1)\ \ |\lambda_n| \geqslant \delta > 0, \qquad (2)\ \ P_m(x) \underset{k}{\mathscr{C}\to} f(x),$$

then, for $\alpha > 1$,

$$\int_1^{\infty} t^{-\alpha} P_m(t+x)\,dt \underset{k}{\mathscr{C}\to} -\sum_{p=1}^{k} \frac{\Gamma(\alpha+p-1)}{\Gamma(\alpha)} \{{}^pF(x+1)\} + \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \int_1^{\infty} t^{-\alpha-k} \{{}^kF(x+t)\}\,dt, \tag{8.1}$$

where ${}^kF(x)$ denotes the k-th integral of $f(x)$. For bounded $f(x)$ this is

$$\int_1^\infty t^{-a} P_m(t+x)\,dt \underset{k}{\mathscr{C}\to} \int_1^\infty t^{-a} f(x+t)\,dt. \tag{8.2}$$

We have in fact proved in Lemma 7 that, if n is an arbitrary positive integer, there is a sequence of trigonometrical polynomials $\{{}^nQ_m(x)\}$ such that

$$\frac{d^n}{dx^n}[{}^nQ_m(x)] = P_m(x),$$

and

$${}^nQ_m \underset{k}{\mathscr{C}\to} \left(\int_0^x dx\right)^n f(x) + c_0 x^{n-1} + c_1 x^{n-2} + \ldots + c_{n-1}. \tag{8.3}$$

Integrating by parts, we get

$$\int_1^\infty t^{-a} P_m(x+t)\,dt = -\sum_{p=1}^{k} \frac{\Gamma(a+p-1)}{\Gamma(a)} \{{}^pQ_m(x+1)\} + \frac{\Gamma(a+k)}{\Gamma(a)} \int_1^\infty t^{-a-k} \{{}^kQ_m(x+t)\}\,dt;$$

and, by (4.3), the last integral converges uniformly in m. The lemma is therefore proved.

9. LEMMA. *If*

$$(1)\ |\lambda_n| \leqslant \tfrac{1}{2}, \qquad (2)\ P_m(x) \underset{k}{\mathscr{C}\to} f(x),$$

then

$$\frac{2\Gamma(k+1)}{\pi} \int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t)\, P_m(x+t) \begin{Bmatrix} \cos t \\ i \sin t \end{Bmatrix} dt \underset{k}{\mathscr{C}\to} \frac{2\Gamma(k+1)}{\pi} \int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t) f(x+t) \begin{Bmatrix} \cos t \\ i \sin t \end{Bmatrix} dt, \tag{9.1}$$

uniformly for all $\omega \subset (0, 1)$.

We consider only the expressions with $\cos t$ when k is even. The other results can be proved in exactly the same way. Substitute for $C_{1+k}(\omega t)$

from (1.3) and (1.4) in the first member of (9.1). We get

$$(9.2)\quad \frac{2\Gamma(k+1)}{\pi}\sum_{p=1}^{m}\frac{(-1)^{p+1}\omega^{k+1-2p}}{\Gamma(k+3-2p)}\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)t^{-2p}P_m(x+t)\cos t\,dt$$

$$+(-1)^n\frac{2\Gamma(k+1)}{\pi}\left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right)t^{-(1+k)}\cos\omega t\,P_m(x+t)\cos t\,dt$$

$$+\frac{2\Gamma(k+1)}{\pi}\int_{-1}^{1}t^{-(1+k)}C_{1+k}(\omega t)\,P_m(x+t)\cos dt$$

$$=I_1+I_2+I_3.$$

Now I_1 converges $\mathscr{C}_k$ uniformly for $\omega\subset(0,1)$, in virtue of Lemma 8. The same is true of I_2 in virtue of Lemma 7. Finally, since $t^{-(1+k)}C_{1+k}(\omega t)$ is bounded, I_3 also converges uniformly $\mathscr{C}_k$ in $\omega\subset(0,1)$. Hence the lemma is proved.

10. LEMMA. *If*

$$(1)\quad |\lambda_n|\leqslant 1,$$

$$(2)\quad P_m(x)\underset{k}{\mathscr{C}}\to f(x),$$

then

$$(10.1)\qquad f^{(k)}(x)=\frac{\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty}t^{-(1+k)}C_{1+k}(t)f(x+t)\begin{Bmatrix}\cos t\\ i\sin t\end{Bmatrix}dt,$$

according as k is even or odd.

This follows immediately from Lemma 9 on putting $\omega=1$ and using Lemma 2 and (6.1).

11. *Definition.* We define the property α of a bounded set $\mathfrak{E}$ in the following way.

Suppose that $P_m(x)$ is any sequence of trigonometrical polynomials whose exponents belong to $\mathfrak{E}$ and which converges $\mathscr{C}_k$ to a bounded function $f(x)$. We say that $\mathfrak{E}$ has the property α if, in the above circumstances, $f(x)$ is necessarily almost periodic.

12. LEMMA. *If the bounded set $\mathfrak{E}$ can be divided into a finite number of closed sets $\mathfrak{E}_1, \mathfrak{E}_2, \ldots, \mathfrak{E}_n$ each of which has the property α, then $\mathfrak{E}$ itself has the property α.*

Since $\mathfrak{E}$ is bounded, we may suppose, without loss of generality, that every $\mathfrak{E}_m$ can be enclosed in a closed interval I_m which has no point in common with any other such interval. Now let a_m be the centre of I_m,

$2\phi_m$ its length. Let us choose ψ_m so that $(a_m-\phi_m-2k\psi_m,\ a_m+\phi_m+2k\psi_m)$ is outside every interval I_l, other than I_m itself.

Then, putting $l=k$ in (5.4), we find

$$(12.1)\quad \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\psi_m t}{\psi_m t}\right)^k \frac{\sin(\phi_m+k\psi_m)t}{t}\, e^{-iat} P_l(x+t)\,dt = \sum_{\lambda_n \subset \mathfrak{S}_m} a_{\lambda_n}^{(l)} e^{i\lambda_n x} = P_{l,m}(x).$$

Suppose that $P_l(x)$ is a sequence of trigonometrical polynomials, whose exponents belong to $\mathfrak{E}$, converging $\mathfrak{C}_k$ to a bounded function $f(x)$. Then, by (12.1) and Lemma 4, the sequence $P_{l,m}(x)$ converges $\mathfrak{C}_k$ to the function

$$(12.2)\qquad f_m(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin\psi_m t}{\psi_m t}\right)^k \frac{\sin(\phi_m+k\psi_m)t}{t}\, f(x+t)\,dt$$

and its exponents belong to $\mathfrak{E}_m$.

Now $f_m(x)$ is bounded, in virtue of (12.2). Further, $\mathfrak{E}_m$ has the property a, and so $f_m(x)$ must be almost periodic. But, from the construction of the integral in (12.1),

$$(12.3)\qquad P_l(x)=\sum_{m=1}^{n} P_{l,m}(x),$$

and therefore

$$f(x)=\sum_{m=1}^{n} f_m(x).$$

Hence $f(x)$ itself is almost periodic, and so $\mathfrak{E}$ has the property a by definition.

13. Let $\mathfrak{E}_\delta$ denote the set obtained by removing from $\mathfrak{E}$ those points of $\mathfrak{E}$ which lie in the interior of $(a-\delta,\ a+\delta)$.

LEMMA. *If, for every positive* δ, $\mathfrak{E}_\delta$ *has the property* a, *then* $\mathfrak{E}$ *itself has the property* a.

Since $\mathfrak{E}$ is bounded, we may suppose, without loss of generality, that $\mathfrak{E}$ is inside $(-\frac{1}{2},\ \frac{1}{2})$ and that the singular point is $a=0$. As before, let $P_m(x)$ be a sequence whose exponents belong to $\mathfrak{E}$ and which converges $\mathfrak{C}_k$ to a bounded function $f(x)$. We now make use of Lemma 9.

By (1.6),

$$(13.1)\quad P_m(x,\omega)=\frac{2\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t)\, P_m(x+t)\begin{Bmatrix}\cos t\\ i\sin t\end{Bmatrix} dt$$

is a polynomial with $\frac{1}{2} > |\lambda_n| \geqslant 1-\omega$.

We take $\omega < 1$ so that there are no λ_n in $(\omega-1,\ 1-\omega)$. By Lemma 9,

$$(13.2)\quad P_n(x,\omega) \overset{\mathcal{C}}{\underset{k}{\to}} \frac{2\Gamma(k+1)}{\pi}\int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t) f(x+t) \begin{Bmatrix} \cos t \\ i \sin t \end{Bmatrix} dt = f(x,\omega).$$

But $f(x,\omega)$ is bounded if $f(x)$ is. Therefore, by (13.2), $f(x,\omega)$ is the bounded limit of a sequence of trigonometrical polynomials whose exponents belong to $\mathfrak{C}_{1-\omega} = \mathfrak{C} - \mathfrak{C}.(\omega-1,\ 1-\omega)$. This implies that $f(x,\omega)$ is almost periodic.

From (9.2)

$$(13.3)\quad f(x,\omega) \to f(x,1)$$

as $\omega \to 1$, uniformly on the whole axis. Therefore $f(x,1)$ is almost periodic.

By Lemma 9,

$$(13.4)\quad P_m(x,1) \overset{\mathcal{C}}{\underset{k}{\to}} f(x,1).$$

But (2.1), and (2.2) give

$$(13.5)\quad P_n(x,1) = P_n^{(k)}(x);$$

also, from (6.1), (13.3), and (13.4) we get

$$f(x,1) = f^{(k)}(x).$$

We have therefore proved that $f^{(k)}(x)$ is almost periodic.

We now make use of the following well-known theorem*.

If the integral of an almost periodic function $\phi(x)$ is bounded, then it is almost periodic itself. Lemma 6 tells us that $f(x)$ has all its derivatives bounded. But we have just proved $f^{(k)}(x)$ to be almost periodic. Therefore the same is true of $f^{(k-1)}(x), f^{(k-2)}(x), \ldots, f(x)$. Hence the set of points $\mathfrak{C}$ has the property α and Lemma 12 is proved.

14. THEOREM. *Any bounded reducible set of points $\mathfrak{C}$ has the property α.*

It is evident that if $\mathfrak{C}$ consists of a single point it has the property α. By Lemma 11 the same is true of a finite set of points.

15. LEMMA. *If any closed set of points $\mathfrak{C}$ of order β (an ordinal number of the first or second class) has the property α, then any closed set of points $\mathfrak{B}$ of order $\beta+1$ has the property α.*

* See Besicovitch, *Almost periodic functions*, 7.

A closed set of points is said to be of order β if its derivative $\mathfrak{C}^{(\beta)}$ of order β consists of a finite number of points. To prove the lemma, observe first that, by Lemma 11, it is sufficient to consider the case in which $\mathfrak{B}^{(\beta+1)}$ consists of one point only, say $x = 0$. Using the notation of Lemma 12, we see that for any $\delta > 0$, $\mathfrak{B}_\delta$ is of order β and therefore has the property a. But then, in virtue of Lemma 12, $\mathfrak{B}$ has the property a. This is the desired result.

16. LEMMA. *Suppose that $\{\beta_k\}$ is an increasing sequence of ordinal numbers of the first or second class, such that any closed set of order β_k has the property a. Let γ be the transfinite number to which $\{\beta_k\}$ converges*. Then every closed set of order γ has the property a.*

Let $\mathfrak{C}$ be a closed set of order γ. We may again, without loss of generality, suppose that $\mathfrak{C}^{(\gamma)}$ consists of only one point, say $x = 0$. Then $\mathfrak{C}_\delta$ is of lower order γ' where $\gamma' < \gamma$. Hence there is a $\beta_m > \gamma'$. But, since all sets of points of order β_m have the property a, the same is true of a set of order γ'.

Hence, for any $\delta > 0$, $\mathfrak{C}_\delta$ has the property a and therefore the same is true of $\mathfrak{C}$. This proves the lemma.

Now every ordinal number of the first or second class can be obtained either by adding unity to some lower number or as the limit of an increasing sequence of ordinal numbers of the first or second class. We have therefore proved the theorem for any closed set whose order is of the first or second class, *i.e.* for any reducible set.

17. We have proved the following result:

THEOREM. *If*

(1) $P_m(x) \underset{k}{\mathfrak{C}\to} f(x)$,

(2) λ_n *belongs to a reducible bounded set of points,*

(3) $f(x)$ *is a bounded function,*

then $f(x)$ is almost periodic.

18. For an unbounded set $\mathfrak{C}$ we have the following

THEOREM. *If $\mathfrak{C}$ is an unbounded reducible set of points and if $P_m(x)$ is a sequence of trigonometrical polynomials whose exponents λ_n belong to $\mathfrak{C}$*

* *I.e.* the lowest transfinite number which is greater than all β_k.

and which converges $\mathfrak{C}_k$ to a bounded UNIFORMLY *continuous function $f(x)$, then $f(x)$ is almost periodic.*

Since $\mathfrak{C}$ is everywhere non-dense, we may suppose, without loss of generality, that $|\lambda_n| \geqslant \delta > 0$. Now let $P_m(x)$ be a sequence satisfying the condition of the theorem; then

$$P_m(x, \omega) = \frac{\Gamma(k+1)}{\pi\omega^k} \int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t)\, P_m(x+t)\, dt$$

are polynomials whose exponents are such that

$$|\lambda| \leqslant \omega,$$

and which converge $\mathfrak{C}_k$ to the function

$$f(x, \omega) = \frac{\Gamma(k+1)}{\pi\omega^k} \int_{-\infty}^{\infty} t^{-(1+k)} C_{1+k}(\omega t) f(x+t)\, dt.$$

This follows, in virtue of Lemma 8, from the decomposition (9.2). Further, $f(x, \omega)$ is bounded and $\mathfrak{C}.(-\omega, \omega)$ has the property a. Therefore $f(x, \omega)$ is almost periodic for all ω. But, since $k \geqslant 1$ and $f(x)$ is bounded and uniformly continuous,

$$\lim_{\omega \to \infty} f(x, \omega) = f(x)$$

uniformly on the whole axis. Hence $f(x)$ is almost periodic and the theorem is proved.

19. *Definition.* We write

$$P_m(x) \underset{k}{\mathfrak{B}\to} f(x),$$

if

$$\frac{1}{N^{2k-1}} \int_{-N}^{N} |f(x) - P_m|^2 dx$$

is bounded for all N and m, and tends to zero for all N as m tends to infinity.

20. LEMMA. *If $D(t) = O(t^{-k-1})$ as $t \to \pm\infty$ and is a continuous function, then*

$$P_m(x) \underset{k}{\mathfrak{B}\to} f(x)$$

implies

$$\int_{-\infty}^{\infty} D(t)\, P_m(x+t)\, dt \underset{k}{\mathfrak{B}\to} \int_{-\infty}^{\infty} D(t) f(x+t)\, dt.$$

21. LEMMA. *If*

$$|\lambda_n| \geqq \delta > 0, \quad \alpha > 1,$$

then

$$P_m(x) \mathfrak{B}\underset{k}{\rightarrow} f(x)$$

implies

$$(21.1) \quad \int_0^\infty t^{-\alpha} P_m(x+t)\,dt \,\mathfrak{B}\underset{k}{\rightarrow} -\sum_{p=1}^{k} \frac{\Gamma(\alpha+p-1)}{\Gamma(\alpha)} \{{}^pF(x+1)\} + \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \int_1^\infty t^{-\alpha-k} \{{}^kF(x+t)\}\,dt$$

with the notation of Lemma 8.

For a function which is bounded $\mathfrak{B}_0$, that is, such that

$$\frac{1}{N} \int_{-N}^{N} |f(x)|^2\,dx \leqq M,$$

the second member of (23.1) can be simplified into

$$\int_1^\infty t^{-\alpha} f(x+t)\,dt.$$

The proofs of these lemmas are analogous to those concerning convergence $\mathfrak{C}_k$. The other lemmas for convergence $\mathfrak{B}_k$ and boundedness $\mathfrak{B}$ are much the same and finally we get the following

22. THEOREM. *If*

(1) $P_m(x) \mathfrak{B}\underset{k}{\rightarrow} f(x)$,

(2) $\lambda_n \subset \mathbb{C}$, *a bounded reducible set of points,*

(3) $f(x)$ *is bounded* $\mathfrak{B}$,

then $f(x)$ *is quasi-periodic in the sense of Besicovitch.*

23. Now let the function $f(x)$ be called continuous $\mathfrak{B}$ if, for an arbitrary $\epsilon > 0$, there can be found a $\delta > 0$ such that, for $|t| \leqq \delta$, we have

$$\frac{1}{N} \int_{-N}^{N} |f(x+t) - f(x)|^2\,dt \leqq \epsilon^2.$$

Then we have the

THEOREM. *If*

(1) $P_m(x)\ \mathfrak{B} \underset{k}{\to} f(x)$,

(2) $\lambda_n \subset \mathfrak{C}$, *a reducible set of points*,

(3) $f(x)$ *is bounded* $\mathfrak{B}$ *and continuous* $\mathfrak{B}$,

then $f(x)$ *is B.a.p., i.e. almost periodic in the sense of Besicovitch.*

24. We now wish to show that the condition imposed on $\mathfrak{C}$ is essential. We shall construct an example of a function which is approximated C_1 by polynomials with exponents out of a set $\mathfrak{C} \subset (0, 1)$, which is dense in itself, and this function will not be almost periodic.

First we show that if $\phi(\lambda)$ is a continuous monotonic function and

$$F(x) = \int_0^1 e^{i\lambda x}\, d\phi(\lambda) \tag{24.1}$$

is almost periodic, then $F(x) \equiv 0$.

We know that for every almost periodic function the limit

$$a(\mu) = \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} F(x)\, e^{-i\mu x}\, dx \tag{24.2}$$

exists and is zero for all μ only if $F(x) \equiv 0$*.

From (24.1) it follows that

$$\begin{aligned} a(\mu) &= \lim_{M\to\infty} \frac{1}{2M^2} \int_0^M du \int_{-u}^{u} F(x)\, e^{-i\mu x}\, dx \\ &= \lim_{M\to\infty} \frac{1}{2M^2} \int_0^1 d\phi(\lambda) \int_0^M du \int_{-u}^{u} e^{i(\lambda-\mu)x}\, dx \\ &= \lim_{M\to\infty} \int_0^1 \frac{1-\cos(\lambda-\mu)M}{(\lambda-\mu)^2 M^2}\, d\phi(\lambda). \end{aligned}$$

Thus the absolute value

$$|a(\mu)| \leqslant \lim_{M\to\infty} O\left(\frac{1}{M^2}\right) + \left| \int_I d\phi(\lambda) \right|.$$

The range of integration I is understood to be the interval common to $(0, 1)$ and $(\mu-\epsilon, \mu+\epsilon)$, where $\epsilon > 0$ is arbitrary. Since $\phi(\lambda)$ is continuous, the absolute value of the integral can be made as small as we please. Hence $a(\mu) = 0$ for all μ and therefore $F(x) \equiv 0$.

* See Besicovitch, *Almost periodic functions*, 12, 20.

25. Suppose now that $\mathfrak{C}$ is an enumerable set which is dense in itself. Avoiding the easier case, let us suppose that $\mathfrak{C}$ is nowhere dense. Further, let $\phi(\lambda)$ be a non-decreasing continuous function, with $\phi(0) = 0$, $\phi(1) = 1$, which is constant in every interval complementary to $\bar{\mathfrak{C}}$, the closure of $\mathfrak{C}$. $\bar{\mathfrak{C}}$ being a perfect set, such a function exists.

Since $F(x) = \int_0^1 e^{i\lambda x} d\phi(\lambda)$ does not vanish identically, it cannot, by the preceding section, be almost periodic, and yet we shall show that it can be approximated, uniformly in any finite interval, by polynomials with exponents λ_n out of $\mathfrak{C}$. This is precisely what we wanted to show.

The function $\phi(\lambda)$ can be uniformly approximated in (0, 1) by non-decreasing step functions $\phi_m(\lambda)$, each of which has its jumps at a finite number of points of $\mathfrak{C}$. It can be supposed that $\phi_m(0) = 0$, $\phi_m(1) = 1$. The polynomials

$$P_m(x) = \int_0^1 e^{i\lambda x} d\phi_m(\lambda)$$

are in absolute value less than

$$\int_0^1 d\phi_m(\lambda) = 1,$$

and the expression

$$P_m(x) = e^{ix} - ix \int_0^1 e^{i\lambda x} \phi_m(\lambda)\, d\lambda,$$

obtained by partial integration, shows that

$$P_m(x) \overset{C}{\underset{1}{\to}} f(x).$$

Matem. ústav Karlovy University,
Prague II, Czechoslovakia.

DUAL INTEGRAL EQUATIONS

By IDA W. BUSBRIDGE.

[Received 16 June, 1937.—Read 17 June, 1937.]

1. *Introduction.*

In his new book on Fourier integrals*, Prof. Titchmarsh gives the formal solution by Mellin transforms of some dual integral equations, which arise in connection with the problem of a flat electrified disc of conducting material†. He suggested to me that I should consider conditions for the validity of his analysis and in this paper I give a solution. Prof. Titchmarsh has very kindly helped me with many suggestions and criticisms and I should here like to express my gratitude to him.

The dual equations are the following:

$$\int_0^\infty y^\alpha f(y)\, J_\nu(xy)\, dy = g(x) \quad (0<x<1), \tag{1.1}$$

$$\int_0^\infty f(y)\, J_\nu(xy)\, dy = 0 \qquad (x>1). \tag{1.2}$$

Titchmarsh's solution is that

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s-\alpha}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\chi(s)\,x^{-s}\,ds, \tag{1.3}$$

where

$$\chi(s) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}w)}\,\frac{\mathfrak{G}(\alpha+1-w)}{w-s}\,dw \quad (\sigma<k), \tag{1.4}$$

and

$$\mathfrak{G}(s) = \int_0^1 g(x)\,x^{s-1}\,dx.$$

* *Introduction to the theory of Fourier integrals* (1937). I shall refer to this as F.I., and throughout this paper I shall use the notation which is used in the book.

† F.I., § 11.16.

If $\alpha > 0$ the solution may be expressed in the form

$$(1.5)\quad f(x) = \frac{(2x)^{1-\frac{1}{2}\alpha}}{\Gamma(\frac{1}{2}\alpha)} \int_0^1 \mu^{1+\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(\mu x)\, d\mu \int_0^1 g(\rho\mu)\, \rho^{\nu+1}(1-\rho^2)^{\frac{1}{2}\alpha-1}\, d\rho.$$

I shall consider the validity of the argument for values of α and ν such that $0 < \alpha < 2$ and $-\nu-1 < \alpha-\frac{1}{2} < \nu+1$. It is then helpful to transform equations (1.1) and (1.2) as follows. If we multiply (1.1) by $x^{\nu+1}$ and integrate formally with respect to x, we have

$$x^{\nu+1} \int_0^\infty J_{\nu+1}(xy)\, y^{\alpha-1} f(y)\, dy = \int_0^x y^{\nu+1} g(y)\, dy \quad (0 < x < 1).$$

Writing

$$h(x) = x^{-\nu-1} \int_0^x y^{\nu+1} g(y)\, dy,$$

so that $h(x)$ is an integral, we have

$$\int_0^\infty J_{\nu+1}(xy)\, y^{\alpha-1} f(y)\, dy = h(x) \qquad (0 < x < 1).$$

Multiplying by $x^{\nu+2}$ and integrating again, we may write this in the form

$$(1.6)\qquad \frac{d}{dx}\left\{x^{\nu+2} \int_0^\infty J_{\nu+2}(xy)\, y^{\alpha-2} f(y)\, dy\right\} = x^{\nu+2} h(x) \quad (0 < x < 1).$$

Similarly (1.2) may be written

$$(1.7)\qquad \frac{d}{dx}\left\{x^{\nu+1} \int_0^\infty J_{\nu+1}(xy)\, y^{-1} f(y)\, dy\right\} = 0 \qquad (x > 1).$$

If we now write ν for $\nu+1$ (so that we now have $-\nu < \alpha-\frac{1}{2} < \nu$), equations (1.6) and (1.7) become equations (1.8) and (1.9) of the following theorem. In it, we have aimed at a simple set of conditions, not at complete generality.

THEOREM 1. *Let $h(x)$ be an integral over $(0, 1)$ and let*

$$\mathfrak{H}(s) = \int_0^1 h(x)^{s-1}\, dx.$$

Let $0 < \alpha < 2$ and $-\nu < \alpha-\frac{1}{2} < \nu$. Then the equations

$$(1.8)\qquad \frac{d}{dx}\left\{x^{\nu+1} \int_0^\infty J_{\nu+1}(xy)\, y^{\alpha-2} f(y)\, dy\right\} = x^{\nu+1} h(x) \quad (0 < x < 1),$$

$$(1.9)\qquad \frac{d}{dx}\left\{x^{\nu} \int_0^\infty J_{\nu}(xy)\, y^{-1} f(y)\, dy\right\} = 0 \qquad (x > 1),$$

have one and only one solution, $f(x)$, in the class of functions whose Mellin transforms are regular for $-\nu<\sigma<\alpha$ and $O(|t|^{\sigma-\alpha+\epsilon})$, for every $\epsilon>0$, in any interior strip. It is given by

$$(1.10)\quad f(x)=\frac{x^{\frac{1}{2}-k}}{2\pi i}\,\text{l.i.m.}_{T\to\infty}\int_{k-iT}^{k+iT}\chi(s)\,2^{s-\alpha+1}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,x^{k-\frac{1}{2}-s}\,ds,$$

where $-\nu<k<\alpha-\frac{1}{2}$ and

$$(1.11)\quad \chi(s)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(\frac{1}{2}\nu+1-\frac{1}{2}\alpha+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\,\mathfrak{H}(\alpha-w)\,\frac{dw}{w-s}\quad(k<c'<\alpha).$$

The proof is in two parts. We first show that if the equations (1.8) and (1.9) have a solution $f(x)$ in the class of functions considered, then this solution must be given by (1.10) and (1.11). We then verify that the function $f(x)$, defined by (1.10) and (1.11), belongs to the above class of functions and satisfies equations (1.8) and (1.9). This is proved in §2.

In §3 we consider the solution (1.5). This solution, valid when $\alpha>0$, breaks down when $\alpha\leqslant 0$, and we obtain another form of solution, which is valid when $\alpha>-2$ and equivalent to (1.5) when $\alpha>0$. A similar difficulty arises when we try to express the solution given by (1.10), (1.11) in terms of $h(x)$. This can also be overcome and we give the second form for this solution in Theorem 2 at the end of §3.

2. *The proof of Theorem* 1.

The proof depends on the following lemma.

LEMMA. *Let $\phi(w)$ be an analytic function of $w=u+iv$, which is regular in a strip containing the line $u=c$; let**

$$(2.1)\qquad |\phi(c+iv)|=O(|v|^{-\beta}),$$

where $0<\beta<1$. Let

$$(2.2)\qquad \psi(s)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\phi(w)\,\frac{dw}{w-s}\quad(\sigma\neq c).$$

Then $\psi(s)$ is regular for $\sigma\neq c$ and

$$(2.3)\qquad |\psi(s)|\leqslant A\,|\sigma-c|^{-\delta}\,|t|^{-\beta+\delta},$$

* Throughout this paper $\phi(x)=O(x^{-\alpha})$ will imply "as $x\to\infty$".

where $0<\delta<\beta$ *and* A *is a constant depending on* δ. *Also, for every* $\epsilon>0$,

$$|\psi(s)| = O(|t|^{-\beta+\epsilon}) \tag{2.4}$$

uniformly for $|\sigma-c|\geqslant\eta>0$.

The integral (2.2) converges uniformly with respect to s for $|\sigma-c|\geqslant\eta$ and $|t|\leqslant T$ (say). Hence $\psi(s)$ is regular for $\sigma\neq c$. Now

$$|\psi(s)|\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|\phi(c+iv)|}{\{(\sigma-c)^2+(v-t)^2\}^{\frac{1}{2}}}\,dv,$$

and by an extension of the theorem of the arithmetic and geometric means, given δ such that $0<\delta<1$, a constant $K=K(\delta)$ exists such that

$$\{(\sigma-c)^2+(v-t)^2\}^{\frac{1}{2}}\geqslant K|\sigma-c|^{\delta}|v-t|^{1-\delta} \tag{2.5}$$

for all σ, v, and t. Hence

$$|\psi(s)|\leqslant\frac{K}{|\sigma-c|^{\delta}}\int_{-\infty}^{\infty}\frac{|\phi(c+iv)|}{|v-t|^{1-\delta}}\,dv.$$

Now since $\phi(w)$ is continuous on $u=c$, it follows from (2.1) that we can determine a constant K, so that

$$|\phi(c+iv)|\leqslant K|v|^{-\beta}$$

for every v. Hence

$$\begin{aligned}|\psi(s)|&\leqslant\frac{K}{|\sigma-c|^{\delta}}\int_{-\infty}^{\infty}\frac{dv}{|v|^{\beta}|v-t|^{1-\delta}}\\&=\frac{K}{|\sigma-c|^{\delta}|t|^{\beta-\delta}}\int_{-\infty}^{\infty}\frac{du}{|u|^{\beta}|u-1|^{1-\delta}}\leqslant\frac{A}{|\sigma-c|^{\delta}|t|^{\beta-\delta}},\end{aligned}$$

if $0<\delta<\beta<1$. This proves (2.3) and (2.4) follows immediately.

The necessity of the form (1.10) *for the solution.* If $h(x)$ is an integral over $(0, 1)$, $\mathfrak{H}(s)$ must be regular for $\sigma>0$ and, on integrating by parts, we see that

$$|\mathfrak{H}(s)| = O(|t|^{-1}),$$

uniformly for $\sigma\geqslant\delta>0$.

We assume that equations (1.8) and (1.9) have a solution $f(x)$, whose Mellin transform $\mathfrak{F}(s)$ is regular for $-\nu<\sigma<a$, and that

$$|\mathfrak{F}(s)| = O(|t|^{\sigma-a+\epsilon}),$$

uniformly in any interior strip. It follows from this, by suitable choice of ϵ, that, if $-\nu<k<a-\frac{1}{2}$, then $\mathfrak{F}(k+it)$ belongs to $L^2(-\infty,\infty)$ and

therefore that $x^{k-\frac{1}{2}}f(x)$ belongs to $L^2(0, \infty)$. Now let

$$\max(-\nu, \alpha-\tfrac{3}{2}) < k < \alpha-\tfrac{1}{2};$$

then $y^{k-\frac{1}{2}}f(y)$ and $y^{-k+\frac{1}{2}}y^{\alpha-2}J_\nu(xy)$ both belong to $L^2(0, \infty)$. We may therefore apply Parseval's theorem* for functions of L^2 to the left-hand side of (1.8) and we have

$$\int_0^\infty J_{\nu+1}(xy)\,y^{\alpha-2}f(y)\,dy = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\,2^{\alpha-s-2}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}{\Gamma(2+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\,x^{s-\alpha+1}\,ds,$$

and therefore

$$(2.6)\quad x^{\nu+1}h(x) = \frac{d}{dx}\left\{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathfrak{F}(s)\,2^{\alpha-s-2}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}{\Gamma(2+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\,x^{s-\alpha+\nu+2}\,ds\right\}$$

$$(0<x<1).$$

The integrand here is regular for $-\nu<\sigma<\alpha$ and it is $O(|t|^{-2+\epsilon})$, so we may take $k=\alpha-\frac{1}{2}$. Let $\phi(x)$ denote the expression in brackets, so that $\phi'(x)=x^{\nu+1}h(x)$ $(0<x<1)$. Let $w=u+iv$ and let $u<\alpha-\frac{1}{2}$. Then

$$\begin{aligned}\mathfrak{H}(\alpha-w) &= \int_0^1 h(x)\,x^{\alpha-w-1}\,dx\\ &= \int_0^1 \phi'(x)\,x^{\alpha-w-\nu-2}\,dx\\ &= \phi(1)+(w-\alpha+\nu+2)\int_0^1\phi(x)\,x^{\alpha-w-\nu-3}\,dx,\end{aligned}$$

since $\lim_{x\to 0} x^{\alpha-\nu-u-2}\phi(x)=0$. Hence, substituting for $\phi(x)$ and inverting the order of integration, which is justified by absolute convergence, we have

$$(2.7)\quad \mathfrak{H}(\alpha-w)$$

$$= \frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\mathfrak{F}(s)\,2^{\alpha-s-2}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}{\Gamma(2+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\left\{1+\frac{w-\alpha+\nu+2}{s-w}\right\}ds$$

$$= \frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\mathfrak{F}(s)\,2^{\alpha-s-1}\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}s)}\,\frac{ds}{s-w}\quad(u<\alpha-\tfrac{1}{2}).$$

Equation (1.9) may be reduced in the same way to a form similar to (2.7). We find that Parseval's theorem for functions of L^2 may be used

* F.I., Theorem 72.

if $\max(-\nu, -\frac{1}{2}) < k < \alpha - \frac{1}{2}$, and we have

$$\frac{d}{dx}\left\{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathfrak{F}(s)\, 2^{-1-s}\, \frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + 1)}\, x^{\nu+s}\, ds\right\} = 0 \quad (x > 1).$$

The integrand here is regular for $-\nu < \sigma < \min(\alpha, \nu)$ and it is $O(|t|^{-\alpha-1+\epsilon})$, so we may again take $k = \alpha - \frac{1}{2}$. Multiplying by $x^{-w-\nu}$, where $u > \alpha - \frac{1}{2}$, and integrating over $(1, \infty)$ we have, after integrating by parts as above and then inverting the order of integration in the double integral,

$$0 = \frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty} \mathfrak{F}(s)\, 2^{-1-s}\, \frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + 1 + \frac{1}{2}s)}\left|1 + \frac{w+\nu}{s-w}\right| ds \quad (u > \alpha - \tfrac{1}{2})$$

and thus

$$\frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty} \mathfrak{F}(s)\, 2^{-s}\, \frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}\, \frac{ds}{s-w} = 0 \qquad (u > \alpha - \tfrac{1}{2}). \tag{2.8}$$

Write

$$\chi(s) = 2^{-s+\alpha-1}\, \mathfrak{F}(s)\, \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\alpha - \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}, \tag{2.9}$$

so that $\chi(s)$ is regular for $-\nu < \sigma < \alpha$; moreover it is $O(|t|^{-\frac{1}{2}\alpha+\epsilon})$, uniformly in any interior strip. Then (2.7) and (2.8) become

$$\frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty} \chi(s)\, \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}s)}\, \frac{ds}{s-w} = \mathfrak{H}(\alpha - w) \quad (u < \alpha - \tfrac{1}{2}), \tag{2.10}$$

$$\frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty} \chi(s)\, \frac{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\alpha - \frac{1}{2}s)}\, \frac{ds}{s-w} = 0 \qquad (u > \alpha - \tfrac{1}{2}). \tag{2.11}$$

Now let $-\nu < c < \alpha - \frac{1}{2} < c' < \min(\alpha, \nu)$ and let $c < u < \alpha - \frac{1}{2}$ in (2.10). Since the integrand in (2.10) is regular in the strip $-\nu < \sigma < \alpha$, except for a simple pole at $s = w$, and since it is $O(|t|^{-2+\epsilon})$, we may move the line of integration to $\sigma = c$; this gives (by Cauchy's theorem)

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \chi(s)\, \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}s)}\, \frac{ds}{s-w} = \mathfrak{H}(\alpha - w) - \chi(w)\, \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)}{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)}. \tag{2.12}$$

Since the left-hand side is regular for $u > c$, so is the right-hand side, and since

$$\frac{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)}$$

is regular for $u > a-\nu-2$ and therefore for $u > c$, it follows, by integrating round a large rectangle to the right of $u = c'$, that if $x < c'$, then

$$(2.13)\quad \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\left\{\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}a+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\,\mathfrak{H}(a-w)-\chi(w)\right\}\frac{dw}{w-z}=0,$$

provided that the integrand is of a suitable order at infinity. To prove this, we use the lemma with

$$\phi(w)=\chi(w)\,\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}a+\frac{1}{2}w)}.$$

This is $O(|v|^{-1+\epsilon})$ when $u=c$; hence it follows from the lemma that the right-hand side of (2.12) is less than

$$A\,|u-c|^{-\delta}\,|v|^{-1+\epsilon+\delta}\quad(0<\delta<1-\epsilon),$$

where A depends on δ. Since

$$\left|\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}a+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\right|=O(|w|^{1-\frac{1}{2}a})$$

uniformly for $|\arg w|\leqslant\frac{1}{2}\pi$, the integrand in (2.13) is less than

$$A\,|u-c|^{-\delta}\,|v|^{-1+\epsilon+\delta}(u^2+v^2)^{-\frac{1}{2}a},$$

where A may now depend on z and δ. If we integrate the function round the rectangle whose corners are the points $c'-iT$, $T-iT$, $T+iT$, $c'+iT$, where $T>|y|$, we have

$$\left|\int_{T-iT}^{T+iT}\right|\leqslant\frac{2A}{(T-c)^{\delta}\,T^{\frac{1}{2}a}}\int_0^T\frac{dv}{v^{1-\epsilon-\delta}}=O(T^{-\frac{1}{2}a+\epsilon}),$$

$$\left|\int_{c'+iT}^{T+iT}\right|\leqslant\frac{A}{T^{1+\frac{1}{2}a-\epsilon-\delta}}\int_{c'}^T\frac{du}{(u-c)^{\delta}}=O(T^{-\frac{1}{2}a+\epsilon}).$$

Similarly

$$\left|\int_{c'-iT}^{T-iT}\right|=O(T^{-\frac{1}{2}a+\epsilon}).$$

Since ϵ may always be chosen less than $\frac{1}{2}a$, all these integrals tend to zero as $T\to\infty$ and (2.13) follows.

Now consider (2.11) when $a-\frac{1}{2}<u<c'$. We transform this in a manner similar to that given for (2.10). The integrand is regular in the strip $-\nu<\sigma<\min(a,\nu)$ except for a simple pole at $s=w$ and it is $O(|t|^{-a-1+\epsilon})$; hence, moving the line of integration to $\sigma=c'$, we have

$$(2.14)\quad \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\chi(s)\,\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}a-\frac{1}{2}s)}\,\frac{ds}{s-w}=\chi(w)\,\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}a-\frac{1}{2}w)}.$$

The left-hand side of this equation is regular for $u < c'$; hence so is the right-hand side, and since

$$\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}a-\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}w)}$$

is regular for $u < \nu+a$, it follows that $\chi(w)$ must be regular for $u < c'$ and therefore for $u < a$. Now

$$\left|\chi(s)\,\frac{\Gamma(\frac{1}{2}\nu-\frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}a-\frac{1}{2}s)}\right| = \begin{cases} O(|t|^{-a+\epsilon}) & (a<1), \\ O(|t|^{-1+\epsilon}) & (a\geq 1) \end{cases}$$

on the line $\sigma = c'$; hence it follows from the lemma with $\beta = \gamma-\epsilon$, where $\gamma = \min(a, 1)$, that the right-hand side of (2.14) is less than

$$A\,|u-c'|^{-\delta}|v|^{-\gamma+\epsilon+\delta} \quad (0<\delta<\gamma-\epsilon).$$

Since $$\left|\frac{\Gamma(\frac{1}{2}\nu+\frac{1}{2}a-\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}w)}\right| = O(|w|^{\frac{1}{2}a}),$$

uniformly for $\frac{1}{2}\pi \leqslant \arg w \leqslant \frac{3}{2}\pi$, we have

$$\left|\frac{\chi(w)}{w-z}\right| \leqslant \frac{A}{|u-c'|^{\delta}|v|^{\gamma-\epsilon-\delta}(u^2+v^2)^{\frac{1}{2}-\frac{1}{4}a}}$$

when $u < x$ (A depending upon z and δ). Let $x > c$; then integrating $\chi(w)/(w-z)$ round the rectangle whose corners are the points $c-iT$, $c+iT$, $-T+iT$, $-T-iT$ $(T > |y|)$, we find that

$$\left|\int_{c+iT}^{-T+iT}\right|, \qquad \left|\int_{-T+iT}^{-T-iT}\right|, \qquad \left|\int_{-T-iT}^{c-iT}\right|$$

are all $O(T^{-(\gamma-\frac{1}{2}a-\epsilon)})$, and therefore, since $\gamma = \min\,(a, 1) > \frac{1}{2}a$ when $a < 2$,

$$\lim_{T\to\infty}\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\chi(w)\,\frac{dw}{w-z} = 0 \quad (x>c). \tag{2.15}$$

Finally, let $c < x < c'$ and let us move the line of integration in (2.15) to $u = c'$. Then we have

$$\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\chi(w)\,\frac{dw}{w-z} = \chi(z), \tag{2.16}$$

the move being justified by the fact that $|\chi(w)|$ is $O(|v|^{-\frac{1}{2}a+\epsilon})$ for $u < a$. Substituting from (2.16) into (2.13), we have

$$\chi(z) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}a+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\,\mathfrak{H}(a-w)\,\frac{dw}{w-z},$$

which is equation (1.11). This will be true for $x < c'$. Equation (1.10) now follows from (2.9) by the theory of Mellin transforms for functions of L^2.

The sufficiency of the conditions. We shall first show that if $f(x)$ is defined by (1.10) and (1.11), then its Mellin transform, $\mathfrak{F}(s)$, satisfies the stated conditions.

Consider (1.11). When $w = c' + iv$, $\mathfrak{H}(\alpha - w) = O(|v|^{-1})$ since $c' < \alpha$, and

$$\left| \frac{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)} \right| = O(|v|^{\frac{1}{2}-\frac{1}{2}\alpha}).$$

It therefore follows from the lemma that $\chi(s)$ is regular for $\sigma < c'$ and thus for $\sigma < \alpha$ (since c' may be taken as near to α as we please); and so, by taking $\beta = \frac{1}{2}\alpha$, we see that

$$|\chi(s)| = O(|t|^{-\frac{1}{2}\alpha + \epsilon}),$$

uniformly for $\sigma \leqslant \alpha - \eta$. Hence

$$\mathfrak{F}(s) = 2^{s-\alpha+1}\chi(s)\frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}\alpha - \frac{1}{2}s)} \tag{2.17}$$

is regular for $-\nu < \sigma < \alpha$, and using Stirling's theorem again we see that $|\mathfrak{F}(s)| = O(|t|^{\sigma-\alpha+\epsilon})$, uniformly in any interior strip.

We next show that this function $f(x)$ satisfies (1.8) and (1.9). The left-hand side of (1.8), when the value of $f(x)$ from (1.10) is substituted, may be reduced to the right-hand side of (2.6), where $\mathfrak{F}(s)$ has the value (2.17), the argument being unchanged. Thus

$$\frac{d}{dx}\left\{x^{\nu+1}\int_0^\infty J_{\nu+1}(xy)\,y^{\alpha-2}f(y)\,dy\right\}$$

$$= \frac{d}{dx}\left\{\frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\chi(s)\,2^{-1}\frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(2 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}s)}\,x^{s-\alpha+\nu+2}\,ds\right\}.$$

If we substitute for $\chi(s)$ from (1.11), the right-hand side of this becomes

$$\frac{d}{dx}\left\{-\frac{1}{4\pi^2}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\frac{2^{-1}\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(2 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}s)}\,x^{s-\alpha+\nu+2}\,ds \times \int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)}\,\mathfrak{H}(\alpha - w)\,\frac{dw}{w - s}\right\} \tag{2.18}$$

$$= \frac{d}{dx}\left\{-\frac{1}{4\pi^2}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)}\,\mathfrak{H}(\alpha - w)\,dw \times \int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\frac{2^{-1}\Gamma(\frac{1}{2}\nu + \frac{1}{2}s)}{\Gamma(2 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}s)}\,x^{s-\alpha+\nu+2}\,\frac{ds}{w - s}\right\},$$

provided that the inversion of the order of integration is justified. Since $c' > \alpha - \frac{1}{2}$, the value of the inner integral when $0 < x < 1$ is

$$-2\pi i \, \frac{2^{-1}\Gamma(\frac{1}{2}\nu + \frac{1}{2}w)}{\Gamma(2 + \frac{1}{2}\nu - \frac{1}{2}\alpha + \frac{1}{2}w)} \, x^{w-\alpha+\nu+2},$$

and we have

$$\begin{aligned}\frac{d}{dx}\left\{x^{\nu+1}\int_0^\infty J_{\nu+1}(xy)\,y^{\alpha-2}f(y)\,dy\right\} &= \frac{d}{dx}\left\{-\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\mathfrak{H}_c(\alpha-w)}{w-\alpha+\nu+2}\,x^{w-\alpha+\nu+2}\,dw\right\}\\ &= \frac{d}{dx}\left\{\frac{1}{2\pi i}\int_{\alpha-c'-i\infty}^{\alpha-c'+i\infty}\frac{\mathfrak{H}_c(s)}{\nu+2-s}\,x^{\nu+2-s}\,ds\right\}\\ &= x^{\nu+1}h(x) \quad (0<x<1),\end{aligned}$$

the last step following in several ways from the theory of Mellin transforms [*e.g.* by the theory for functions of L^2, since $\mathfrak{H}_c(s)$ belongs to $L^2(-\infty, \infty)$ when $\sigma > 0$]. The inversion of the order of integration in (2.18) is justified by absolute convergence, since, by the inequality (2.5), the modulus of the integrand is less than a constant multiple of

$$|v|^{-\frac{1}{2}\alpha}|t|^{\frac{1}{2}\alpha-2}(c'-\alpha-\tfrac{1}{2})^{-\delta}|v-t|^{-1+\delta}x^{\nu+\frac{3}{2}} \quad (0<\delta<1),$$

when v and t are large. The substitution $t = v\xi$ now shows that the double integral is absolutely convergent when $\alpha < 2$. This proves that $f(x)$ satisfies (1.8) when $0 < \alpha < 1$.

The proof that $f(x)$ satisfies (1.9) is similar. By Parseval's theorem we have

$$\begin{aligned}(2.19)\quad &\frac{d}{dx}\left\{x^\nu\int_0^\infty J_\nu(xy)\,y^{-1}f(y)\,dy\right\}\\ &= \frac{d}{dx}\left\{\frac{1}{2\pi i}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\chi(s)\,\frac{2^{-\alpha+1}\Gamma(\frac{1}{2}\nu-\frac{1}{2}s)}{(\nu+s)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,x^{\nu+s}\,ds\right\}\\ &= \frac{d}{dx}\left\{-\frac{1}{4\pi^2}\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\frac{2^{-\alpha+1}\Gamma(\frac{1}{2}\nu-\frac{1}{2}s)}{(\nu+s)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,x^{\nu+s}\,ds\right.\\ &\qquad\left.\times\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\,\mathfrak{H}_c(\alpha-w)\,\frac{dw}{w-s}\right\}\\ &= \frac{d}{dx}\left\{-\frac{1}{4\pi^2}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}w)}\,\mathfrak{H}_c(\alpha-w)\,dw\right.\\ &\qquad\left.\times\int_{\alpha-\frac{1}{2}-i\infty}^{\alpha-\frac{1}{2}+i\infty}\frac{2^{-\alpha+1}\Gamma(\frac{1}{2}\nu-\frac{1}{2}s)\,x^{\nu+s}}{(\nu+s)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\frac{ds}{w-s}\right\},\end{aligned}$$

the inversion of the order of integration being again justified by absolute convergence. The value of the inner integral is, when $x > 1$,

$$2\pi i \frac{2^{-\alpha+1}\Gamma(\nu)}{\Gamma(\nu+\frac{1}{2}\alpha)} \frac{1}{w+\nu},$$

from the residue at $s = -\nu$. Since this is independent of x, the last expression in (2.19) is zero when $x > 1$ and $f(x)$ therefore satisfies (1.9). This completes the proof.

3. *The second form for the solution.*

We first find a form for the solution of the equations (1.1) and (1.2), which is valid when $\alpha > -2$. In the following analysis, which is purely formal, we make repeated use of Parseval's theorem for Mellin transforms. The results could be obtained by repeated inversions of double and triple integrals, coupled with suitable changes in the variables, but our method makes the justification of the process easier when justification is required, since the applications of Parseval's theorem are usually covered by known theorems*.

Consider first the integral for $\chi(s)$ given by (1.4). We may write this as

$$(3.1) \quad \chi(s) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}{2\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}w)} \mathfrak{G}(\alpha+1-w)\left\{1+\frac{s+\nu+1}{w-s}\right\} ds.$$

If $\alpha > -2$, the following pairs of functions are Mellin transforms :

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\alpha+\frac{1}{2}w)}{2\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}w)} \; ; \quad \begin{cases} \frac{1}{\Gamma(1+\frac{1}{2}\alpha)} y^{\nu-\alpha+1}(1-y^2)^{\frac{1}{2}\alpha} & (0<y<1), \\ 0 & (y>1); \end{cases}$$

$$G(\alpha+w) \; ; \quad \begin{cases} y^\alpha g(y) & (0<y<1), \\ 0 & (y>1); \end{cases}$$

$$\frac{G(\alpha+w)}{1-w-s} \; ; \quad \begin{cases} y^{s-1}\int_0^y g(u)u^{\alpha-s}du & (0<y<1), \\ 0 & (y>1); \end{cases}$$

the last of these being easily verified. Hence applying Parseval's theorem

* *E.g.*, F.I., Theorems 42, 43, 72, and 87.

to each part of (3.1), we have

$$\chi(s) = \frac{1}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\,dy + \frac{s+\nu+1}{\Gamma(1+\frac{1}{2}\alpha)} \times \int_0^1 y^{\nu-\alpha+s}(1-y^2)^{\frac{1}{2}\alpha}\,dy \int_0^y g(u)\,u^{\alpha-s}\,du$$

$$= \frac{1}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\,dy + \frac{s+\nu+1}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha}\,dy \times \int_0^1 g(yu)\,u^{\alpha-s}\,du$$

$$= \chi_1(s)+\chi_2(s),$$

where

$$\chi_1(s) = \frac{1}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\,dy, \tag{3.2}$$

$$\chi_2(s) = \frac{s+\nu+1}{\Gamma(1+\frac{1}{2}\alpha)}\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha}\,dy \int_0^1 g(yu)\,u^{\alpha-s}\,du, \tag{3.3}$$

so that $\chi_1(s)$ is a constant.

Now let $f(x)$ be defined by (1.3). Then

$$f(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s-\alpha}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\{\chi_1(s)+\chi_2(s)\}\,x^{-s}\,ds,$$

and so

$$\int_0^x y^{\nu+\alpha} f(y)\,dy = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s-\alpha-1}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\{\chi_1(s)+\chi_2(s)\}\,x^{\nu+\alpha+1-s}\,ds.$$

This may be written

$$\int_0^x y^{\nu+\alpha} f(y)\,dy \tag{3.4}$$

$$= 2^{-\frac{1}{2}\alpha} x^{\nu+\alpha+1}\left\{\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s-\frac{1}{2}\alpha-1}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\chi_1(s)\,x^{-s}\,ds + \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} 2^{s-\frac{1}{2}\alpha}\,\frac{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\,\frac{\chi_2(s)}{s+\nu+1}\,x^{-s}\,ds\right\}$$

$$= 2^{-\frac{1}{2}\alpha} x^{\nu+\alpha+1}\{I_1+I_2\} \quad \text{(say)}.$$

Since $\chi_1(s)$ is a constant,

$$(3.5)\qquad I_1 = \chi_1(s)\, x^{-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(x)$$

$$= \frac{1}{\Gamma(1+\frac{1}{2}\alpha)}\, x^{-\frac{1}{2}\alpha} J_{\nu+1+\frac{1}{2}\alpha}(x) \int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\, dy,$$

by (3.2). From (3.3) we have

$$\frac{\chi_2(1-s)}{2+\nu-s} = \frac{1}{\Gamma(1+\frac{1}{2}\alpha)} \int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} dy \int_0^1 g(yu)\, u^{\alpha+s-1} du$$

$$= \frac{1}{\Gamma(1+\frac{1}{2}\alpha)} \int_0^1 u^{\alpha+s-1} du \int_0^1 g(yu)\, y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} dy.$$

Hence the functions

$$\frac{\chi_2(1-s)}{2+\nu-s}; \qquad \begin{cases} \dfrac{y^\alpha}{\Gamma(1+\frac{1}{2}\alpha)} \displaystyle\int_0^1 g(yu)\, u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} du & (0<y<1), \\ 0 & (y>1) \end{cases}$$

are Mellin transforms and so are

$$2^{s-\frac{1}{2}\alpha} \frac{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}s)}{\Gamma(\frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}\alpha-\frac{1}{2}s)}\, x^{-s}; \quad (xy)^{1-\frac{1}{2}\alpha} J_{\nu+2+\frac{1}{2}\alpha}(xy).$$

Hence

$$(3.6)\quad I_2 = \frac{1}{\Gamma(1+\frac{1}{2}\alpha)} \int_0^1 y^\alpha (xy)^{1-\frac{1}{2}\alpha} J_{\nu+2+\frac{1}{2}\alpha}(xy)\, dy \int_0^1 g(yu)\, u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} du$$

$$= \frac{1}{\Gamma(1+\frac{1}{2}\alpha)} \int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} du \int_0^1 g(yu)\, y^\alpha (xy)^{1-\frac{1}{2}\alpha} J_{\nu+2+\frac{1}{2}\alpha}(xy)\, dy.$$

Equations (3.4), (3.5), and (3.6) now give

$$\int_0^x y^{\nu+\alpha} f(y)\, dy = \frac{2^{-\frac{1}{2}\alpha}}{\Gamma(1+\frac{1}{2}\alpha)} \Big\{ x^{\nu+\frac{1}{2}\alpha+1} J_{\nu+\frac{1}{2}\alpha+1}(x) \int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha} g(y)\, dy$$

$$+ \int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha} du \int_0^1 \frac{g(yu)}{y^{\nu+1}} (xy)^{\nu+2+\frac{1}{2}\alpha} J_{\nu+2+\frac{1}{2}\alpha}(xy)\, dy \Big\}.$$

Differentiating this, and using the recurrence relation

$$\frac{d}{dz}\left\{z^n J_n(z)\right\} = z^n J_{n-1}(z),$$

we have

$$(3.7)\quad f(x)=\frac{2^{-\frac{1}{2}\alpha}x^{-\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\left\{x^{\frac{1}{2}\alpha+1}J_{\nu+\frac{1}{2}\alpha}(x)\int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha}g(y)\,dy\right.$$
$$\left.+\int_0^1 u^{\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\,du\int_0^1 g(yu)(xy)^{2+\frac{1}{2}\alpha}J_{\nu+1+\frac{1}{2}\alpha}(xy)\,dy\right\}.$$

If $g(y)$ is integrable over $(0, 1)$ and of a suitable order as $y\to 0$, this solution will hold for $\alpha>-2$, ν being suitably restricted. When $\alpha>0$ it can be reduced to the form (1.5), but the reduction is tedious.

Examples. Take $g(y)=y^\nu$; then, if $\nu+2+\frac{1}{2}\alpha>0$,

$$\int_0^1 g(yu)(xy)^{2+\frac{1}{2}\alpha}J_{\nu+1+\frac{1}{2}\alpha}(xy)\,dy=\left(\frac{u}{x}\right)^\nu\int_0^1(xy)^{2+\nu+\frac{1}{2}\alpha}J_{\nu+1+\frac{1}{2}\alpha}(xy)\,dy$$
$$=u^\nu x^{1+\frac{1}{2}\alpha}J_{\nu+2+\frac{1}{2}\alpha}(x).$$

Hence

$$(3.8)\quad f(x)=\frac{2^{-\frac{1}{2}\alpha}x^{-\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\{x^{1+\frac{1}{2}\alpha}J_{\nu+\frac{1}{2}\alpha}(x)+x^{1+\frac{1}{2}\alpha}J_{\nu+2+\frac{1}{2}\alpha}(x)\}\int_0^1 u^{2\nu+1}(1-u^2)^{\frac{1}{2}\alpha}\,du$$
$$=\frac{2^{-\frac{1}{2}\alpha}x^{1-\frac{1}{2}\alpha}}{\Gamma(1+\frac{1}{2}\alpha)}\,\frac{2(\nu+1+\frac{1}{2}\alpha)}{x}\,J_{\nu+1+\frac{1}{2}\alpha}(x)\,\frac{\Gamma(\nu+1)\,\Gamma(\frac{1}{2}\alpha+1)}{2\Gamma(\nu+2+\frac{1}{2}\alpha)}$$
$$=\frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\frac{1}{2}\alpha)}\,(2x)^{-\frac{1}{2}\alpha}J_{\nu+1+\frac{1}{2}\alpha}(x).$$

If $\alpha=1$ and $\nu=0$, so that $g(y)=1$, this becomes

$$f(x)=\frac{2}{\pi}\left(\frac{\sin x}{x^2}-\frac{\cos x}{x}\right),$$

an example which Titchmarsh deduces from (1.5). If $\alpha=-1$ and $\nu=0$, (3.8) becomes

$$f(x)=\frac{2}{\pi}\sin x.$$

If we write $x^{-1}f(x)=F(x)$, we see that

$$F(x)=\frac{2}{\pi}\,\frac{\sin x}{x}$$

is a solution of the dual equations

$$\int_0^\infty F(u)\,J_0(xu)\,du=g(x)\quad(0<x<1),$$

$$\int_0^\infty F(u)\,uJ_0(xu)\,du=0\qquad(x>1),$$

when $g(x) = 1$. The general solution is, from (3.7),

$$F(x) = \frac{2}{\pi}\cos x \int_0^1 y(1-y^2)^{-\frac{1}{2}} g(y)\,dy + \frac{2}{\pi}\int_0^1 y(1-y^2)^{-\frac{1}{2}}\,dy \int_0^1 g(yu)\,xu \sin xu\,du.$$

The form analogous to (3.7), for the solution to the equations (1.8) and (1.9), is obtained from (1.10) and (1.11) in a similar way. The analysis may be justified fairly easily with the help of Theorems 87 and 42 in Titchmarsh's book. The following theorem gives the solution.

THEOREM 2. *Let $h(x)$, α and ν satisfy the conditions of Theorem* 1. *Then the solution of the dual equations* (1.8) *and* (1.9), *which belongs to the class of functions considered in Theorem* 1, *is given by*

$$f(x) = \frac{2^{1-\frac{1}{2}\alpha} x^{1-\alpha}}{\Gamma(\frac{1}{2}\alpha)} \left\{ x^{\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha-1}(x) \int_0^1 y^{\nu+1} h(y)(1-y^2)^{\frac{1}{2}\alpha-1}\,dy + \int_0^1 y^{\nu+1}(1-y^2)^{\frac{1}{2}\alpha-1}\,dy \int_0^1 (xu)^{1+\frac{1}{2}\alpha} J_{\nu+\frac{1}{2}\alpha}(xu)\, h(uy)\,du \right\}.$$

St. Hugh's College,
Oxford.

THE BICIRCULAR QUINTIC CURVE

By WILLIAM P. MILNE.

[Received 5 May, 1937.—Read 20 May, 1937.]

1. *Introduction.*

The plane binodal quintic curve derives its importance from the fact that it is the simplest plane curve of genus 4. Without loss of generality, the nodes can be placed at the circular points at infinity, in which case the curve is called the "bicircular" quintic curve. All curves passing once through each of the circular points at infinity will, as usual, be called "circular" curves. The object of the present paper is to obtain by direct means the chief fundamental properties of the bicircular quintic Γ.

2. *The tritangent circles.*

It is known from the general theory of the Abelian integrals that there are* $2^{4-1}(2^4-1) = 120$ conics passing through the two nodes of a binodal quintic and touching it at three further points. The bicircular quintic Γ will therefore possess 120 tritangent circles. If also we take a pair of such tritangent circles, we know* that there are in all $2^{3-1}(2^3-1) = 28$ pairs of tritangent circles mutually congruent, where two such pairs of tritangent circles are said to be "congruent", if a bicircular quartic Q passes through their points of contact with Γ. Adopting the method of representation used in my paper on the "Relation between the quadritangent hyperplanes of Noether's canonical curve for $p = 5$ and the 5-tangent conics of the plane quintic curve"† we begin by taking any two tritangent circles S_1, S_2

* A. Krazer, *Lehrbuch der Thetafunktionen* (1903), 303, § 36.

† W. P. Milne, *Proc. London Math. Soc.* (2), 36 (1934), 217–234.

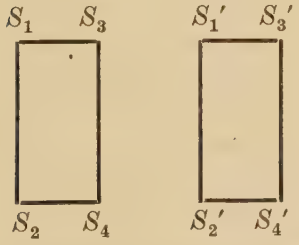
at random and represent them by the points at the ends of a straight line, thus:

We know* that there are 54 other tritangent circles S_3 whose points of contact with Γ lie on a bicircular quartic Q through the points of contact of S_1 and S_2. The residual points of intersection of Q with Γ are the points of contact of a tritangent circle S_4. There are 27 such pairs $S_3 S_4$ and we represent them diagrammatically by the points at the ends of 28 equal parallel lines thus:

... ... (28 such straight lines).

Thus we have obtained 28 pairs of tritangent circles which are mutually congruent. We call S_1, S_2, S_3, S_4 a congruent tetrad of tritangent circles.

Let us now combine together any two of the above straight lines, *e.g.* $S_1 S_2$ and $S_3 S_4$, to form the rectangle $S_1 S_2 S_4 S_3$. We see that the pair $S_2 S_3$ is congruent with $S_1 S_4$ and that $S_3 S_1$ is congruent with $S_2 S_4$. We call $S_1 S_2 S_4 S_3$ a "congruence rectangle". From the above 28 straight lines, we can construct six mutually congruent rectangles* in



... ... (6 such rectangles)

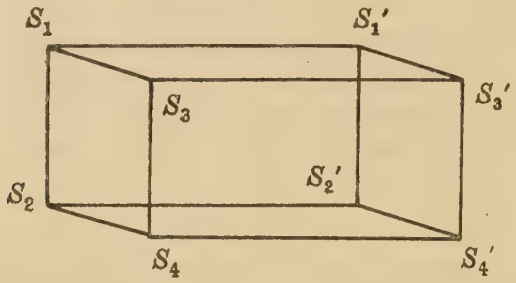
* A. Krazer, *loc .cit.*

the sense that two pairs of these tritangent circles are congruent if they are represented by points

(i) lying on parallel sides of two of the rectangles,

(ii) lying on diagonals of two of the rectangles.

Finally, we can combine any two of these six congruence rectangles to form a "congruence parallelepiped" and only one such can be constructed when once the two rectangles have been chosen.



In the above rectangular parallelepiped, two pairs of tritangent circles are congruent with one another if they are represented by points lying

(i) on two parallel edges,

(ii) on two diagonals of parallel faces,

(iii) on two diagonals of the parallelepiped.

Consider now the pencil of octavics

$$S_1 S_2 S_3 S_4 - \lambda Q^2 = 0. \qquad (1)$$

We can so choose λ that this octavic intersects Γ in any arbitrary point of Γ in addition to the eight coincident points at each of the circular points at infinity A, B and the points of contact of each of the tritangent circles S_1, S_2, S_3, S_4, *i.e.* $1+2\times 8+4\times 3\times 2 = 41$ points in all. Hence, for an ascertainable value of λ (say $\lambda = 1$) the above octavic must contain Γ as part of itself in addition to a residual cubic curve, having double points at A, B and touching S_1, S_2, S_3, S_4 each at the residual point in which Q intersects them. This cubic must therefore degenerate into the straight line joining A, B, *i.e.* the line at infinity, and a conic S, passing through A, B and touching S_1, S_2, S_3, S_4 each at the residual point where they are met by Q. Hence, *the four tritangent circles S_1, S_2, S_3, S_4 of a congruent tetrad are touched by the same circle S.*

Let us take the line AB as $z=0$ and S as

$$x=\theta^2, \quad y=1, \quad z=2\theta. \tag{2}$$

Let S_i $(i=1, 2, 3, 4)$ touch S at α_i and let the tangent to S at α_i be $t_i=0$, where

$$t_i \equiv 2x+2\alpha_i^2 y-2\alpha_i z. \tag{3}$$

Let $t_{ij}=0$ denote the line joining α_i, α_j, so that

$$t_{ij} \equiv 2x+2\alpha_i\alpha_j y-(\alpha_i+\alpha_j)z. \tag{4}$$

Put $$S_i \equiv \lambda_i^2 S+zt_i \quad \text{and} \quad S_{ij} \equiv \lambda_i\lambda_j S+zt_{ij}. \tag{5}$$

Then we have

$$t_i t_j - t_{ij}^2 \equiv -(\alpha_i-\alpha_j)^2 S, \tag{6}$$

$$S_i S_j - S_{ij}^2 \equiv Sz\, P_{ij}, \tag{7}$$

where $$P_{ij} \equiv \lambda_i^2 t_j+\lambda_j^2 t_i-2\lambda_i\lambda_j t_{ij}-(\alpha_i-\alpha_j)^2 z. \tag{8}$$

P_{ij} is obviously a common tangent of the circles S_i, S_j passing through the same centre of similitude as t_{ij}. Also, S_{ij} is the circle passing through the points of contact of S and P_{ij} with S_i, S_j.

We shall denote the other common tangent to S_i, S_j passing through the same centre of similitude as P_{ij} by P'_{ij}, and the circle through the points of contact of S and P'_{ij} with S_i, S_j by S'_{ij}. We then have

$$S'_{ij} \equiv -\lambda_i\lambda_j S+zt_{ij}, \tag{9}$$

$$S_i S_j - S'^2_{ij} \equiv Sz\, P'_{ij},$$

where $$P'_{ij} \equiv \lambda_i^2 t_j+\lambda_j^2 t_i+2\lambda_i\lambda_j t_{ij}-(\alpha_i-\alpha_j)^2 z. \tag{10}$$

We may take Q in the form

$$Q \equiv \lambda_1\lambda_2\lambda_3\lambda_4 S^2+\tfrac{2}{3}z^2 S'+NSz=0, \tag{11}$$

where N is linear in x, y, z, and S' is the rectangular hyperbola through the points α_1, α_2, α_3, α_4, so that

$$S' \equiv 6x^2+6s_4 y^2+\tfrac{3}{2}s_2 z^2-3s_3 yz-3s_1 zx=0$$

$$(s_1 \equiv \Sigma\alpha_i, \ s_2 \equiv \Sigma\alpha_i\alpha_j, \ s_3 \equiv \Sigma\alpha_i\alpha_j\alpha_k, \ s_4 \equiv \alpha_1\alpha_2\alpha_3\alpha_4). \tag{12}$$

Also $$t_{ij}t_{kl}-\tfrac{2}{3}S' \equiv -(\alpha_i\alpha_j+\alpha_k\alpha_l)S. \tag{13}$$

We thus obtain immediately three systems of contact bicircular quartics of

$$S_1 S_2 S_3 S_4 - Q^2 \equiv Sz\,\Gamma=0, \tag{14}$$

viz. $$\sigma^2 S_2 S_3 + 2\sigma Q + S_1 S_4 = 0, \tag{15}$$

$$\sigma^2 S_3 S_1 + 2\sigma Q + S_2 S_4 = 0, \tag{16}$$

$$\sigma^2 S_1 S_2 + 2\sigma Q + S_3 S_4 = 0. \tag{17}$$

The bicircular quartic (15) touches Γ at six points and S at two points, the line joining which always passes through the point of intersection of the lines t_{23}, t_{14}. For twenty-eight values of σ, (15) breaks up into a pair of tritangent circles, two such pairs being S_2, S_3, and S_1, S_4. Similar results hold with regard to (16) and (17).

For one value of σ, the bicircular quartic (15) degenerates into the line at infinity and a circular cubic curve R_1, passing through the point H where Γ cuts the line at infinity (other than at the circular points), and touching Γ at five further points. A circular cubic R_{14} passes through the points of contact of the circles S_1, S_4 and the cubic R_1 with Γ, and similarly with regard to the cubic R_{23} defined in the same way by S_2, S_3.

We have also,

$$Q - S_{23} S_{14} \equiv L_1 Sz, \tag{18}$$

where $$L_1 \equiv -\lambda_2\lambda_3 t_{14} - \lambda_1\lambda_4 t_{23} + (a_2 a_3 + a_1 a_4) z + N, \tag{19}$$

and $$Q - S'_{23} S'_{14} \equiv L_1' Sz, \tag{20}$$

where $$L_1' \equiv \lambda_2\lambda_3 t_{14} + \lambda_1\lambda_4 t_{23} + (a_2 a_3 + a_1 a_4) z + N, \tag{21}$$

etc.

Q intersects S_{23} and S_{14} in four collinear points (other than a_2, a_3 and a_1, a_4) which lie on the straight line L_1.

Now (14) can be written

$$S_1 S_2 S_3 S_4 - (S_{23} S_{14} + L_1 Sz)^2 = 0,$$

i.e. $$(S_2 S_3 - S_{23}^2) S_1 S_4 + S_{23}^2 (S_1 S_4 - S_{14}^2) - 2 S_{23} S_{14} L_1 Sz - L_1^2 S^2 z^2 = 0,$$

i.e. $$P_{23} S_1 S_4 + P_{14} S_{23}^2 - 2 L_1 S_{23} S_{14} - L_1^2 Sz = 0,$$

i.e. $$P_{23}(S_{14}^2 + P_{14} Sz) + P_{14} S_{23}^2 - 2 L_1 S_{23} S_{14} - L_1^2 Sz = 0,$$

i.e. $$\begin{vmatrix} -P_{14} & -L_1 & S_{14} \\ -L_1 & -P_{23} & S_{23} \\ S_{14} & S_{23} & Sz \end{vmatrix} = 0. \tag{22}$$

We thus get in one form an equation for the bicircular quintic Γ.

The curves

$$P_{23}Sz+S_{23}^2\equiv S_2S_3=0,\quad P_{14}Sz+S_{14}^2\equiv S_1S_4=0,\quad P_{14}P_{23}-L_1^2\equiv U_1=0$$

are in the usual way curves that touch Γ wherever they meet it, other than at the circular points at infinity. U_1 is plainly a 5-tangent conic of Γ.

The curves

$$P_{14}S_{23}-L_1S_{14}\equiv C_{14}=0,\quad P_{23}S_{14}-L_1S_{23}\equiv C_{23}=0,$$

$$S_{23}S_{14}+L_1Sz\equiv Q=0$$

pass respectively through the points of contact with Γ of S_1S_4 and U_1, S_2S_3 and U_1, S_2S_3, and S_1S_4.

U_1 is a 5-tangent conic of Γ, through whose points of contact with Γ and those of the tritangent circles S_2, S_3 passes the circular cubic C_{23} and similarly with regard to S_1, S_4, and C_{14}. The common tangents P_{23} of S_2, S_3 and P_{14} of S_1, S_4 each touch U_1 and the line L_1 passes through the points of contact of P_{23} and P_{14} with U_1. The circular cubic C_{23} passes through the points of contact of P_{23} with S_2, S_3, and U_1. It also passes through the points of intersection of the circles S_{23}, S_{14} and through the points common to L_1, S_{14}, and Q. Similar properties hold with regard to the circular cubic C_{14}.

The nine points of intersection of C_{23} and C_{14} with each other and their twelve points of intersection with Q are thus accounted for.

By proceeding in the same way with regard to S_2, S_3, S'_{23}, P'_{23}, etc., we obtain a contact conic $U_1'\equiv P'_{14}P'_{23}-L_1'^2=0$ and a circular cubic C'_{23} passing through the points of contact of S_2, S_3, U_1' with Γ. We thus see that of the pencil of circular cubics through the six points of contact of S_2 and S_3 with Γ, two members C_{23} and C'_{23} each intersect Γ in five residual points that are the points of contact with Γ of the contact conics U_1 and U_1' respectively. Similar results hold with regard to C_{14} and C'_{14} through the points of contact of S_1 and S_4 with Γ. After some rather tedious algebra it can be shown that *the tangents to C_{23}, C'_{23} at A and B harmonically separate the nodal tangents to Γ and the tangents to S_2, S_3 at these points, and similarly with regard to C_{14} and C'_{14}.*

More generally, $\sigma^2P_{23}+2\sigma L_1+P_{14}=0$ *and* $\sigma^2P'_{23}+2\sigma L_1'+P'_{14}=0$ *are each bitangents of the bicircular quartic* $\sigma^2S_2S_3+2\sigma Q+S_1S_4=0$, *the points of contact of each of which lie on a circle. They therefore form a "corresponding pair" of bitangents of the bicircular quartic, there being four such corresponding pairs. The bitangent* $\sigma^2P_{23}+2\sigma L_1+P_{14}=0$ *envelops*

the conic U_1 *and* $\sigma^2 P'_{23}+2\sigma L_1'+P'_{14}=0$ *envelops* U_1'. *These two bitangents intersect each other in a point whose locus is a cubic curve. In particular, the twenty-eight centres of similitude already found belonging to the twenty-eight degenerate circle-pairs* $S_2 S_3$, $S_1 S_4$, *etc. of the family* $\sigma^2 S_2 S_3+2\sigma Q+S_1 S_4=0$ *lie on a cubic curve.*

3. *The contact conics.*

Let us arrange the contact conics as previously defined in the following manner:

$$\begin{matrix} U_1 & U_2 & U_3 \\ U_1' & U_2' & U_3'. \end{matrix}$$

Let the line AB meet the quintic Γ again in H, so that the tangent at H constitutes its proper asymptote. It can then be shown that *a circular cubic* V_1 *passes through the points of contact of* U_1, U_1' *with* Γ_1 *and also through* H; *also that* V_1 *intersects* U_1, U_1' *each in a residual point at which they are each touched by the same line through* H. *Corresponding properties hold with regard to* V_2 *and* V_3 *similarly defined.*

In other words, *each of the conic-pairs* U_1, U_1'; U_2, U_2'; U_3, U_3' *possesses a common tangent parallel to the asymptotes of* Γ, V_1, V_2, V_3.

It is easy to show that V_1 can be expressed in either of the forms:

$$(\lambda_1\lambda_4 P_{23}-\lambda_2\lambda_3 L_1) S_{14}+(\lambda_2\lambda_3 P_{14}-\lambda_1\lambda_4 L_1) S_{23}\equiv\lambda_1\lambda_4 C_{23}+\lambda_2\lambda_3 C_{14}=0,$$

$$(\lambda_1\lambda_4 P'_{23}-\lambda_2\lambda_3 L_1') S'_{14}+(\lambda_2\lambda_3 P'_{14}-\lambda_1\lambda_4 L_1') S'_{23}\equiv\lambda_1\lambda_4 C'_{23}+\lambda_2\lambda_3 C'_{14}=0. \qquad (23)$$

Also the line parallel to the asymptote of Γ and touching both U_1 and U_1' can be expressed in either of the forms

$$\lambda_2^2\lambda_3^2 P_{14}-2\lambda_1\lambda_2\lambda_3\lambda_4 L_1+\lambda_1^2\lambda_4^2 P_{23}=0,$$

$$\lambda_2^2\lambda_3^2 P'_{14}-2\lambda_1\lambda_2\lambda_3\lambda_4 L_1'+\lambda_1^2\lambda_4^2 P'_{23}=0. \qquad (24)$$

The circular cubic V_1 *passing through the points of contact of* U_1 *and* U_1' *with* Γ *passes also through the points of intersection of the circles* S_{23}, S_{14}, *and* S'_{23}, S'_{14}. *It passes also through the nine points of intersection of* C_{23}, C_{14}, *and* C'_{23}, C'_{14}. *Similar properties hold with regard to* V_2 *and* V_3.

The line (24) *touches the cubic* R_1. *For one value of* σ, *the pencil* $\sigma S_{23}+S_{14}=0$ *degenerates into the line at infinity and a line passing through the point of contact of* R_1 *with the line* (24).

It can also be proved by the use of some rather tedious algebra that *cubic curves pass through the points of contact with* Γ *of the triads of contact conics* $U_1U_2U_3$, $U_1U_2'U_3'$, $U_1'U_2U_3'$, $U_1'U_2'U_3$. *The cubic curve through the points of contact of* $U_1U_2U_3$ *cuts the conics of the triad respectively in residual points at which they are all touched by the same line, and similarly with regard to the other triads of contact conics.*

The cubic through the points of contact of $U_1U_2U_3$ with Γ can have its equation expressed in any of the three forms

$$\Delta_2\Delta_3 C_{23}+\Delta_1\Delta_4 C_{14}+(L_1-L_2-L_3)U_1=0,$$

$$\Delta_3\Delta_1 C_{31}+\Delta_2\Delta_4 C_{24}+(L_2-L_3-L_1)U_2=0,$$

$$\Delta_1\Delta_2 C_{12}+\Delta_3\Delta_4 C_{34}+(L_3-L_1-L_2)U_3=0,$$

where Δ_1, Δ_2, Δ_3, Δ_4 are the constituent determinants of

$$\begin{Vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 1 & 1 & 1 & 1 \end{Vmatrix} \tag{25}$$

each with its proper sign.

The common tangent to the triad U_1, U_2, U_3 can have its equation expressed in either of the forms:

$$\Delta_1^2\Delta_4^2 P_{14}-2\Delta_1\Delta_2\Delta_3\Delta_4 L_1+\Delta_2^2\Delta_3^2 P_{23}=0,$$

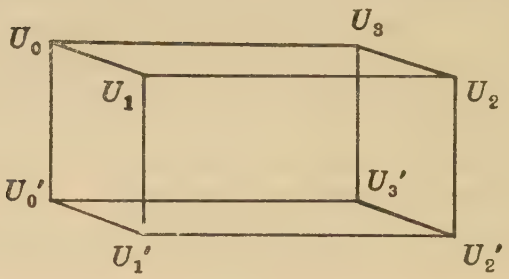
$$\Delta_2^2\Delta_4^2 P_{24}-2\Delta_1\Delta_2\Delta_3\Delta_4 L_2+\Delta_3^2\Delta_1^2 P_{31}=0,$$

$$\Delta_3^2\Delta_4^2 P_{34}-2\Delta_1\Delta_2\Delta_3\Delta_4 L_3+\Delta_1^2\Delta_2^2 P_{12}=0. \tag{26}$$

It is thus seen that the six conics U_1, U_1', U_2, U_2', U_3, U_3' together with the degenerate conic consisting of the line AB taken twice over together constitute a particular instance of a set of seven contact conics of Γ of the type first noted by F. P. White*. They possess the property that the points of contact with Γ of every two conics of the set lie on a cubic which passes through the points of contact of a third conic of the set. This property can be represented diagrammatically as explained in my previous paper*. Let U_0' represent the line at infinity AB taken twice over, and let U_0 represent any two coincident straight lines regarded as a degenerate

* W. P. Milne, *loc. cit.*

conic. We then have the rectangular parallelepiped



possessing the property that four contact conics form a congruent tetrad (*i.e.* a quartic passes through their points of contact with Γ) if they are represented by two pairs of points lying

(i) on two parallel edges of the parallelepiped;

(ii) on two diagonals of parallel faces;

(iiii) on two diagonals of the parallelepiped.

It will be noted that the quartic through the points of contact of the conics of a congruent tetrad containing U_0 degenerates into a cubic and the straight line which taken twice over constitutes the degenerate conic U_0. We can now make use of the general properties known for such a configuration*.

If U_0, U_0' both enter into a congruent tetrad, we obtain the tetrads $U_0U_0'U_1'U_1$, $U_0U_0'U_2'U_2$, $U_0U_0'U_3'U_3'$; i.e. the conic-pairs U_1U_1', U_2U_2', U_3U_3' are such that a circular cubic passes through the points of contact with Γ of every pair and has its asymptote parallel to that of Γ. Every two such conic-pairs U_2U_2' and U_3U_3', etc. are such that a quartic W_1, etc. passes through their points of contact with Γ. W_1 intersects U_2, U_2', U_3, U_3' each in three residual points at which they are all touched by the same cubic Φ_1, and similarly with regard to Φ_2 and Φ_3. Φ_1, Φ_2, Φ_3 have three collinear points in common.

If U_0, but not U_0', enters into every congruent tetrad, we can consider the conic-pair U_0U_1 as constituting a type of the resulting combinations. We then have the congruent tetrads $U_0U_1U_2U_3$, $U_0U_1U_2'U_3'$, $U_0U_1U_1'U_0'$, i.e. the triads of conics $U_1U_2U_3$, $U_1U_2'U_3'$, $U_1U_1'U_0'$, through whose points of contact with Γ cubics pass. Quartics X, Y, W_1 pass through the

* W. P. Milne, *loc. cit.*

points of contact of $U_0'U_1'U_2'U_3'$, $U_0'U_1'U_2U_3$, $U_2U_3U_2'U_3'$. *X and Y are circular quartics having each an asymptote parallel to that of* Γ.

Thus cubics pass through the points of contact of the triads of contact conics $U_1U_2U_3$, $U_1U_2'U_3'$, $U_1'U_2U_3'$, $U_1'U_2'U_3$, whereas circular quartics, each with an asymptote parallel to that of Γ, pass through the points of contact of the complementary triads $U_1'U_2'U_3'$, $U_1'U_2U_3$, $U_1U_2'U_3$, $U_1U_2U_3'$.

4. *A more extended configuration.*

In § 2, we referred to the existence of congruent octads of tritangent circles such as S_1, S_2, S_3, S_4, S_1', S_2', S_3', S_4'. If we take S_1 in turn along with each of S_2, S_3, S_4, S_1', S_2', S_3', S_4', we obtain seven pairs of tritangent circles, each of which will define a pair of contact-conics of Γ of the type U_1, U_1'. We thus obtain a configuration of seven pairs of contact-conics, through whose points of contact with Γ a circular cubic passes. Their main properties will be evident from the investigations of § 3.

The University,
Leeds.

ON A DIOPHANTINE EQUATION OF THE FOURTH DEGREE

By R. D. JAMES.

[Received 2 June, 1937.—Read 17 June, 1937.]

Introduction.

This paper is concerned with the solution of the equation

$$N = \sum_{j=1}^{16} x_j^4 - Mx_{17}^4, \tag{1.1}$$

where M is some fixed positive integer and $x_1, \ldots, x_{17}$ are integers. The number N may be a positive or negative integer or zero. The special case in which $N = 0$ leads to results on the representation of a rational number as a sum of rational fourth powers.

We prove the following theorems:

THEOREM 1. *If N is any positive or negative integer or zero then the equation* (1.1) *always has at least one solution.*

THEOREM 2. *Every positive rational number is the sum of sixteen rational fourth powers. There is an infinite number of integers which are not the sum of less than fifteen rational fourth powers.*

THEOREM 3. *The diophantine equation*

$$\sum_{j=1}^{s} x_j^4 = Mx_{s+1}^4, \quad x_j \neq 0 \quad (j = 1, \ldots, s+1) \tag{1.2}$$

has a solution for every positive integer M when $s \geqslant 16$. If, however, $s \leqslant 15$, then there is an infinite number of integers M for which the equation (1.2) *has no solution in integers x_j different from zero.*

The proof of Theorem 1 differs only slightly from the proof given by Davenport and Heilbronn (1) for the fourth power Waring's problem. Theorems 2 and 3 follow readily from Theorem 1. Theorem 2 is a new

result since nothing was known previously about rational k-th powers except when $k = 2$ or $k = 3$. Whether there exist rational numbers which are not the sum of less than sixteen rational fourth powers is still an open question.

Notation.

We follow as closely as possible the notation used by Davenport and Heilbronn (1). Let N be any positive or negative integer or zero. Let P be any sufficiently large number such that

$$-3P^4 < N < 7P^4.$$

Let

$$T(\alpha, P) = \sum_{P<n<2P} e(n^4\alpha) \quad [e(x) = e^{2\pi i x}],$$

$$T_i(\alpha) = T(\alpha, 2^{-i}P^{\frac{3}{4}i}) \quad (i = 0, 1, 2, 3),$$

$$U(\alpha) = \sum_{M^{-\frac{1}{4}}P<n<2M^{-\frac{1}{4}}P} e(-Mn^4\alpha),$$

$$Q(\alpha) = T_1(\alpha)\,T_2(\alpha)\,T_3^2(\alpha) = \sum_n r_4'(n)\,e(n\alpha),$$

$$R(\alpha) = T_0(\alpha)\,Q(\alpha) = \sum_n r'_5(n)\,e(n\alpha),$$

$$U(\alpha)\,T_0^8(\alpha)\,Q^2(\alpha) = \sum_n r'_{17}(n)\,e(n\alpha),$$

where $r'_{17}(n)$ denotes the number of solutions of equation (1.1) with restrictions on $x_1, \ldots, x_{17}$.

We divide the interval $0 \leqslant \alpha \leqslant 1$ into Farey arcs in the usual way. We write

$$\alpha = \frac{a}{q} + \beta,$$

where

$$1 \leqslant q \leqslant P^3, \quad 0 \leqslant a < q, \quad (a, q) = 1,$$

$$-\vartheta_1 q^{-1} P^{-3} \leqslant \beta \leqslant \vartheta_2 q^{-1} P^{-3}, \quad \tfrac{1}{2} \leqslant \vartheta_i \leqslant 1.$$

Those arcs for which $1 \leqslant q \leqslant P$ are called major arcs $\mathfrak{M}$ and those for which $P < q \leqslant P^3$ are called minor arcs $\mathfrak{m}$. Each major arc is divided into three parts as follows:

(I) $|\beta| \leqslant \frac{1}{2}P^{-4}$,

(II) $\frac{1}{2}P^{-4} \leqslant |\beta| \leqslant \frac{1}{2}q^{-\frac{4}{5}}P^{-\frac{16}{5}}$,

(III) $|\beta| \geqslant \frac{1}{2}q^{-\frac{4}{5}}P^{-\frac{16}{5}}$.

Let
$$S_{a,q} = \sum_{v=1}^{q} e\left(v^4 \frac{a}{q}\right),$$

$$V_{a,q} = \sum_{v=1}^{q} e\left(-Mv^4 \frac{a}{q}\right),$$

$$A_q(n) = q^{-9} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-n \frac{a}{q}\right) V_{a,q} S_{a,q}^8,$$

$$\mathfrak{S}_9(n) = \sum_{q=1}^{\infty} A_q(n),$$

$$\psi_{a,q}(\alpha) = \tfrac{1}{4} \frac{S_{a,q}}{q} \sum_{P^4 < j < (2P)^4} \frac{\Gamma(j+\frac{1}{4})}{j!} e(j\beta),$$

$$\phi_{a,q}(\alpha) = \frac{1}{4M^{\frac{1}{4}}} \frac{V_{a,q}}{q} \sum_{P^4 < j < (2P)^4} \frac{\Gamma(j+\frac{1}{4})}{j!} e(-j\beta).$$

We use ϵ to denote an arbitrarily small positive number, and c_1, c_2, ... to denote positive absolute constants. The constants implied by the O-symbols may involve M, but it is assumed that M, is fixed throughout.

Preliminary lemmas.

We observe first that all the results obtained in (**1**) for $T_0(\alpha)$ and $\psi_{a,q}(\alpha)$ hold also for $U(\alpha)$ and $\phi_{a,q}(\alpha)$, respectively. The reason is that

$$|U(\alpha)| = |U(-\alpha)|,$$

$$U(-\alpha) = \sum_n e(Mn^4\alpha)$$

$$= O\{P^\epsilon q^\epsilon (Pq^{-\frac{1}{8}} + P^{\frac{7}{8}} + P^{\frac{1}{2}} q^{\frac{1}{8}})\},$$

by Landau (**2**), Theorem 5. Also, if $M = \delta M_1$, $q = \delta q_1$, where $(M_1, q_1) = 1$, then $e\{-M(a/q)\}$ is a primitive q_1-th root of unity and may be written in the form $e(a_1/q_1)$, where $a_1 \equiv -M_1 a \pmod{q_1}$. Then

$$V_{a,q} = \frac{q}{q_1} \sum_{v=1}^{q_1} e\left(v^4 \frac{a_1}{q_1}\right)$$

$$= \delta S_{a_1, q_1}.$$

Since $\delta \leqslant M$ and $M^{-1}q \leqslant q_1 \leqslant q$, the results for $V_{a,q}$ and $\phi_{a,q}(\alpha)$ are consequences of the results obtained for S_{a_1,q_1} and $\psi_{a_1,q_1}(\alpha)$.

Some of the lemmas which follow are stated without proof because of their great similarity to the corresponding lemmas in (**1**).

Lemma 1. [(1), Lemma 2.]

$$\sum_{\mathfrak{m}} \int_{\mathfrak{m}} |U(\alpha)\, T_0^{6}(\alpha)\, R^2(\alpha)|\, d\alpha = O\{P^{5-\frac{1}{12}+\epsilon}\, Q^2(0)\}.$$

Lemma 2. [(1), Lemma 7.] *On* (I)

$$U(\alpha)\, T_0^{8}(\alpha) - \phi_{a,q}(\alpha)\, \phi_{a,q}^{8}(\alpha) = O(q^{-1-\frac{1}{4}+\epsilon}\, P^8).$$

Proof. We write

$$(2.1)\quad U(\alpha)\, T_0^{8}(\alpha) - \phi_{a,q}(\alpha)\, \psi_{a,q}^{8}(\alpha)$$

$$= U(\alpha)\{T_0(\alpha) - \psi_{a,q}(\alpha)\} \sum_{m=0}^{7} T_0^{7-m}(\alpha)\, \psi_{a,q}^{m}(\alpha)$$

$$+ \psi_{a,q}^{8}(\alpha)\{U(\alpha) - \phi_{a,q}(\alpha)\}.$$

Lemmas 3 and 4 of (1) and the corresponding results for $U(\alpha)$ and $\phi_{a,q}(\alpha)$ show that each term on the right side of (2.1) is $O(q^{-1-\frac{1}{4}+\epsilon}\, P^8)$.

Lemma 3. [(1), Lemma 8.] *On* (II)

$$U(\alpha)\, T_0^{8}(\alpha) - \phi_{a,q}(\alpha)\, \psi_{a,q}^{8}(\alpha) = O(q^{-1-\frac{1}{4}+\epsilon}\, P^4\, |\beta|^{-1}).$$

Lemma 4. [(1), Lemma 9.] *On* (III)

$$U(\alpha)\, T_0^{8}(\alpha) - \phi_{a,q}(\alpha)\, \psi_{a,q}^{8}(\alpha) = O(q^{-\frac{3}{8}}\, P^{4+\frac{1}{2}+\epsilon}\, |\beta|^{-\frac{3}{8}}).$$

Lemma 5. [(1), Lemma 10.] *We have*

$$\sum_{\mathfrak{M}}{}' \int_{\mathfrak{M}} |U(\alpha)\, T_0^{8}(\alpha) - \phi_{a,q}(\alpha)\, \psi_{a,q}^{8}(\alpha)|\, |Q^2(\alpha)|\, d\alpha = O\{P^{5-\frac{1}{10}+\epsilon}\, Q^2(0)\},$$

where the summation is over all major arcs with $q \leqslant P^{\frac{1}{2}}$.

Lemma 6. [(1), Lemma 11.] *If* $\overline{\mathfrak{M}}_{a,q}$ *denotes the part of the interval* (0, 1) *not belonging to* $\mathfrak{M}_{a,q}$, *then*

$$\sum_{q \leqslant P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\overline{\mathfrak{M}}_{a,q}} |\phi_{a,q}(\alpha)\, \psi_{a,q}^{8}(\alpha)\, Q^2(\alpha)|\, d\alpha = O\{P^{5-\frac{1}{2}}\, Q^2(0)\}.$$

Lemma 7. [(1), Lemma 12.]

$$\sum_{\mathfrak{M}}{}'' \int_{\mathfrak{M}} |U(\alpha)\, T_0^{6}(\alpha)\, R^2(\alpha)|\, d\alpha = O\{P^{5-\frac{1}{12}+\epsilon}\, Q^2(0)\},$$

where the summation is over all major arcs with $q > P^{\frac{1}{2}}$.

The singular series.

Our treatment of the singular series follows that given by Landau (**4**).

LEMMA 8. [(**1**), Lemma 13.]

$$\int_0^1 e(-n\alpha)\,\phi_{a,q}(\alpha)\,\psi_{a,q}^8(\alpha)\,d\alpha = q^{-9}\,V_{a,q}\,S_{a,q}^8\,e\left(-\frac{na}{q}\right)F(n),$$

where for $-4P^4 < n < 7P^4$ *we have*

$$c_2\,P^5 < F(n) < c_3\,P^5.$$

Proof. We have

$$\int_0^1 e(-n\alpha)\,\phi_{a,q}(\alpha)\,\psi_{a,q}^8(\alpha)\,d\alpha = q^{-9}\,V_{a,q}\,S_{a,q}^8\,e\left(-\frac{na}{q}\right)F(n),$$

where

$$F(n) = (\tfrac{1}{4})^9\,M^{-\frac{1}{4}} \sum_{\substack{j_1+j_2+\ldots+j_8-j_9=m \\ P^4<j_i<(2P)^4}} \frac{\Gamma(j_1+\frac{1}{4})\ldots\Gamma(j_9+\frac{1}{4})}{j_1!\ldots j_9!}.$$

Since
$$c_4 j^{-\frac{3}{4}} j! < \Gamma(j+\tfrac{1}{4}) < c_5 j^{-\frac{3}{4}} j!,$$

we have

$$(3.1) \qquad c_6(P^{-3})^9 \sum_{\substack{j_1+\ldots+j_8-j_9=n \\ P^4<j_i<(2P)^4}} 1 < F(n) < c_7(P^{-3})^9 \sum_{\substack{j_1+\ldots+j_8-j_9=n \\ P^4<j_i<(2P)^4}} 1.$$

Also, the summation for the j_i in these inequalities gives the number of solutions of the equation

$$(3.2) \qquad j_1+\ldots+j_8-j_9 = n, \quad P^4 < j_i < (2P)^4.$$

In this equation $j_1, j_2, \ldots, j_8$ may take independently the values from P^4 to $(2P)^4$ and then there is at most one value for j_9. The number of solutions of equation (3.2) is therefore less than $c_8(P^4)^8$. If $j_1, j_2, \ldots, j_8$ take independently the values from P^4 to $\frac{3}{2}P^4$ then there is always a value for j_9 since

$$-4P^4 < n < 7P^4,$$

$$P^4 < j_1+\ldots+j_8-n < (2P)^4.$$

Hence the number of solutions of equation (3.2) is greater than $c_9(P^4)^8$. The lemma now follows from (3.1).

LEMMA 9. *We have*

$$\sum_{q \mid p^l} A_q(n) = p^{-8l}\,W(p^l),$$

where $W(p^l)$ is the number of solutions of the congruence

$$\sum_{j=1}^{8} u_j^4 - Mu_9^4 \equiv n \pmod{p^l} \quad (0 \leqslant u_i < p^l).$$

If $(m_1, m_2) = 1$, *then*

$$\sum_{q \mid m_1 m_2} A_q(n) = \sum_{q \mid m_1} A_q(n) \sum_{q \mid m_2} A_q(n).$$

Proof. The method of proof is exactly the same as the proof of Theorem 1 in (**4**).

Lemma 10. *We have*

$$p^{-8l} W(p^l) \geqslant p^{-8} \quad \text{if} \quad p > 2, \quad l \geqslant 1;$$

$$2^{-8l} W(2^l) \geqslant 2^{-32} \quad \text{if} \quad l \geqslant 1, \quad n \equiv 1, 2, \ldots, 8 \pmod{16}.$$

Proof. By Landau (**3**), Theorem 301(2), the congruence

$$\sum_{j=1}^{8} u_j^4 - Mu_9^4 \equiv n \pmod{p} \quad (0 \leqslant u_i < p),$$

where $p > 2$, has a solution with u_1 prime to p and $u_9 \equiv 0 \pmod{p}$. Hence there are at least $p^{8(l-1)}$ solutions of the congruence

$$u_1^4 \equiv -\sum_{j=2}^{8} u_j^4 + Mu_9^4 + n \pmod{p}, \quad p \nmid u_1$$
$$(0 \leqslant u_1 < p, \quad 0 \leqslant u_i < p^l, \quad i = 2, \ldots, 9).$$

By Landau (4), Theorem 3, there are at least $p^{8(l-1)}$ solutions of thecongruence

$$u_1^4 \equiv -\sum_{j=2}^{8} u_j^4 + Mu_9^4 + n \pmod{p^l} \quad (0 \leqslant u_i < p^l).$$

This proves the first inequality of the lemma.

The second inequality follows in the same way, but we start with the fact that the congruence

$$\sum_{j=1}^{8} u_j^4 - Mu_9^4 \equiv n \pmod{16}$$

has a solution with u_1 odd if $n \equiv 1, 2, \ldots, 8 \pmod{16}$.

Lemma 11. [(1), Lemma 14.] $\mathfrak{S}_9(n) \geqslant 0$ *for all* n *and* $\mathfrak{S}_9(n) > c_{10}$ *provided that* n *is congruent to one of* $1, 2, \ldots, 8 \pmod{16}$.

Proof. Since $S_{a,q}$ and $V_{a,q}$ are both $O(q^{\frac{1}{2}})$ we have

$$A_q(n) = O\left\{\sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-\frac{3}{4}}\right\} = O(q^{-\frac{5}{4}}).$$

Hence the singular series converges absolutely. Moreover, we have

$$\begin{aligned}\sum_{q|p^l} A_q(n) &= 1+\sum_{\lambda=1}^{l} A_{p^\lambda}(n)\\ &> 1-c_{11}\sum_{\lambda=1}^{\infty} p^{-\frac{5}{4}\lambda}\\ &= 1-c_{11}(p^{\frac{5}{4}}-1)^{-1}.\end{aligned}$$

Then, if p_l denotes the l-th prime, we have

$$\mathfrak{S}_9(n) = \lim_{l\to\infty} \prod_{p\leq p_l} \sum_{q|p^l} A_q(n).$$

Using Lemmas 9 and 10 we obtain

$$\mathfrak{S}_9(n) \geqslant 0 \quad \text{for all } n;$$

$$\mathfrak{S}_9(n) \geqslant 2^{-32} \prod_{p>2} \max\{p^{-8}, 1-c_{11}(p^{\frac{5}{4}}-1)^{-1}\},$$

if n is congruent to one of 1, 2, ..., 8 (mod. 16).

The proof of Theorem 1.

We follow closely the proof given in (**1**). By Lemmas 1, 5, 6, 7,

$$\begin{aligned}r_{17}'(N) &= \int_0^1 e(-N\alpha)\, U(\alpha)\, T_0{}^8(\alpha)\, Q^2(\alpha)\, d\alpha\\ &= \sum_{\mathfrak{M}}{}' \int_0^1 e(-N\alpha)\, \phi_{a,q}(\alpha)\, \psi_{a,q}^8(\alpha)\, Q^2(\alpha)\, d\alpha + O\{P^{5-\frac{3}{2}+\epsilon}\, Q^2(0)\}.\end{aligned}$$

Using Lemma 8 and the definition of $Q(\alpha)$ we have

$$\begin{aligned}&\sum_{\mathfrak{M}}{}' \int_0^1 e(-N\alpha)\, \phi_{a,q}(\alpha)\, \psi_{a,q}^8(\alpha)\, Q^2(\alpha)\, d\alpha\\ &= \sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2) \sum_{\mathfrak{M}}{}' \int_0^1 e\{-(N-n_1-n_2)\alpha\}\, \phi_{a,q}(\alpha)\, \psi_{a,q}^8(\alpha)\, d\alpha\\ &= \sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2)\, F(N-n_1-n_2)\\ &\qquad \times \sum_{q\leq P^{\frac{1}{2}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9}\, V_{a,q}\, S_{a,q}^8\, e\left\{-(N-n_1-n_2)\,\frac{a}{q}\right\}.\end{aligned}$$

Also,

$$\sum_{q\leqslant P^{\frac{1}{2}}} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-9} V_{a,q} S_{a,q}^{8} e\left\{-(N-n_1-n_2)\frac{a}{q}\right\}$$

$$= \sum_{q\leqslant P^{\frac{1}{2}}} A_q(N-n_1-n_2) = \mathfrak{S}_9(N-n_1-n_2)+O\left\{\sum_{q>P^{\frac{1}{2}}} q^{-\frac{5}{4}}\right\}$$

$$= \mathfrak{S}_9(N-n_1-n_2)+O(P^{-\frac{1}{8}}).$$

From the definitions of P and $Q(\alpha)$ we have

$$-3P^4 < N < 7P^4,$$

$$\tfrac{1}{8}A < n_1+n_2 < 2A,$$

where $$A = P^3+\tfrac{1}{16}P^2+\tfrac{1}{128}P^{\frac{27}{16}}.$$

Hence $$-4P^4 < N-n_1-n_2 < 7P^4,$$

so that Lemma 8 gives

$$r'_{17}(N) = \sum_{n_1}\sum_{n_2} r_4'(n_1)\, r_4'(n_2)\, F(N-n_1-n_2)\, \mathfrak{S}_9(N-n_1-n_2)+O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}.$$

We define t to be 0 if $N \equiv 1, 2, \ldots, 8 \pmod{16}$ and to be 1 if

$$N \equiv 9, 10, \ldots, 16 \pmod{16}.$$

Then if $n_1 \equiv n_2 \equiv 4t \pmod{16}$ we have $N-n_1-n_2 \equiv 1, 2, \ldots, 8 \pmod{16}$. Also

$$\sum_{n_1\equiv 4t} r_4'(n_2) > c_{13}\sum_{n_1} r_4'(n_1) = c_{13}\, Q(0).$$

Hence, by Lemmas 8 and 11, we have

$$r'_{17}(N) > c_2 P^5 c_{10}\left\{\sum_{n_1\equiv 4t} r_4'(n_1)\right\}^2 + O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}$$

$$> c_2 P^5 c_{10} c_{13}^2\, Q^2(0)+O\{P^{5-\frac{1}{32}+\epsilon}\, Q^2(0)\}$$

$$> 0.$$

This completes the proof of Theorem 1.

The proofs of Theorems 2 and 3.

Let p/q be any rational number. In Theorem 1 let $N = 0$, $M = pq^3$. Then there is always a solution of the equation

$$\sum_{j=1}^{16} x_j^4 = pq^3 x_{17}^4$$

which may be written in the form

$$\sum_{j=1}^{16} \left(\frac{x_j}{qx_{17}}\right)^4 = \frac{p}{q}.$$

This proves the first part of Theorem 2.

The second part is easily proved. Let M be any integer which is congruent to 15 (mod 16). If M were the sum of less than fifteen rational fourth powers there would be a solution of the equation

$$\sum_{j=1}^{s} x_j^4 = Mx_{s+1}^4,$$

with $s \leqslant 14$. Since every fourth power is congruent to 0 or 1 (mod 16), it follows that if x_{s+1} is even so are $x_1, x_2, \ldots, x_s$. After division by the appropriate power of 2 we obtain the equation

$$\sum_{j=1}^{s} y_j^4 = My_{s+1}^4$$

in which y_{s+1} is odd. The right side of this equation is congruent to 15 (mod 16) whereas the left side is congruent to some integer b where $0 \leqslant b \leqslant 14$. It follows that this equation is impossible and hence that M is not the sum of less than fifteen rational fourth powers.

The first part of Theorem 3 is obtained by letting $N = 0$ in Theorem 1 and the second part is just another way of stating the second part of Theorem 2.

References.

1. H. Davenport and H. Heilbronn, "On Waring's problem for fourth powers", *Proc. London Math. Soc.* (2), 41 (1936), 143–150.
2. E. Landau, "Über die neue Winogradoffsche Behandlung des Waringschen Problems", *Math. Zeitschrift*, 31 (1930), 319–338.
3. ———, *Vorlesungen über Zahlentheorie*, 1 (Leipzig, 1927).
4. ———, "Zum Waringschen Problem, Dritte Abhandlung", *Math. Zeitschrift*, 32 (1930), 699–702.

University of California,
Berkeley, California, U.S.A.

SINGULARITIES OF SPACE ALGEBRAIC CURVES

By J. G. SEMPLE.

[Received 16 July, 1937.—Read 18 November, 1937.]

In contrast to the fairly complete theory that we possess at present of singular algebroid branches in the plane or, what amounts to the same thing, on a linear surface branch, the theory of algebroid curve-branches in space, or on a linear threefold branch, is admittedly incomplete. Enriques' account† of the subject is a preliminary survey which requires considerable amplification to make it a systematic account. More particularly, the problem of the relation between the succession of multiple points on a space branch and that on its general plane projection has never been cleared up; and recently Zariski‡ has emphasised this difficulty by giving a simple example of a space branch whose resolution is different from that of its plane projection. We attempt in this paper to carry the investigation of space branches a stage further; and we discuss in particular the origin of the new singular points arising by projection of a space branch, and point out that these latter cannot properly be regarded as projections of any definite points or groups of points actually belonging to the space branch itself.

1. *Formal characterization of space-branches.* The formal characterization of a space-branch type is essentially equivalent to the characterization of a consecutive§ sequence of neighbouring points,

$$E_1, E_2, \ldots, E_k,$$

through the last of which the branch is supposed to pass simply and freely. The simplest method of characterizing a succession of neighbouring points

† Enriques-Chisini (1).

‡ Zariski (1), 10–12.

§ By a *consecutive* sequence is understood a sequence in which each point is in the first neighbourhood of its predecessor.

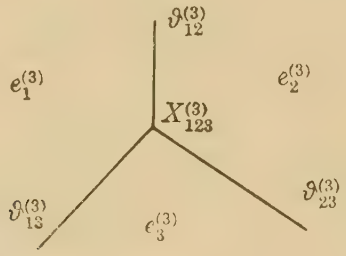
in the plane (or on a linear surface branch) was first stated formally by Du Val†; and his method and notation admit of immediate extension to similar sequences in space, not only of three but of any number of dimensions.

When a simple point E_1 of a three-fold V is resolved, it gives rise exceptionally, on the transformed manifold $V^{(1)}$, to a plane $e_1^{(1)}$, abstractly identical, from the point of view of linear equivalence, with the point E_1 of V. When a point E_2 of $e_1^{(1)}$ is again resolved, we obtain in $V^{(2)}$ a plane $e_2^{(2)}$, representing E_2, and a surface $e_1^{(2)}$, representing $E_1 - E_2$, these two meeting in a line $\vartheta_{12}^{(2)}$. Evidently $E_1 \equiv e_1^{(2)} + e_2^{(2)}$, and any point of $\vartheta_{12}^{(2)}$ is double in E_1.

Now we say that a point E_i of any sequence is *proximate* to a point E_a if E_i is either (a) any point of the first neighbourhood of E_a or (b) any point of any (irreducible) diminished neighbourhood of E_a, obtained by subtracting from it (resolving) a set of points of itself. In particular in a consecutive sequence, every point is proximate to its *immediate ancestor*; and it may be also proximate to one or more *remote ancestors*.

In the present instance we may choose E_3 freely in $e_2^{(2)}$, in which case it is a *free point* (one of a double infinity) proximate only to its immediate ancestor E_2; or we may choose it in $\vartheta_{12}^{(2)}$, in which case we call it a *semi-satellite* (one of a simple infinity), proximate both to E_1 and E_2.

Suppose E_3 is chosen on $\vartheta_{12}^{(2)}$ and let it be resolved. We obtain then in $V^{(3)}$ a plane $e_3^{(3)}$ representing E_3, a surface $e_2^{(3)}$ representing $E_2 - E_3$, and a surface $e_1^{(3)}$ representing $E_1 - E_2 - E_3$, these intersecting in curves $\vartheta_{12}^{(3)}$, $\vartheta_{23}^{(3)}$, $\vartheta_{13}^{(3)}$ concurrent in a point $X_{123}^{(3)}$.

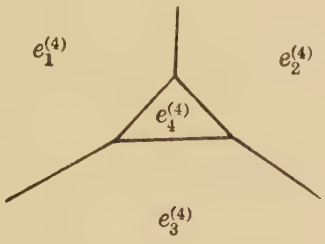


We may now choose E_4 (successor to E_3) either (1) freely in $e_3^{(3)}$, proximate only to E_3, or (2) on $\vartheta_{13}^{(3)}$, semi-satellite, proximate to E_1 and E_3, or (3) on

† Du Val (1). In a recent memoir (not yet published), on singularities of surfaces, which I have been privileged to see in ms., Du Val extends his own results to higher space; but the results obtained in his memoir do not materially overlap the results obtained here.

$\vartheta_{23}^{(3)}$, semi-satellite, proximate to E_2 and E_3, or (4) at $X_{123}^{(3)}$, *proper satellite* proximate to E_1, E_2, E_3.

If E_4 then is now chosen at the proper satellite $X_{123}^{(3)}$ we get, on resolving it, the following configuration:



and there are now no less than *seven* possibilities for E_5. In fact, E_5 may be free (in $e_4^{(4)}$, of course), or semi-satellite proximate to E_4 and to any one of E_1, E_2, E_3, or (proper) satellite proximate to E_4 and to any two of E_1, E_2, E_3. In the figure, that is to say, E_5 may be chosen freely in $e_4^{(4)}$, or on any one of the sides of the triangle shown in that plane, or at any one of the vertices of the same triangle.

Proceeding in this way then and defining a *free* point as one which is proximate to no remote ancestor, a *semi-satellite* as one which is proximate to one remote ancestor, and a (proper) *satellite* as one which is proximate to two remote ancestors (evidently the maximum number possible in a V_3), we see that the first neighbourhood of any free point (other than E_1) contains (besides free points) a line of semi-satellites; the first neighbourhood of a semi-satellite point contains, besides free points, two lines of semi-satellites (of different types) and one proper satellite, while the first neighbourhood of a satellite contains, besides free points, three lines of semi-satellites (of different types) and three distinct proper satellites.

1.1. We may now extend Du Val's matrix characterization of a two-dimensional sequence of points† and characterize similarly any space sequence by a triangular proximity matrix $\mathbf{m}$. If the sequence of the points themselves is $\mathbf{E} \equiv (E_1, E_2, \ldots, E_k)$, and the sequence of their diminished neighbourhoods (after the last point E_k has been resolved) is $\mathbf{e} \equiv (e_1, e_2, \ldots, e_k)$, then

$$e_i \equiv E_i - \overset{(i)}{\sum_j} E_j, \quad (i = 1, \ldots, k),$$

the summation $\overset{(i)}{\Sigma}$ extending to all points E_j proximate to E_i; and regarding

† Du Val (1).

$\mathbf{E}$ and $\mathbf{e}$ as symbolic vectors, and denoting the transposition of a matrix by the symbol $\sim$, we may write

$$\tilde{\mathbf{e}} = \mathbf{m} \cdot \tilde{\mathbf{E}},$$

where $\mathbf{m}$ is a matrix with units down the leading diagonal, zeros to the left, and -1 or zero in every place to the right. This proximity matrix $\mathbf{m}$ characterizes the sequence, telling us which points are proximate to which. It will have at most three -1's in any column (for any three-dimensional point-sequence), and, if the sequence is consecutive, every element immediately to the right of the leading diagonal will be -1.

On reversing the above equations to express each of the points E_a as a sum of its own diminished neighbourhood e_a and multiples of those of succeeding points of the sequence, we obtain a relation

$$\tilde{\mathbf{E}} = \mathbf{n} \cdot \tilde{\mathbf{e}},$$

where $\mathbf{n}$ also is triangular, having units along the leading diagonal, zeros to the left, and positive integers or zeros in all places to the right†. Evidently the i-th column of $\mathbf{n}$ is, except for its unit element on the leading diagonal, the sum of such previous columns as correspond to points to which E_i is proximate, this being actually the rule by which one would naturally construct $\mathbf{n}$, step by step, following the resolutions. Since also $\mathbf{m}$ and $\mathbf{n}$ are both unit matrices, we have

$$\mathbf{n} = \mathbf{m}^{-1}.$$

A primitive curve branch γ, passing freely through E_k, meets the irreducible surface e_k in one point of general position on e_k, and it has no intersection with any other diminished neighbourhood $e_i (i < k)$. Thus since, [writing $\mathbf{n} = (n_{ij})$],

$$E_i = \sum_{a=1}^{a=k} n_{ia} e_a,$$

γ has n_{ik} intersections with E_i; and it represents, in the original manifold V, a singular branch whose multiplicity at E_i is n_{ik}.

Thus the elements of the last column of the matrix $\mathbf{n}$ *represent the multiplicities at* $E_1, \ldots, E_k$ *of a primitive branch through* E_k.

More generally the element n_{ij} represents, in an obvious sense, the multiplicity of E_j in E_i; so that $\mathbf{n}$ may appropriately be called the multiplicity matrix of the sequence $E_1, \ldots, E_k$.

† If the sequence is consecutive, there will be positive integers in all places to the right of the leading diagonal.

2. *The analytic form of a space-branch.* The equations of a space branch, referred to axes of general orientation at the origin of the branch, may be written in the form

$$x = t^{\nu},$$

$$y = a_1 t^{\nu} + a_2 t^{\nu+\nu'} + a_3 t^{\nu+\nu'+\nu''} + \ldots,$$

$$z = \alpha_1 t^{\nu} + \alpha_2 t^{\nu+\nu'} + \alpha_3 t^{\nu+\nu'+\nu''} + \ldots,$$

and a general projection, in this case, may be taken to be the branch in the plane $z = 0$ defined by the first two equations. To resolve the branch any transformation with an isolated simple (or multiple) base point at the origin will do: and the most convenient by far is that introduced by Enriques†, namely

$$x' = x; \quad y' = \frac{y}{x}; \quad z' = \frac{z}{x},$$

(or a similar one in which y or z plays the special part above assigned to x). The naturalness of this resolution becomes more apparent when, instead of using [3], we work with a linear three-fold branch. For we must then regard x, y, z as parameters in terms of which this branch is analytic at the origin; and after a resolution, we require just some such transformation of parameters as the above to restore regularity to the representation of the threefold branch at a new origin, representing a point proximate to the old.

The resolution of the above branch, which we may call γ, by a series of such transformations, is discussed at length by Enriques (*loc. cit.*); but from our point of view we require not only the mere multiplicity sequence but also the character of the sequence of successive points $E_1, E_2, \ldots$ up to the last satellite or semi-satellite point of the branch. This is, in fact, very easy to obtain either by deduction from the multiplicity sequence itself (starting from the last multiple point and using the fact that the multiplicity of any point is equal to the sum of the multiplicities of those proximate to it), or by observing, at each stage in the resolution, the location of the new origin E_i relative to diminished neighbourhoods of previous points $E_{i-\alpha}$. In fact, the transformation

$$\begin{pmatrix} x & y & z \\ x & y/x & z/x \end{pmatrix}$$

leaves the planes $y = 0$ and $z = 0$ unaltered, sends the plane $x = 0$ to

† Enriques-Chisini (1).

infinity, and transforms the origin (*i.e.* its neighbourhood) into the plane $x=0$; and we find therefore that the diminished neighbourhoods actually required are always in fact planes of reference, the three such planes, in the case of a proper satellite point E_i, being the neighbourhoods, whole or diminished, of the three points to which E_i is proximate, and two of them performing a similar role if E_i is a semi-satellite.

We now give some important particular cases of characteristically space branches and their analytic expressions.

3. *Particular space branches.*

(i) *Zariski's example.* The simplest possible type of space branch with a proper satellite in its resolution is that of Zariski, referred to in our introduction. We saw (§ 2) that a succession of four points E_1, E_2, E_3, E_4 is possible in which E_4 is proximate to E_1, E_2, E_3, this requiring that E_3 be proximate to E_1 and E_2. Zariski's branch may be defined then as a primitive branch through the fourth point E_4 of such a succession, the latter being evidently the nearest type of satellite point accessible from any initial point E_1. For the sequence we have

$$\begin{aligned} e_1 &= E_1 - E_2 - E_3 - E_4 & E_1 &= e_1 + e_2 + 2e_3 + 4e_4 \\ e_2 &= \quad E_2 - E_3 - E_4 & E_2 &= \quad e_2 + e_3 + 2e_4 \\ e_3 &= \quad\quad E_3 - E_4 & E_3 &= \quad\quad e_3 + e_4 \\ e_4 &= \quad\quad\quad E_4 & E_4 &= \quad\quad\quad e_4. \end{aligned}$$

The multiplicities of the branch at E_1, E_2, E_3, E_4, the coefficients of e_4 in the expressions on the right, are 4, 2, 1, 1 respectively. The most general analytic form of the branch is

$$x = t^4,$$

$$y = a_1 t^4 + a_2 t^6 + \sum_1^\infty a_{2+i} t^{6+i},$$

$$z = \alpha_1 t^4 + \alpha_2 t^6 + \sum_1^\infty \alpha_{2+i} t^{6+i},$$

where it is assumed that $a_2 \alpha_3 - a_3 \alpha_2 \neq 0$.

(ii) *The three proper satellites immediately following Zariski's satellite.* We may now choose, as primitive origin of a branch, any one of the three

proper satellites E_5 following the Zariski satellite E_4, described above (*cf.* § 2). We consider each of the three branch types so defined in turn.

(*a*) If E_5 is proximate to E_1, E_2, E_4, then

$$\tilde{\mathbf{e}} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ & 1 & -1 & -1 & -1 \\ & & 1 & -1 & \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{E}}; \qquad \tilde{\mathbf{E}} = \begin{pmatrix} 1 & 1 & 2 & 4 & 6 \\ & 1 & 1 & 2 & 3 \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{e}}.$$

The branch in question is of order 6, with multiplicities 6, 3, 1, 1, 1 at $E_1, \ldots, E_5$, and its analytic form is found to be

$$x = t^6,$$

$$y = a_1 t^6 + a_2 t^9 + \sum_1^\infty a_{2+i} t^{9+i},$$

$$z = \alpha_1 t^6 + \alpha_2 t^9 + \sum_1^\infty \alpha_{2+i} t^{9+i}.$$

(*b*) If E_5 is proximate to E_1, E_3, E_4, then

$$\tilde{\mathbf{e}} = \begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ & 1 & -1 & -1 & \\ & & 1 & -1 & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{E}}; \qquad \tilde{\mathbf{E}} = \begin{pmatrix} 1 & 1 & 2 & 4 & 7 \\ & 1 & 1 & 2 & 3 \\ & & 1 & 1 & 2 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{e}}.$$

The multiplicities are 7, 3, 2, 1, 1; and the analytic form is

$$x = t^7$$

$$y = a_1 t^7 + a_2 (t^{10} + \lambda t^{11}) + a_3 t^{12} + \sum_1^\infty a_{3+i} t^{12+i},$$

$$z = \alpha_1 t^7 + \alpha_2 (t^{10} + \lambda t^{11}) + \alpha_3 t^{12} + \sum_1^\infty \alpha_{3+i} t^{12+i}.$$

(*c*) If E_5 is proximate to E_2, E_3, E_4, then

$$\tilde{\mathbf{e}} = \begin{pmatrix} 1 & -1 & -1 & -1 & \\ & 1 & -1 & -1 & -1 \\ & & 1 & -1 & -1 \\ & & & 1 & -1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{E}}; \qquad \tilde{\mathbf{E}} = \begin{pmatrix} 1 & 1 & 2 & 4 & 7 \\ & 1 & 1 & 2 & 4 \\ & & 1 & 1 & 2 \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \tilde{\mathbf{e}}.$$

The multiplicities are 7, 4, 2, 1, 1, and the analytic form is

$$x = t^7,$$

$$y = a_1 t^7 + a_2(t^{11} + \lambda t^{12}) + \sum_1^\infty a_{2+i} t^{12+i},$$

$$z = \alpha_1 t^7 + \alpha_2(t^{11} + \lambda t^{12}) + \sum_1^\infty \alpha_{2+i} t^{12+i}.$$

(iii) *Sequence of points all proximate to E_1, E_2, followed by a further sequence all proximate to E_1 or all proximate to E_2.* We next analyse a more general type of space branch, that namely admitting of the following comparatively simple analytical form:

$$x = t^\nu$$

$$y = a_1 t^\nu + a_2 t^{\nu+\nu_1} + \sum_1^\infty a_{2+i} t^{\nu+\nu_1+i},$$

$$z = \alpha_1 t^\nu + \alpha_2 t^{\nu+\nu_1} + \sum_1^\infty \alpha_{2+i} t^{\nu+\nu_1+i},$$

where we assume, as usual, that $a_2 \neq 0$, $\alpha_2 \neq 0$, $a_2\alpha_3 - a_3\alpha_2 \neq 0$, and also, in addition†, that $\nu_1 < \nu$.

After effecting the usual resolution, which we may refer to for convenience as $(x, y/x, z/x)$, and change of origin, we obtain the branch

$$x = t^\nu,$$

$$y = a_2 t^{\nu_1} + a_3 t^{\nu_1+1} U_1,$$

$$z = \alpha_2 t^{\nu_1} + \alpha_3 t^{\nu_1+1} U_2,$$

any symbol U_i denoting here, and below, an ascending power series in t whose constant term is unity. The new plane $x = 0$ represents the origin E_1 of the original branch.

† We could generalize this example slightly by taking, instead of the suggested form, a branch given by

$$x = t^\nu,$$

$$y = \sum_1^k a_{1i} t^{i\nu} + b_1 t^{k\nu+\nu_1} + \sum_1^\infty c_i t^{k\nu+\nu_1+i},$$

$$z = \sum_1^k \alpha_{1i} t^{i\nu} + \beta_1 t^{k\nu+\nu_1} + \sum_1^\infty \gamma_i t^{k\nu+\nu_1+i},$$

this including the case $\nu_1 \geqslant \nu$ excluded in the text; but this generalization merely prefixes a string of $k-1$ free ν-fold points to a branch of the kind discussed in the text.

A second resolution $(x/y,\ y,\ z/y)$, and displacement along the z axis, leaves $x=0$ unaltered, transforms E_2 into $y=0$ (so that the new origin is proximate to E_1 and E_2), and gives a resolved branch of the form

$$x = \frac{1}{a_2}\, t^{\nu-\nu_1} U_3,$$

$$y = a_2 t^{\nu_1} + a_3 t^{\nu_1+1} U_1,$$

$$z = \frac{a_3 a_2 - a_2 a_3}{a_2 a_2}\, t\, U_1.$$

This has a simple point at the new origin E_3, and there are two cases to consider according as $\nu \geqslant 2\nu_1$ or $\nu \leqslant 2\nu_1$.

(*a*) Suppose that $\nu \geqslant 2\nu_1$. Then the above (linear) branch evidently contains $\nu_1 - 1$ additional points, $E_4, \ldots, E_{\nu_1+2}$, consecutive to E_3 on the axis OZ, all of which, accordingly, are proximate to E_1 and E_2 and therefore proper satellites; and following after these it evidently contains $\nu - 2\nu_1$ further points, $E_{\nu_1+3}, \ldots, E_{\nu-\nu_1+2}$, lying in the plane $x=0$, and semi-satellite therefore, proximate to E_1. Hence if $\nu \geqslant 2\nu_1$, the resolution of the branch is indicated by the symbol

$$[\nu^1,\ \nu_1{}^1,\ 1^1\,(E_1),\quad 1^{\nu_1-1}(E_1,\ E_2),\quad 1^{\nu-2\nu_1}(E_1),\ 1^\infty],$$

where here, and in the sequel, all *remote* proximities are indicated in brackets after the term denoting a sequence of points possessing them, the nomenclature $E_1, E_2, \ldots$ for successive points on a branch being implicitly assumed, and a term such as $1^{\nu_1-1}(E_1, E_2)$, for example, denoting a sequence of $\nu_1 - 1$ simple points all (remotely) proximate to E_1 and E_2.

The complete divergence of the above resolution from that of a general plane projection of the branch should be noted. If, for example, ν_1 is prime to ν, the multiplicity sequence of the projection is indicated by the symbol

$$[\nu^1,\ \nu_1^{k_1},\ \nu_2^{k_2},\ \ldots,\ \nu_{g-1}^{k_{g-1}},\ 1^{k_g},\ 1^\infty],$$

where the numbers k_i, ν_i are those which arise in the equations

$$\nu = k_1\nu_1 + \nu_2,\quad \nu_1 = k_2\nu_2 + \nu_3,\quad \ldots,\quad \nu_{g-1} = k_{g-1}1,$$

incidental to the process of expressing $\nu_1 : \nu$ as a continued fraction.

(*b*) Suppose now that $\nu \leqslant 2\nu_1$. Then the resolved branch which has origin at E_3 has $\nu - \nu_1 - 1$ consecutive points after E_3 on OZ, proper satellites proximate to E_1, E_2, and, following after these, $2\nu_1 - \nu$ further

points in $y = 0$, semi-satellites, therefore, proximate to E_2. Thus in this case the character of the branch is indicated by the symbol

$$[\nu^1,\ \nu_1{}^1,\ 1^1(E_1),\ 1^{\nu-\nu_1-1}(E_1, E_2),\ 1^{2\nu_1-\nu}(E_2),\ 1^{\nu}].$$

We shall return later in this paper to further consideration of the above branches (§§ 7, 8).

(iv) *An extension of the preceding case.* Any case of much greater generality than the preceding seems to lead, inevitably, to a ternary process in the resolution, which makes it virtually impossible to state the result in explicit terms. To illustrate the manner in which this occurs and the influence of the actual coefficients on the character of the branch, we may consider briefly the following apparently innocent extension of the previous case, the type of branch namely of the form

$$x = t^\nu,$$

$$y = a_1 t^\nu + \sum_0^{\nu_2-1} a_{2+i} t^{\nu+\nu_1+i} + b t^{\nu+\nu_1+\nu_2} U_1,$$

$$z = a_1 t^\nu + \lambda \sum_0^{\nu_2-1} a_{2+i} t^{\nu+\nu_1+i} + \beta t^{\nu+\nu_1+\nu_2} U_2,$$

where $a_2 \neq 0$, $\beta \neq \lambda b$, and U_i represents as usual a power series in t with unit constant term.

After applying the resolution $(x,\ y/x,\ z/x)$, changing the origin, and rotating the axes about OX, we obtain the equations of the transformed branch in the form

$$x = t^\nu,$$

$$y = c t^{\nu_1} U_3,$$

$$z = \gamma t^{\nu_1+\nu_2} U_4,$$

where $c \neq 0$, $\gamma \neq 0$, and the plane $x = 0$ still represents the neighbourhood of the origin E_1 of the original branch.

Taking, as before, $\nu_1 < \nu$, we effect the resolution $(x/y,\ y,\ z/y)$, and we obtain

$$x = \frac{1}{c} t^{\nu-\nu_1} U_5,$$

$$y = c t^{\nu_1} U_3,$$

$$z = \frac{\gamma}{c} t^{\nu_2} U_6.$$

where $x=0$ and $y=0$ represent E_1 and E_2 respectively. If now ν_2 is large, the resolution may proceed to the end as a plane resolution of x against y: and this corresponds to the case in which the original branch had a plane-like singularity, unaffected by projection. In general, however, any one of the numbers $\nu-\nu_1$, ν_1, ν_2 may be least and the resolution will proceed by a ternary process of reduction of these numbers. In particular, if ν_2 is least, E_4 is the first of a series of proper satellites, following each other in a more or less arbitrary sequence of proximity types according to the variable order of magnitude of the three leading indices, and immediately succeeded by a semi-satellite or free point.

Examples (ii) b and (ii) c are simple examples of the general type discussed above.

In conclusion then, it is clear that the character of a space branch in general depends on possible relations between the coefficients in its analytical expression, the series of *indices* which occur in these expansions determining, in itself, only the character of the plane projection; and furthermore, when the character of the general plane projection is assigned, the space singularity appears to vary from a minimum, when the coefficients in its canonical analytical expression are perfectly general, to a maximum (as regards multiplicities) which is that of the assigned general projection, the passage from the one case to the other being characterised by the eventual replacement of all proper satellites by series of semi-satellites.

4. *Surfaces with assigned multiplicities at points of a sequence.* Let $E_1, E_2, \ldots, E_k$ be a succession of points (arranged in a normal order) on a three-fold V, such that their common ancestor E_1 is a simple point of V. On resolving all the points, they will be represented on a transformed manifold V' by combinations of multiples of k irreducible surfaces $e_1, \ldots, e_k$, the diminished neighbourhoods of $E_1, \ldots, E_k$.

We denote by $|F|$ the system of prime sections of V, or their equivalent transforms on V', and consider possible irreducible systems of surfaces of the form

$$|F^*| \equiv |nF - \sum_1^k \rho_\alpha E_\alpha|,$$

(for sufficiently large n) representing systems of irreducible surfaces in V' without singularities on e_i, or systems of irreducible surfaces in V whose only singularities, near the point E_1, are at the points E_i.

For $|F^*|$ to be irreducible it is evidently required that, for every i, the system of curves $|\theta_i|$, cut by $|F^*|$ on e_i, should exist effectively. (It may, of course, be a zero system, or consist, in whole or in part, of multiples of the

fixed intersection curves δ_{ij} of the surfaces e_i.) Now

$$\theta_i \equiv e_i(nF - \sum_1^k \rho_\alpha E_\alpha)$$

$$\equiv -e_i \sum_1^k \rho_\alpha E_\alpha$$

$$\cdot -(E_i - \overset{(i)}{\Sigma} E_\alpha) \Sigma \rho_\alpha E_\alpha,$$

where the summation $\overset{(i)}{\Sigma}$ extends to such points of the set as are proximate to E_i. A symbol of the form $E_\alpha E_\beta$ is equivalent to zero if $\alpha \neq \beta$ (as in the two-dimensional case); and a symbol such as E_α^2, a characteristic curve of E_α, represents the negative of a line of the plane, π_α say, obtained on first resolving E_α. Thus if we write $E_\alpha^2 = -\xi_\alpha$, we obtain

$$\theta_i = \rho_i \xi_i - \overset{(i)}{\Sigma} \rho_\alpha \xi_\alpha.$$

The surface e_i, on which these curves θ_i lie, is a birational transform of the plane π_i into which E_i was first resolved; and it is a transform, namely, on which there have been resolved into *curves* precisely those points of the set which lay on it, in other words, those points of the set which are proximate to E_i. Thus e_i contains in all a basic homaloidal net of curves ξ_i, and a finite set of (composite) exceptional curves ξ_α, representing the total superficial neighbourhoods (in π_i) of points E_α proximate to E_i. We note also, incidentally, that a surface e_α meets e_i in a curve $\delta_{i\alpha}$ only if one or other of E_i, E_α is proximate to the other. In fact

$$e_i e_\alpha \equiv (n_{i\lambda} E_\lambda)(n_{\alpha\lambda} E_\lambda),$$

where (n_{ij}) is the proximity matrix of the set; so that, if E_i is proximate to E_α,

$$\delta_{i\alpha} \equiv -E_i^2 + \overset{(i,\alpha)}{\Sigma} E_j^2,$$

the summation $\overset{(i,\alpha)}{\Sigma}$ extending to such points of the set as are proximate to E_i and E_α; while if E_α is proximate to E_i, then

$$\delta_{i\alpha} \equiv -E_\alpha^2 + \overset{(i,\alpha)}{\Sigma} E_j^2;$$

and interpreting these as curves *on* e_i, we have, in the former case,

$$\delta_{i\alpha} = \xi_i - \overset{(\alpha)}{\Sigma} \xi_j;$$

while in the latter

$$\delta_{i\alpha} \equiv \mathfrak{E}_\alpha - \overset{(\alpha)}{\Sigma} \mathfrak{E}_j \equiv \epsilon_\alpha,$$

where ϵ_α is the diminished *superficial* neighbourhood of the point E_α in π_i.

The curves θ_i, then, must be *effective* curves on e_i, and this means that they must consist of transforms of actual curves in π_i, together possibly with a sum of multiples of the diminished superficial neighbourhood ϵ_α of points E_α proximate to E_i. Hence

The restrictions (*analogous to the proximity inequalities in the plane*) *on the sequence of effective multiplicities*, $\rho_1, \ldots, \rho_k$, *that an irreducible surface can have at an assigned set of neighbouring points*, $E_1, \ldots, E_k$, *are that each* ρ_i *must be suffiiciently large to ensure the existence in* E_i, *regarded as resolved into a plane, of curves of order* ρ_i *which contain*, at least formally, *all points* E_α, *proximate to* E_i, *to their assigned multiplicities* ρ_α.

4.1. We may apply the preceding analysis, in particular, to the problem of finding the simplest type of algebroid surface branch through a given type of space branch.

For example, the cubical cusp

$$x = t^3, \quad y = a_4 t^4 + a_5 t^5 + \ldots, \quad z = a_5 t^5 + a_6 t^6 + \ldots$$

has resolution $[3^1, 1^1, 1^2(E_1), 1^\infty]$; so that in the plane π_1 (first resolution of E_1) there lie three linearly consecutive points, E_2, E_3, E_4, of the branch. The cubical cusp lies on a *linear* surface branch therefore if, and only if, E_2 E_3, E_4 lie on a line of π_1. The analytical condition for this, as is easy to verify, is $a_5 = 0$. A general cubical cusp imposes a conical double point on any surface required to contain it. *Such a branch is the simplest example of a space-branch, containing no proper satellites, which does not, in general, lie on a linear surface branch*†.

In a similar way we may verify that the *general* branch with resolution of the form $[\nu^1, 1^1, 1^{\nu-1}(E_1), 1^\infty]$ imposes a μ-fold point on containing surfaces, where μ is the least integer satisfying $\nu \leqslant \frac{1}{2}\mu(\mu+3)$.

† The point E_4, in this case, when collinear with E_2, E_3, is apparently the type of semi-satellite which Enriques prefers to regard as a satellite, calling it a *plane-satellite*, because it is uniquely determined by E_2 and E_3. But since its distinguishing property is not a proximity relation we prefer to regard it as a (special) semi-satellite. (*Cf.* Enriques 1, 562.)

Zariski's branch $[4^1, 2^1, 1^1(E_1), 1^1(E_1, E_2), 1^\infty]$ imposes an ordinary binode B_3 on containing surfaces, the identity corresponding to this fact being

$$\begin{aligned}\vartheta_1 &\equiv 2\mathfrak{E}_1-\mathfrak{E}_2-\mathfrak{E}_3-\mathfrak{E}_4\\ &= 2(\mathfrak{E}_1-\mathfrak{E}_2)+\mathfrak{E}_2-\mathfrak{E}_3-\mathfrak{E}_4\\ &= 2(\mathfrak{E}_1-\mathfrak{E}_2)+\epsilon_2.\end{aligned}$$

Similarly the singular branch iii(a) (§ 3), if we write $\nu = k\nu_1+\nu_2$, where $0 \leqslant \nu_2 < \nu_1$, imposes a k or a $(k+1)$-fold point on containing surfaces, according as $\nu_2 = 0$ or $\nu_2 > 0$, the nodal cone in either case consisting entirely of planes through the tangent to the branch†.

5. *The general two-dimensional resolution of a curve-branch.* The fundamental property of a *branch-form* γ, the name that we may give to a branch which is primitive at the last point E_k of an assigned sequence $E_1, \ldots, E_k$, but subsequently unrestricted, is that it has a definite intersection multiplicity at its origin with any surface F which passes through this origin, namely $\sum_1^k \rho_i r_i$, where ρ_i is the multiplicity of γ at E_i, and r_i is the multiplicity of F at E_i, as calculated by successive resolution. We may, however, envisage a second general type of "resolution" of γ, namely that which shall determine directly for us its intersection multiplicity with an arbitrary singular curvilinear surface branch which passes through its origin.

There is, besides the point-resolution employed so far, a second fundamental type of resolution of a three-fold V which does not introduce new singularities on the transformed manifold V', namely the *curvilinear resolution*, say of a simple curve C of V, by primals of a certain order through C; and it is by means of a sequence of curvilinear resolutions that one can define a succession of neighbouring curves on V, and, eventually, *superficial curvilinear branch-forms* (linear and singular). In a sequence of such resolutions, the neighbourhood of the original curve C is transformed into a series of ruled surfaces (the transform of a ruled surface being still regarded as ruled) counted to appropriate multiplicities, and the individual points of C are transformed into chains of generators of these surfaces‡.

† The relevant identity, if $\nu_2 = 0$, is

$$k\mathfrak{E}_1-\sum_2^{\nu+\nu_1+2}\mathfrak{E}_i = k(\mathfrak{E}_1-\mathfrak{E}_2)+(k-1)\epsilon_2+\sum_1^{\nu_1} i(k-2)\epsilon_{2+i}+\sum_1^{\nu-2\nu_1-1}\{\nu_1(k-2)-i\}\epsilon_{\nu_1+2+i},$$

while, if $\nu_2 > 0$, it is

$$(k+1)\mathfrak{E}_1-\sum_2^{\nu+\nu_1+2}\mathfrak{E}_i \equiv (k+1)(\mathfrak{E}_1-\mathfrak{E}_2)+k\epsilon_2+\sum_1^{\nu_1} i(k-1)\epsilon_{2+i}+\sum_1^{\nu-3\nu_1}\{\nu_1(k-1)-i\}\epsilon_{\nu_1+2+i}.$$

‡ *Cf.* Semple (1), 55–59.

The resolution of a curve C, however, destroys completely the character of a sequence of points $E_1, \ldots, E_k$, originating on C, as such, the set of branches which are primitive at E_i, for example, being scattered, on the transformed manifold, into a complicated system depending on relations of the original individual branches to the sequence of points following E_1 on C. Thus the resolution of C destroys the hierarchy of points (as such) successive to any point of C, and it creates in V' new points and new hierarchies not corresponding to anything analogous on V.

We envisage the general curvilinear resolution of a branch γ as follows. We choose in V a simple curve C through E_1 having no essential relation to γ (its tangent at E_1, in particular, being different from that of γ), and we resolve C into a ruled surface R. the branch γ being thereby transformed into a branch γ_1 having origin O_1 on R. We then choose on R (and successive therefore to C) a curve C_1, passing through O_1, but otherwise generally related to γ_1, and resolve it in turn, obtaining γ_2 in place of γ_1; and so on. The process comes to an end, as far as the singularity of the branch is concerned, when we arrive at a branch γ_2 which is linear and does not touch any of the surfaces representing whole or partial neighbourhoods of the resolved curves C_i. It will be noted, however, that the resolution of the general plane projection $\overline{\gamma}$ of γ can be regarded as a particular (but permissible) form of the process outlined above, the curves of space which are successively resolved in this case being those lines which join the general vertex of projection to successive points of $\overline{\gamma}$. *Thus the general curvilinear resolution of γ includes the resolution of the projection of γ as a particular case; and its significance is relative to curvilinear surface branches drawn from any curve C passing through the origin of γ but otherwise essentially unrelated to γ.*

The proximity character of a sequence of curves in a threefold will evidently be *two-dimensional*, a curve C_i being proximate to at most two preceeding curves of the sequence. We may thus refer to the type of resolution described above as the two-dimensional resolution of a branch as distinct from its ordinary, or three-dimensional, resolution.

The essential problem, evidently, in relating the two-dimensional to the three-dimensional resolutions, is that proposed by Beppo Levi† as long ago as 1899, namely:

Given a surface F with any arbitrary curvilinear singularity along a curve C, to find the multiplicities of F at the points of any consecutive succession (itinerario), originating in a point of C.

Beppo Levi was more particularly interested in sequences *initially* along C; but our concern primarily is with sequences initially transverse to C,

† Beppo Levi, (1), § 1.

but possibly containing proper satellites at which the behaviour of F is bound to be peculiar.

6. *The general curvilinear surface singularity.* We consider a single singular curvilinear branch† of a surface F, whose origin is C_1; and we denote the sequence of curves successive to C_1 on F by C_2, C_3 .... The *type* of singularity involved, formally similar to the type possessed by a general plane section of F at one of its intersections with C_1, we denote by S_1; and the various derivatives of S_1 (first, second, third, ..., resolutions) we denote by S_2, S_3, S_4, ..., these being all, like S_1, two-dimensional singularity types.

Suppose that we now resolve a general point E_1 of C_1 into a plane e_1. The transform F' of F has (in addition to its transformed curvilinear singularity, which is still of type S_1) a singular line $l_2^{(2)}$ in e_1 corresponding to the singular neighbourhood of E_1 on F. This line-singularity of F' is actually of type S_2, as is easily seen, for example, by considering its section by any linear surface branch which is the transform of a plane through E_1. Thus on F' there is a succession of lines $l_2^{(2)}$, $l_3^{(2)}$, ..., with the same proximity matrix as the sequence C_2, C_3, ...; and as many of these lines lie *in* e_1 as correspond to curves C_i proximate to C_1.

We may now resolve a *general* point E_2 of $l_2^{(2)}$ into a plane e_2 meeting the transform e_1' of e_1 in a line λ. The singular neighbourhood of E_2 on F' gives rise to a singular line $l_3^{(3)}$, of type S_3, of F'', and this singular line lies along λ if C_3 is proximate to C_1 (since in that case $l_3^{(2)}$ must have lain in e_1).

Again choosing E_3 to be a *general* point of $l_3^{(3)}$ and resolving it, we get a singular line $l_4^{(4)}$, of type S_4, of F'''; and so on. We define in this way what we may call *the general transverse path across a general curvilinear sequence* (much more general than a section by a linear surface branch), originating in a point E_1 of C_1, and proceeding regularly outwards in the most general manner possible at each step. Such a path has evidently the same proximity character as the curvilinear sequence itself, and its points can fairly be regarded as *lying on* and corresponding uniquely with the curves of the sequence: and conversely the curves of the sequence can be fairly regarded as *drawn separately through* the points of the path. The general transverse path as above defined contains no proper satellites; but it is evident, on the other hand, from the generality of the construc-

† It is, of course, understood that we confine ourselves to consideration of a finite stretch of the multiple curve of F. The possibility that F may have α sheets along C, forming an *irreducible* first neighbourhood, in no way precludes us from isolating one sheet for consideration at a particular point of C.

tion, that it is a quite general space sequence of the type which has a proximity matrix of a two-dimensional kind. In other words, any space sequence of the latter type can be regarded as a regular path across a congruent curvilinear sequence; *i.e.* we can draw curves of such a sequence *separately* through the points of the point sequence. Thus

If an algebroid curve branch has a two-dimensional proximity character (containing no proper satellites), then its general two-dimensional resolution (equivalent to that of its projection) is exactly parallel to its space resolution.

Thus difficulty arises only in the case of space sequences containing proper satellites; and the difficulty arises because any attempt to draw successive curves through the points of such a sequence breaks down owing to the fact that some of the curves pass automatically through points of the sequence successive to those through which they are drawn.

We proceed now to investigate the manner in which this happens in certain important particular cases.

7. *Certain sequences of multiple points on a curvilinear surface singularity.*

In considering the geometry of a curvilinear surface branch at a general point E_1 of its origin C_1, we may evidently, without loss of generality, replace the singularity in the neighbourhood of E_1 by a *cylindrical* surface branch whose origin is the tangent line to C_1 at E_1. We consider then the general cylindrical branch

$$y = c_1 x + c_2 x^{(\mu+\mu_1)/\mu} + c_3 x^{(\mu+\mu_1+\mu_2)/\mu} + \dots$$

as being a typical curvilinear singularity in the neighbourhood of any particular point of OZ; and, taking the origin O as initial point E_1 of a space sequence (E_i), we investigate the multiplicity of the branch at E_i.

We need consider only cylindrical branches which are *immediately cuspidal*, *i.e.*, such that, if L_1 is the origin OZ of the branch, and if L_2, L_3 are its first two successors, then L_3 is already proximate to L_1; and we limit ourselves here to a consideration of branches which have only one (uninterrupted) collection of satellite lines, *i.e.* essentially to branches of the form

$$y = ax + bx^{(\mu+\mu_1)/\mu} \quad (\mu_1 < \mu,\ a \neq 0,\ b \neq 0),$$

which form the first class of approximations to immediately cuspidal cylindrical branches in general.

We denote the surface with the equation just written down by K, and we indicate the resolution of the singular line of K into a series of multiple

and simple lines by the symbol

$$[\mu^1, \quad \mu_1^{k_1}, \quad \mu_2^{k_2}, \quad \ldots, \quad \mu_{\lambda-1}^{k_{\lambda-1}}, \quad 1^{k_\lambda}, \quad 1^\infty],$$

where the numbers k_i and μ_i, as usual, are the quotients and remainders occurring in the reduction of $\mu_1 : \mu$ to a continued fraction (μ_1 being supposed prime to μ), and the term 1^{k_λ} indicates the terminating cluster of (simple) satellite lines immediately preceding the infinite sequence of free lines.

We consider now the intersections of K with any space branch γ of the form

$$x = t^\nu,$$

$$y = at^\nu + a_1 t^{\nu+\nu_1} + \sum_1^\infty a_{1+i} t^{\nu+\nu_1+i},$$

$$z = at^\nu + a_1 t^{\nu+\nu_1} + \sum_1^\infty a_{1+i} t^{\nu+\nu_1+i},$$

where $\nu_1 < \nu$, $a_1 \neq 0$, and $a_1 a_2 - a_2 a_1 \neq 0$. This branch, as we have already seen (§ 3, iii), has a resolution of the form

$$[\nu^1, \quad \nu_1{}^1, \quad 1^1(E_1), \quad 1^{\nu_1-1}(E_1, E_2), \quad 1^{\nu-2\nu_1}(E_1), \quad 1^\infty],$$

if $\nu \geqslant 2\nu_1$, or of the form

$$[\nu^1, \quad \nu_1{}^1, \quad 1^1(E_1), \quad 1^{\nu-\nu_1-1}(E_1, E_2), \quad 1^{2\nu_1-\nu}(E_2), \quad 1^\infty],$$

if $\nu \leqslant 2\nu_1$; and it touches K at the origin (and is also immediately cuspidal). We denote the initial semi-satellite of γ, immediately following E_1, E_2, by E_3, the subsequent group of proper satellites by S, and the final group of semi-satellites by S'.

The equation of K may be written in the form

$$(y - ax)^\mu = b^\mu x^{\mu+\mu_1},$$

and the multiplicity I of its intersection with γ is evidently the smaller of the two numbers $\mu(\nu+\nu_1)$ and $\nu(\mu+\mu_1)$, or say $\mathfrak{L}(\mu\nu+\mu\nu_1, \mu\nu+\mu_1\nu)$. Thus

$$I = \mu\nu + \mathfrak{L}(\mu\nu_1, \nu\mu_1).$$

It is quite clear that K has multiplicity μ at E_1 and multiplicity μ_1 at E_2 (since $a \neq 0$); and in virtue of its singularity it may pass automatically, with certain multiplicities, through some or all of the points E_3, S, S'; but it certainly does not pass *automatically* through any subsequent *free* point of γ, seeing that no irreducible surface can pass through the

whole double infinity of points in the first neighbourhood of any point. *Thus the whole intersection of K with γ takes place, in general, at the points E_1, E_2, E_3, and at some or all of the points of S and S'.*

7.1. Suppose first that $\nu = 2\nu_1$, so that S' is empty. Then

$$I = \mu\nu + \mathfrak{L}(\mu\nu_1,\ 2\mu_1\nu_1),$$

and, since $\mu = k_1\mu_1 + \mu_2$, we see that, if $k_1 \geqslant 2$,

$$I = \mu\nu + \mu_1\nu_1 + \mu_1\nu_1;$$

while, if $k_1 = 1$,

$$I = \mu\nu + \mu_1\nu_1 + \mu_2\nu_1.$$

By putting $\nu_1 = 1, 2, 3, \ldots$ in succession, we find that E_3 and every point of S is μ_1-fold on K if $\mu \geqslant 2\mu_1$, and μ_2-fold on K if $\mu < 2\mu_1$. Thus

If the initial curves C, C', C'' of a curvilinear surface-branch f have multiplicities μ, μ', μ'', where $\mu' < \mu$ and $\mu'' \leqslant \mu'$, and if E_1 is any point of C, and E_2 any neighbouring point of f, not on C, then f passes with multiplicity μ'' through every successor to E_2 which is proximate both to E_1 and E_2.

7.2. Suppose that $\mu \geqslant 2\mu_1$. We denote then by I' that part of the intersection-multiplicity of K and γ which arises (in the general case) only from the semi-satellite group S'. Thus, if $\nu < 2\nu_1$,

$$\begin{aligned} I' &= I - \mu\nu - \mu_1\nu_1 - \mu_1(\nu - \nu_1) \\ &= \mathfrak{L}(\mu\nu_1 - \mu_1\nu,\ 0) \\ &= 0; \end{aligned}$$

so that, if $\mu \geqslant 2\mu_1$, K does not pass through any of the points S' when these are proximate to E_2. On the other hand, if $\nu \geqslant 2\nu_1$,

$$\begin{aligned} I' &= I - \mu\nu - \mu_1\nu_1 - \mu_1\nu_1 \\ &= \mathfrak{L}\{\nu_1(\mu - 2\mu_1),\quad \mu_1(\nu - 2\nu_1)\}. \end{aligned}$$

Here we may regard ν_1 as fixed, and ν as increasing from the starting point $2\nu_1$, so that the number $\nu - 2\nu_1$ of points in S' increases from zero. The same argument as before shows then that, following the last point of S, there is a number σ of points S' (proximate to E_1) which are μ_1-fold for K, and one further point of S' which is τ-fold for K, where σ and τ are quotient and remainder when $\nu_1(\mu - 2\mu_1)$ is divided by μ_1.

7.3. Suppose next that $\mu < 2\mu_1$. Then $\mu = \mu_1 + \mu_2$; and, if $\nu < 2\nu_1$, so that $\nu = \nu_1 + \nu_2$, we find

$$\begin{aligned} I' &= I - \mu\nu - \mu_1\nu_1 - \mu_2(\nu - \nu_1) \\ &= \mathcal{L}\{\mu_2(\nu_1 - \nu_2),\ \nu_2(\mu_1 - \mu_2)\}. \end{aligned}$$

We can now regard ν_2 (the number of points in S increased by unity) as fixed, and we may put $\nu_1 = \nu_2 + 1$, $\nu_2 + 2$, ... in turn. We see then that K has a number σ' of points S' (proximate to E_2) as μ_2-fold points, and one further point S' as τ'-fold point, where σ' and τ' are quotient and remainder when $\nu_2(\mu_1 - \mu_2)$ is divided by μ_2.

On the other hand, if $\nu \geqslant 2\nu_1$,

$$\begin{aligned} I' &= I - \mu\nu - \mu_1\nu_1 - \mu_2\nu_1 \\ &= \mathcal{L}\{0,\ \nu\mu_1 - \mu\nu_1\} \\ &= 0, \end{aligned}$$

so that, if $\mu < 2\mu_1$, K does not contain points S' proximate to E_1.

In conclusion then:

If the three initial curves C, C', C'' of a curvilinear surface branch f have multiplicities μ, μ', μ'', where $\mu' < \mu$, and $\mu'' \leqslant \mu'$, and if $E_{\lambda+2}$ is the last of any general consecutive sequence of λ points, proximate to a point E_1 of C and to a neighbouring point E_2 of f not on C, then, of any consecutive sequence of points following $E_{\lambda+2}$ and all proximate to E_1, f contains none unless $\mu'' = \mu'$, and then a number σ to multiplicity μ', and one more to multiplicity τ, where σ and τ are quotient and remainder when $\lambda(\mu - 2\mu')$ is divided by μ'; while of any similar sequence of points all proximate to E_2, f contains none unless $\mu'' < \mu'$, and then a number σ' to multiplicity μ'', and one more to multiplicity τ', where σ' and τ' are quotient and remainder when $\lambda(\mu' - \mu'')$ is divided by μ''.

8. *Projection of space-branches.* Results of the above kind enable us to understand better the relation between the singularity of a space branch γ and that of its projection γ', when these differ formally from each other.

It is certainly not the case, as has been surmised, that every multiple point of γ' can be regarded as the projection either of one definite point, of equal multiplicity, of γ, or of a definite set of points of γ whose multiplicity sum is equal to the multiplicity of the point of γ' in question.

To fix the ideas consider, for example, the space branch γ whose space resolution is

$$[\nu^1,\quad \nu^1{}_1,\quad 1^1(E_1),\quad 1^{\nu_1-1}(E_1,\ E_2),\quad 1^{\nu-2\nu_1}(E_1),\quad 1^\infty];$$

and suppose that ν_1 is prime to ν, so that the general projection γ' (keeping to our usual notation) has resolution

$$[\nu^1, \quad \nu_1^{k_1}, \quad \nu_2^{k_2}, \quad \ldots, \quad \nu_{g-1}^{k_{g-1}}, \quad 1^{k_g}, \quad 1^\infty].$$

We may suppose that γ' is the orthogonal projection of γ in the plane $z = 0$; and we note that any vertical cylinder must have the same intersection multiplicity with γ as with γ' at the origin.

The general (vertical) approximative vertical cylinder to γ', which passes freely through any one of its satellite points, has resolution of the form

$$[\mu^1, \quad \mu_1^{k_1}, \quad \mu_2^{k_2}, \quad \ldots, \quad \mu_{\lambda-1}^{k_{\lambda-1}}, \quad 1^\theta, \quad 1^\infty],$$

where $\lambda \leqslant g$, $2 \leqslant \theta \leqslant k_\lambda + 1$ (excluding $\lambda = g$, $\theta = k_g + 1$); and the total intersection multiplicity of this cylinder with γ' at satellite points of the latter must be exactly balanced by an equal total intersection multiplicity of the cylinder with γ at its semi-satellite and satellite points, which are all *simple* on γ. The theorems of the last section enable us to see, in the present instance, exactly how the cylinders effect this balance, namely by passing, shuttlewise, through groups of the semi-satellite and satellite points of γ with gradually increasing multiplicity, until, in the case of those passing freely through the last satellite point of γ', they attain the multiplicity ν_1 at every one of the semi-satellite and satellite points of γ. It is, perhaps, unnecessary to enter into further detail; and we only note, therefore, the identity corresponding to this last remark, namely,

$$\nu^2 + k_1 \nu_1^2 + k_2 \nu_2^2 + \ldots + k_g \equiv \nu^2 + \nu_1^2 + \nu_1 + (\nu_1 - 1)\nu_1 + (\nu - 2\nu_1)\nu_1,$$

which reduces to the well-known identity

$$\Sigma k_i \nu_i^2 = \nu\nu_1,$$

and remark that there is a similar identity for each lower type of approximative cylinder.

9. *Effect of a single curvilinear resolution.* Let C be a simple curve on a three-fold V, and let E_1 be any point of C and E_2 the neighbouring point to E_1 on C. The resolution of C into a ruled surface R on a transformed manifold V' may be effected in two stages, by first resolving E_1 into a plane π on a transform V_1 of V, and by then projecting V_1 into V' from the curve C_1 which is the transform of C on V_1. In this projection the plane π is projected into a generator r of R from the point E_2' (corresponding to E_2), in which it is met by C_1. *Thus in the resolution of C, the neighbourhood of any point E_1 of C may be regarded, in a manner, as being*

projected into the neighbourhood of the corresponding generator r of R from the point E_2 consecutive to E_1 on C.

To all directions in a plane through the tangent, t say, to C at E_1, there correspond the same point of r: and, more precisely, to the individual directions in such a plane there correspond sections of the neighbourhood of the corresponding point of r by planes through r (lying in the tangent solid to V' at the point in question). Thus the individual directions (in V) through E_1 are associated with the ∞^2 tangent planes to V' at points of r. By projecting V' from r, for example, it is easy to see that all the tangent planes to V' along r which lie in a prime through r correspond to the set of directions, through E_1, which lie on a quadric cone through t. Thus the tangent planes of V' along r form a cubic-scroll line-cone, whose vertex is r, whose "generators" are the tangent solids to V' at points of r, and whose directrix is the solid of tangent planes to R along r. Hence

When any simple curve C of a three-fold V is resolved into a ruled surface R, the tangent solids to the transformed manifold V' at points of a generator r of R generate a cubic-scroll line-cone, whose ∞^2 planes are associated with the ∞^2 directions in V through a certain point of C.

9.1. We are concerned, in this section, more particularly with the effect on a general space branch γ of a single curvilinear resolution of the containing space; and for this purpose we take the equations of the branch in their most general form, and the resolution the simplest possible in which there is a simple base line, namely

$$\begin{pmatrix} x & y & z \\ x & y/x & z \end{pmatrix}.$$

The use of this very special line-resolution of space is only justified if the branch is assumed to have no special relation, of an invariantively significant character, to the line OZ which is being resolved.

We may write the equations of a general algebroid expansion in t, with non-vanishing constant term, in the canonical form†

$$\begin{aligned} R(t) = a_{10} &+ \sum_1^{k_1} a_{1i} t^{i\nu_1} + b_1 t^{k_1\nu_1 + n_{11}\nu_2} \\ &+ \sum_1^{k_2} a_{2i} t^{k_1\nu_1 + (n_{11}+i)\nu_2} + b_2 t^{k_1\nu_1 + (n_{11}+k_2)\nu_2 + n_{21}\nu_3} \\ &+ \ldots, \end{aligned}$$

† *Cf.* Zariski (1), 7.

where

$$\nu_1 = n_1 \nu_2, \quad 0 < n_{11} < n_1, \quad \text{and} \quad n_{11} \text{ is prime to } n_1,$$
$$\nu_2 = n_2 \nu_3, \quad 0 < n_{21} < n_2, \quad \text{and} \quad n_{21} \text{ is prime to } n_2,$$
$$\text{etc.},$$

and neither a_{10} nor any b_i is supposed to vanish. These latter conditions define a type of canonical expansion, which we may denote by

$$R \equiv \left[\nu_1 \binom{1}{k_1}, \quad \nu_2 \binom{n_{11}}{n_{11}+k_2}, \quad \nu_3 \binom{n_{21}}{n_{21}+k_3}, \quad \ldots, \quad \nu_g \binom{n_{g-1,1}}{n_{g-1,1}+k_g}, \quad 1(\infty) \right],$$

and we note that any positive or negative power of an expansion of this type is of the same type, as is also, in general, the product of any two such expansions.

We may write the equations of any space branch γ, then, in the canonical form

$$x = t^{\nu_1},$$
$$y = \sum_1^{k_1} a_{1,i} t^{i\nu_1} + b_1 t^{k_1\nu_1 + n_{11}\nu_2} S_1,$$
$$z = \sum_1^{k_1} \alpha_{1,i} t^{i\nu_1} + \beta_1 t^{k_1\nu_1 + n_{11}\nu_2} S_2,$$

where $b_1 \neq 0$, $\beta_1 \neq 0$, and S_1, S_2 are of type S, indicated by

$$S \equiv \left[\nu_2 \binom{1}{k_2}, \quad \nu_3 \binom{n_{21}}{n_{21}+k_3}, \quad \ldots, \quad \nu_g \binom{n_{g-1,1}}{n_{g-1,1}+k_g}, \quad 1(\infty) \right].$$

On applying the line-resolution $(x, y/x, z)$ to γ, we obtain, after a change of origin, a branch γ' of the form

$$x = t^{\nu_1},$$
$$y = a_{12} t^{\nu_1} + \ldots + a_{1,k_1} t^{(k_1-1)\nu_1} + b_1 t^{(k_1-1)\nu_1 + n_{11}\nu_2} S_1,$$
$$z = \alpha_{11} t^{\nu_1} + \ldots + \alpha_{1,k_1-1} t^{(k_1-1)\nu_1} + \alpha_{1,k_1} t^{k_1\nu_1} + \beta_1 t^{k_1\nu_1 + n_{11}\nu_2} S_2,$$

in which the characteristic indices in z are all greater by ν_1 than those in y. We wish to discuss the proximity character of the sequence of points on γ'.

A set of $k_1 - 1$ ordinary resolutions, $(x, y/x, z/x)$, with changes of origin, reduces γ' to the branch given by

$$x = t^{\nu_1},$$
$$y = b_1 t^{n_{11}\nu_2} S_1,$$
$$z = \alpha_{1,k_1} t^{\nu_1} + \beta_1 t^{\nu_1 + n_{11}\nu_2} S_2,$$

which, when we replace $z - a_{1,k_1}x$ by z, takes the form

$$x = t^{\nu_1},$$

$$y = b_1 t^{n_{11}\nu_2} S_1,$$

$$z = \beta_1 t^{\nu_1 + n_{11}\nu_2} S_2.$$

From this point the resolution proceeds for some time as between x and y, without formal intervention from z, *i.e.*, as a plane resolution. We write $n_1 = k_{11}n_{11} + n_{12}$, $n_{11} = k_{12}n_{12} + n_{13}$, ..., $n_{1,q_1-2} = k_{1,q_1-1}n_{1,q_1-1} + 1$, these being the equations incidental to the expression of $n_{11} : n_1$ as a continued fraction.

Then after k_{11} resolutions of the type $(x/y, y, z/y)$ we obtain

$$x = \text{const.}\, t^{n_{12}\nu_1} S_{(1)},$$

$$y = \text{const.}\, t^{n_{11}\nu_2} S_{(2)},$$

$$z = \text{const.}\, t^{(n_{11}+n_{12})\nu_2} S',$$

where here, and in the sequel, we use the symbols $S_{(1)}$, $S_{(2)}$ (varying from resolution to resolution) to denote a pair of expansions of type S, while we use S' similarly to denote an expansion of the form $S_2 S_1^{-N}$, which is usually, but not always, of type S.

After k_{12} further resolutions of type $(x, y/x, z/x)$, we obtain

$$x = \text{const.}\, t^{n_{12}\nu_2} S_{(1)},$$

$$y = \text{const.}\, t^{n_{13}\nu_2} S_{(2)},$$

$$z = \text{const.}\, t^{(n_{12}+n_{13})\nu_2} S',$$

and so continuing.

Finally, we obtain the branch

$$x = \text{const.}\, t^{\nu_2} S_{(1)},$$

$$y = \text{const.}\, t^{n_{1,q_1-1}\nu_2} S_{(2)},$$

$$z = \text{const.}\, t^{(n_{1,q_1-1}+1)\nu_2} S',$$

and, on resolving this $n_{1,q_1-1} - 1$ times, we obtain

$$x = \text{const.}\, t^{\nu_2} S_{(1)},$$

$$y = \text{const.}\, t^{\nu_2} S_{(2)},$$

$$z = \text{const.}\, t^{2\nu_2} S',$$

We may change the parameter by the substitution

$$\tau^{\nu_2} = t^{\nu_2} S_{(1)},$$

so that $t = \tau S^*(\tau)$, where S^* is also of type S; and this enables us to write the equations of the branch last obtained in the form

$$x = \text{const.}\, \tau^{\nu_2},$$

$$y = \text{const.}\, \tau^{\nu_2} S(\tau),$$

$$z = \text{const.}\, \tau^{2\nu_2} S'(\tau).$$

Now up to the present the resolution has certainly been two-dimensional as regards its proximity character, having been, after the initial free points, a resolution of x against y, without formal intervention by z. And this last branch has essentially the same kind of peculiarity as the branch γ' from which we started. The possible absence (through multiplication) of certain characteristic terms in S' can only aid, rather than hinder, the continued (formal) non-intervention of z. Thus z, as before, will continue to affect only the positions of the various stretches of *free* points in the resolution of γ' but not its formal character. Hence

If any space branch γ is transformed by a curvilinear resolution of space, whose origin C passes through the origin of γ but is otherwise not specially related to γ, then the transformed branch γ' has a two-dimensional proximity character.

9.2. We may note, in conclusion, the possibility of wide variation in the character of γ' which can arise when there is any *essential* (invariantively significant) special relation of the curve C which is being resolved to γ. For example, if γ is an ordinary cusp (of order 2) whose origin is O, then the resolution of a general line of space through O resolves γ into a linear branch; but the resolution of any line through O which lies in the osculating plane of γ at O transforms γ into a branch which is still cuspidal.

In the case of the resolution $(x, y/x, z)$, the type of branch (at the origin) which resolves into the most general type of singular branch is evidently that represented by equations of the form

$$x = t^{\nu_1}, \quad y = t^{2\nu_1} R_1(t), \quad z = t^{\nu_1} R_2(t),$$

where R_1, R_2 are expansions of the same type (with non-zero constant terms). It will be noted, however, that the line OZ (or OX, but not OY) has a special relation to such a branch, in that the projection of the branch

from any point of it (other than O) has an extra ν_1-fold point, as compared with the projection from an arbitrary point.

Thus when a curve C is resolved into a ruled surface R it appears that the general singular branch, with a characteristically three-dimensional proximity matrix, originating in a point of R, arises as a *special* curvilinear resolution of a branch which has only a two-dimensional proximity matrix, but which is specially related to C.

List of books and papers referred to.

Enriques-Chisini (1). *Lezioni sulla teoria geometrica delle equazioni*, 2 (1918), 551–577. This contains Enriques' work on satellite points, originally published in *Rend. Acc. Lincei* (1917).

Zariski (1). *Algebraic surfaces* (Berlin, 1935).

Du Val (1). "Reducible exceptional curves," *American J. of Math.*, 58 (1936), 285.

Beppo-Levi (1). "Intorno alla composizione dei punti generici delle linee singolari delle superficie algebriche", *Annali di Mat.* (3), 2 (1899), 127–138.

Semple (1). "Contact conditions for surfaces", *Proc. Royal Irish Acad.*, 43 (1936), 49–71.

King's College,
University of London.

ANALYTIC QUASI-PERIODIC CURVES OF DISCONTINUOUS TYPE ON A TORUS

By T. M. CHERRY.

[Received 22 April, 1937.—Read 22 April, 1937.]

1. *Introduction and summary.*

1.1. This paper is concerned with topological properties of a family of curves on a regular surface of genus 1 (torus, anchor ring). If x, y are angular coordinates on the torus, of periods 1, c say, such a family may be specified as the trajectories (solution-curves, characteristics) of differential equations

$$\frac{dx}{dt} = X(x, y), \qquad \frac{dy}{dt} = Y(x, y), \tag{1}$$

in which X, Y are periodic in x (period 1) and y (period c), and are assumed to be analytic or to satisfy some less stringent condition of regularity.

Attention has in the past been directed chiefly to the case in which the family is without singularity, the condition for this being that X, Y are regular and never simultaneously zero for real x, y. Here it has been shown by Poincaré, **9**† and Levi, **7** that the nature of the family is primarily dependent on the value of a certain constant α, hereafter called the *type-number*, which is, roughly speaking, the ratio of the mean rates of increase of x, y/c along any trajectory; the same ratio being derived from all the trajectories. If α is rational, one or more of the trajectories is periodic, and conversely. If α is irrational, some or all of the trajectories are *quasi-periodic* or *pervasive*‡, *i.e.* when followed in either sense they cut any meridian of the torus in a set of points $\mathfrak{E}$ which is dense in itself.

† References will be found at the end of the paper.

‡ In a recent paper (Cherry, **3**) I have described such curves as *pervasive*, a word that is less cumbersome than *quasi-periodic*, *stable à la Poisson*, *quasi-recurrent*, which have been used by various authors in the same sense.

In elementary cases the sets $\mathfrak{E}$ are everywhere dense on the meridian, and the pervasive trajectories are described as being of *continuous type*; *e.g.* this is so for the trajectories defined by

$$\frac{dx}{dt} = a, \quad \frac{dy}{dt} = bc,$$

where a, b are constants whose ratio is irrational. It is, however, a topological possibility that the sets $\mathfrak{E}$ (of which no two points are coincident) should be nowhere dense on the meridian, in which case the trajectories are described as of *discontinuous type* (Poincaré, **9**, Birkhoff, **1**); but it has been proved recently by Denjoy, **4** that this case cannot occur when X, Y are analytic, and that every trajectory must then be everywhere dense on the surface.

This paper is concerned with the case in which the family of trajectories has real singular points, at which X, Y simultaneously vanish. Since one at least of these points is in general a node or focus, no trajectory can be everywhere dense on the torus, and the possibility is apparent that there may exist recurrent trajectories of discontinuous type. My main object is to show how this possibility may be simply realized.

Two conditions are necessary for this. The first is that there must exist trajectories which spiral indefinitely round the torus without encountering a singular point, on which one or both of X, Y is unbounded. When this is so, a type-number α can be defined, of analogous significance to that in the non-singular case. The second condition then is that α is irrational. These two conditions are, I think, to be described as "primitive" relative to their level of generality, in the sense that there are no equivalent conditions of equal generality which are in any specific case more easily to be verified. We can hope only to give special criteria, of less generality, for their satisfaction. I have not attempted to give more than one such criterion: I have taken a special form for X, Y for which the discussion can be carried through completely to the desired conclusion: *that pervasive trajectories of discontinuous type on a torus can be specified by differential equations* (1) *in which* X, Y *are analytic.* The analogue of Denjoy's result for the non-singular case is that *the pervasive trajectories are non-uniformly pervasive.*

At a certain stage of the investigation it becomes essential to regard X, Y as depending on a parameter λ; the type-number α depends on λ, and it is by proper choice of λ that we secure the irrationality of α. Such a parameter arises most naturally as a constant of integration, the system (1) being then the result of reducing a system of the third order by means of a known integral which represents a family of tori. By taking the

variables u, v, w of this third order system

$$\frac{du}{dt} = U(u, v, w), \qquad \frac{dv}{dt} = V(u, v, w), \qquad \frac{dw}{dt} = W(u, v, w) \tag{2}$$

to be rectangular coordinates in space, and by using the fact that a family of tori can be specified by a rational equation in u, v, w, we prove that *pervasive trajectories of discontinuous type on a torus can be specified by differential equations* (2) *in which* U, V, W *are polynomials.*

The paper concludes with a specific example of such equations.

1.2. *Plane representation.* The angular coordinates x, y are multiple-valued functions of position on the torus. To avoid the inconvenience which may thereby be caused we take a plane in which x, y are rectangular cartesian coordinates. To any point (x, y) on the torus correspond the ∞^2 *congruent* points $(x+m, y+nc)$ in the plane, where m, n are integers; and to any curve on the torus correspond ∞^2 *congruent* curves in the plane, which may exceptionally reduce to ∞^1 sets each consisting of ∞^1 coincident curves. From a fact proved for the plane the corresponding fact for the torus follows by regarding congruent points as coincident. We use the plane representation almost exclusively in §§ 2-5.

If x, y are properly chosen curvilinear coordinates on the torus the plane representation is conformal. This fact is useful in the example of § 8.

2. *The differential equations.*

2.1. *The guide-curves. Let* $z(x, y)$ *be an analytic function of* x, y, *real, regular and single-valued for all real* x, y, *and possessing the pseudo-periodic properties*

$$\left.\begin{aligned} z(x+1, y) &= z(x, y)+1, \\ z(x, y+c) &= z(x, y). \end{aligned}\right\} \tag{3}$$

Let $\partial z/\partial x > 0$ *when* $y = (n+\frac{1}{2})c$, n *integral.*

At a later stage we regard z as depending also on a parameter λ, but this is not at present relevant.

The curves

$$z(x, y) = \text{constant}$$

in the xy-plane or on the torus will be called (*complete*) *guide-curves*; their infinite branches (see below) will be called *simple guide-curves.*

On account of (3), the guide-curves form a family built up of congruent repetitions of the unit within the fundamental rectangle

$$|x| \leqslant \tfrac{1}{2}, \quad |y| \leqslant \tfrac{1}{2}c.$$

From its regularity z is bounded in the fundamental rectangle, and, by (3), $z \sim m$ uniformly in the strip between the lines $x = m \pm \frac{1}{2}$. Hence each guide-curve lies in a strip of finite breadth bounded by lines $x = \text{const.}$, and cuts each line $y = \text{const.}$ a finite odd number of times (tangencies being counted with the proper multiplicity). In particular it cuts the lines $y = (n+\frac{1}{2})c$ once only. Hence the portion of the curve between the lines $y = (n \pm \frac{1}{2})c$ consists of one branch which crosses between the lines together with a finite number (possibly zero) of cycles† (simple closed curves) which are in general detached; and a sub-family of cycles must contain at least one centre† (acnode) and be limited externally by a loop united with other branches at a col† or singular point of higher order.

We make the further hypothesis that *the fundamental rectangle contains just two first-order singular points of the family of guide-curves*, these being necessarily a col (a, a') and a centre, and that $z(x, y)$ *has a local minimum at the centre*. Then at the col (a, a') there are united four branches; two of them form together a curve which crosses without further singularity from $y = -\frac{1}{2}c$ to $y = \frac{1}{2}c$, and the other two join to form a loop in the strip between these lines. This loop encloses a sub-family of cycles terminating at a centre, (b, b') say. We describe the col, loop, and centre as mutually *associated*. From (3), all the cols and centres fall into similarly associated pairs, and all the cols $(a, a'+nc)$ lie on the same guide-curve, $\mathcal{G}_a$ say. The guide-curves not congruent to $\mathcal{G}_a$ have an infinite branch without singularity, and some or all of them have also a set of detached cycles.

We call the infinite branch of any guide-curve a *simple guide-curve*. The simple guide-curve derived from $\mathcal{G}_a$ has angular points at the cols thereon. Each simple guide-curve is topologically equivalent to a straight line. We define the *positive sense* on it as that which leads from any point to a congruent point whose ordinate y is greater.

The four branches through the col (a, a') divide its neighbourhood into four compartments of which three lie on one side of the simple guide-curve and one on the other. In two opposite compartments z is less than at the col, and in the other two greater. Of the former, one is the interior of the associated loop since z is a minimum at the associated centre; the other must extend up to the lines $y = \pm\frac{1}{2}c$, which it must cut to the left of $\mathcal{G}_a$ since there $\partial z/\partial x > 0$. Hence the loop associated with (a, a') lies between the simple guide-curves $z = z(a, a')$, $z = z(a+1, a') = z(a, a')+1$; and the strip between the *complete* guide curve $z = z(a, a')$ and the *simple* guide-curve $z = z(a, a')+1$ is filled by a sub-family of simple guide-curves.

† The terms *cycle, col, node, focus, centre* are used with the meaning introduced by Poincaré, 8.

These topological features are illustrated in Fig. 1 for the case in which z is an even function of y.

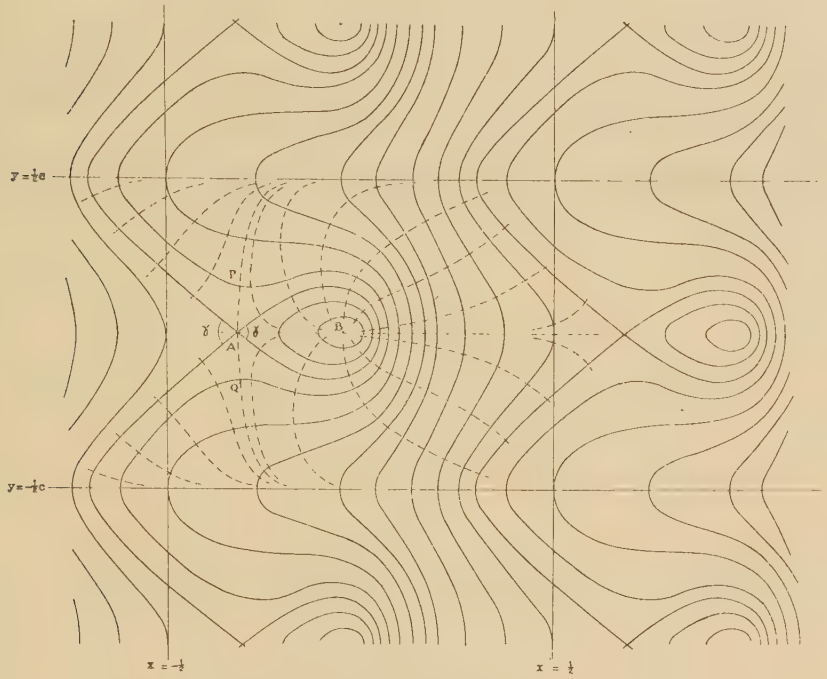


Fig. 1. *The guide-curves in the plane.*

Guide-curves ——, and their orthogonal trajectories – – – – (the lines $y = \pm\frac{1}{2}c$ also are orthogonal trajectories, to which their neighbours are asymptotic as $x \to \infty$.) The particular case sketched is the plane representation of Fig. 2. A fundamental unit only of the orthogonal trajectory field is shown.

The corresponding curves on the torus are illustrated in Fig. 2. There is one col, at which is united a branch embracing the surface with one which does not. The simple guide-curves are simple closed curves embracing the surface, on which may be defined a positive sense corresponding to that in the plane.

An example of a function satisfying the conditions laid down for $z(x, y)$ is given in § 8.1.

2.2. *The differential equations.* The differential equations whose trajectories are to be discussed are

$$\frac{dx}{dt} = \frac{\partial z}{\partial x}\cos\beta - \frac{\partial z}{\partial y}\sin\beta, \qquad \frac{dy}{dt} = \frac{\partial z}{\partial y}\cos\beta + \frac{\partial z}{\partial x}\sin\beta, \tag{4}$$

where β is a constant between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$. (The case $\frac{1}{2}\pi < \beta < \frac{3}{2}\pi$ is reduced to this by the transformation $t = -t'$.) At a later stage β will be taken to be a function of the parameter λ referred to in § 2 . 1.

The trajectories of (4) in the xy-plane are the curves which cut the guide-curves everywhere at a clockwise angle $\frac{1}{2}\pi - \beta$. If the torus is in conformal correspondence with the xy-plane, this property of equiangular section holds also for the trajectories of (4) thereon.

The system (4) is not soluble by quadratures, but its intimate connection with the guide-curves elucidates the qualitative features of the trajectories.

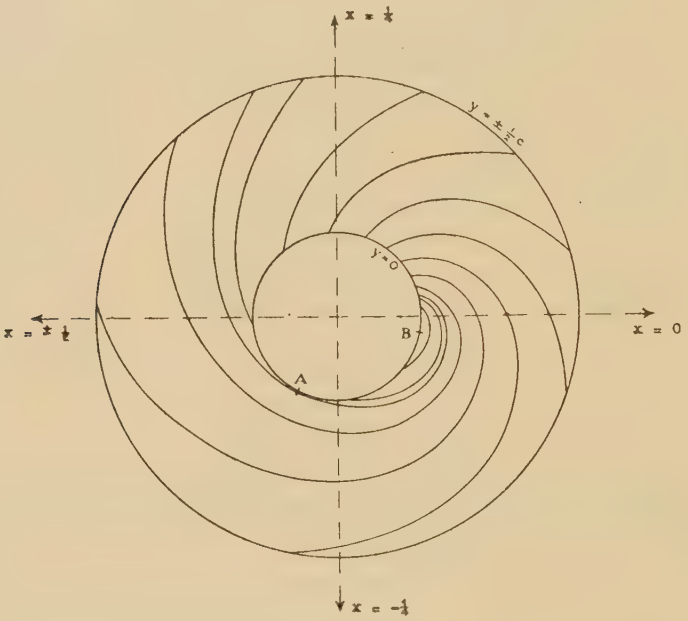


Fig. 2. *The guide-curves on the torus.*

The equatorial plane of the torus is shown, with the curves, which are symmetrical about it, projected thereon. The centre is at B and the col at A. The curves are those of the example of § 8.

3. *First properties of the trajectories.*

3 . 1. Since $\partial z, \partial x, \partial z/\partial y$ are regular† and (being periodic in x, y) bounded for all real x, y, all real solutions

$$x = x(t), \quad y = y(t)$$

† *I.e.* regular-analytic.

of (4) are defined and regular for all real t. Since (4) lead to

$$\frac{dz}{dt} = \left\{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right\} \cos\beta, \tag{5}$$

where $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, $z\big(x(t), y(t)\big)$ increases with t except at points where $\partial z/\partial x = \partial z/\partial y = 0$. The only points where this occurs are (a, a'), (b, b') and congruent points; these are the singular points of (4), and each furnishes a degenerate point-trajectory. Thus on any proper trajectory z is an increasing function of t, and tends to limits, finite or infinite, as $t \to \pm\infty$. If z tends to a finite limit, $dz/dt \to 0$; hence, from (5), $\partial z/\partial x \to 0$ and $\partial z/\partial y \to 0$, and $\big(x(t), y(t)\big)$ has asymptotic approach to a singular point. According as the approach is for $t \to \infty$ or $t \to -\infty$, we shall say that the trajectory concerned *enters* or *leaves* the singular point.

If, as $t \to \infty$ $(-\infty)$, $z \to \infty(-\infty)$, then from (3) $x \to \infty$ $(-\infty)$, and the trajectory concerned can be continued regularly to x-positive (negative) infinity.

3.2. *The singular points.* These are (a, a'), (b, b') and congruent points. Their nature may be investigated by the standard method, starting from the local expansions of $z(x, y)$ in which the linear terms are absent. The results, which are for the most part intuitive from the geometrical interpretation of (4) and the fact that z is a minimum at (b, b'), are as follows.

The point (a, a') is a col, the four asymptotic directions whereat lie one in each of the angles between the four guide-curve branches through the point. As β increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$ these directions rotate counter-clockwise through these respective angles, which are bisected for $\beta = 0$. For the asymptotic direction which points into the loop associated with the col, and for the opposite direction, the positive sense is towards the col: x, y are to the first order proportional to $e^{\nu_1 t}$, $\nu_1 < 0$. For the other two asymptotic directions the positive sense is away from the col: x, y proportional to $e^{\nu_2 t}$, $\nu_2 > 0$. By calculation of the exponents ν_1, ν_2 we find:

$|\nu_1/\nu_2| \gtreqless 1$ *according as* $\gamma \gtreqless \frac{1}{2}\pi$, *where* γ *is the angle at the col between the two guide-curve branches which unite to form the associated loop.*

The point (b, b') is a focus unless $\beta = 0$, when it is a node; we shall call it systematically a focus. The positive sense on neighbouring trajectories is away from it; they spiral out from it clockwise or counter-clockwise according as $-\frac{1}{2}\pi < \beta < 0$ or $0 < \beta < \frac{1}{2}\pi$.

3.3. When the facts of §§3.1, 3.2 are considered in conjunction with the properties of the guide-curves, §2.1, the truth of the following statement becomes evident (Fig. 3).

THEOREM I. (i) *Every proper trajectory* $\mathfrak{T}$ *of the equations* (4) *can be continued regularly over a certain finite or infinite open range of* z, *and cuts the complete guide-curve corresponding to any value in this range once and once only.*

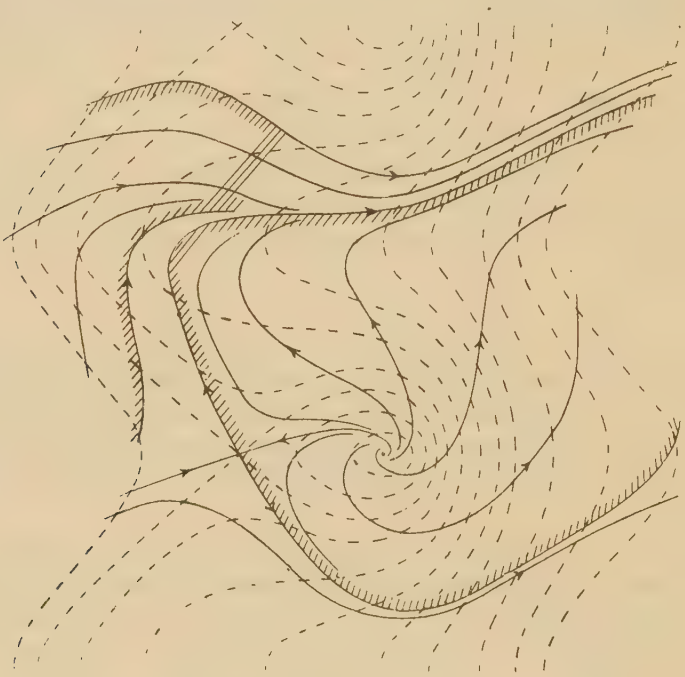


Fig. 3. *A fundamental unit of the trajectories for* $\beta = \frac{1}{4}\pi$.

The shading indicates portions of two black trajectories (§ 6.1). The positive sense is shown by arrow-heads.

(ii) *On continuation in the positive sense, which is that of* z *increasing,* $\mathfrak{T}$ *either enters a col or goes to* x*-positive infinity, having left either a col or focus or having come from* x*-negative infinity. The cols and foci are respectively at* (a, a'), (b, b') *and congruent points.*

(iii) *For the trajectory through any point of the loop associated with any col the positive sense is outwards, and the trajectory leaves the associated focus; all trajectories leaving the focus are thus accounted for, except one which enters the col.*

(iv) *For the trajectory through any non-singular point of the complete guide-curve* $z = z(a, a')$, *and for the two trajectories which leave any col thereon, all points for which* $z > z(a, a')$ *are on simple guide-curves. One of these enters each of the* ∞^2 *cols* $(a+m, a'+nc)$, $m > 0$, *and those of the remaining uncountable aggregate can be continued regularly to x-positive infinity.*

(v) *The trajectory field is invariant under the translations* $x \to x+1$, $y \to y+c$.

3.4. *The transformation-relation*†. Let $\mathcal{G}_0$ be any simple guide-curve $z = z_0$, other than one through a col, and let s be the (algebraic) arc-length in the positive sense from a fixed point (x_0, y_0) to any point P thereon. Let $\mathcal{G}_1$ be the congruent curve $z = z_0+1$ through (x_0+1, y_0), and s_1 the arc-length thereon from this point. Let $(a, a'+nc)$ be the cols between $\mathcal{G}_0$, $\mathcal{G}_1$.

The trajectory through any point P on $\mathcal{G}_0$ can be continued regularly to a unique point of intersection with $z = z_0+1$ unless it enters a col between the two curves; and in the latter case we can regard it as irregularly continued to the latter curve by the two trajectories which leave the col; in all cases the intersections with the latter curve are not on a detached cycle, but on the simple guide-curve $\mathcal{G}_1$ (Theorem I). Hence a transformation from $\mathcal{G}_0$ to $\mathcal{G}_1$:

$$s_1 = f(s)$$

is determined. From Theorem I and elementary theorems,

$$f(s+S) = f(s)+S,$$

where S is the arc-length along $\mathcal{G}_0$ from any point to the next congruent point; and $f(s)$ is regular with $f'(s) > 0$ for all real s except the singular values

$$\sigma+nS \quad (\sigma \text{ constant}, \ n = 0, \ \pm 1, \ \pm 2, \ \ldots),$$

which correspond to trajectories which enter cols $(a, a'+nc)$ between $\mathcal{G}_0$ and $\mathcal{G}_1$.

The trajectory which enters the col (a, a') and the two which leave it delimit two compartments of the neighbourhood of the col, in each of which there is one-sided continuity of the trajectory field at the boundary (and of course complete continuity in the interior). Hence $f(\sigma-0), f(\sigma+0)$, which

† The results of § 3.4 are not required except in §§ 5.2, 7.

exist because of the monotony of $f(s)$, correspond to the points in which $\mathcal{C}_1$ is met by the two trajectories which leave the col. From the one-sided continuity and the non-intersection of trajectories at a non-singular point there follows

$$f(\sigma+0) > f(\sigma-0).$$

The inverse function s is not defined for values of s_1 between $f(\sigma-0)$ and $f(\sigma+0)$, which correspond to trajectories asymptotic to the focus (b, b'). (See Fig. 3.)

More precise results as to the behaviour of $f(s)$ near σ may be obtained from the work of Dulac, **5** and **6**, and Birkhoff **2**† on integration for the neighbourhood of a col, as follows:

$$\left.\begin{aligned} &\text{As } s \to \sigma+0, \\ &\frac{f(s)-f(\sigma+0)}{(s-\sigma)^{\nu}} \to K, \qquad \frac{f'(s)}{(s-\sigma)^{\nu-1}} \to \nu K, \qquad \frac{f''(s)}{(s-\sigma)^{\nu-2}} \to \nu(\nu-1)K; \\ &\text{as } s \to \sigma-0, \\ &\frac{f(\sigma-0)-f(s)}{(\sigma-s)^{\nu}} \to K', \qquad \frac{f'(s)}{(\sigma-s)^{\nu-1}} \to \nu K', \qquad \frac{f''(s)}{(\sigma-s)^{\nu-2}} \to -\nu(\nu-1)K'; \end{aligned}\right\} \quad (6)$$

here K, K' are positive constants, and we have written $-\nu_1/\nu_2 = \nu > 0$. From § 3.2, $\nu \gtreqless 1$ according as $\gamma \gtreqless \frac{1}{2}\pi$. Thus

$f(s)$ has limiting gradients on each side of a discontinuity which are 0, *positive finite, or* $+\infty$ *according as γ is greater than, equal to, or less than* $\frac{1}{2}\pi$. (See Fig. 4.)

Further, if $\nu \neq 1$, the results (6) show that the gradient of $\log f'(s)$ tends to $+\infty$ or $-\infty$ as s tends to σ from either side; since $\log f'(s)$ is regular in

$$\sigma < s < \sigma+S,$$

we have

if $\gamma \neq \frac{1}{2}\pi$, $\log f'(s)$ has only a finite number of turning points in the open range $(\sigma, \sigma+S)$.

† Dulac does not consider the derivatives of $f(s)$, but his two papers provide an effective basis for the deduction of the results given. These results are easy corollaries of Birkhoff's work, but this is restricted to the case in which ν is irrational.

If $\nu = 1$, (6) give no information as to the limiting gradients of $\log f'(s)$ at $\sigma - 0$, $\sigma + 0$, but from the work of Dulac it may be shown† that *the conclusion just stated is valid also if* $\gamma = \frac{1}{2}\pi$.

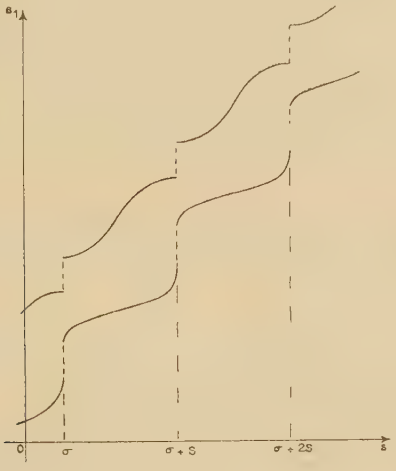


Fig. 4. *Graph of* $s_1 = f(s)$.
Upper curve : $\gamma > \frac{1}{2}\pi$.
Lower curve : $\gamma < \frac{1}{2}\pi$.

4. *The type-number.*

4.1. The property dealt with in this section rests simply upon the fact that there exist trajectories which proceed from the finite part of the plane to x-positive infinity, together with the principles:

LEMMA I. *The trajectories through any two congruent points are congruent curves, mutually transformable by pure translation.* [Theorem I (v)].

LEMMA II. *Two trajectories cannot have a point in common without coinciding entirely.*

† If the formal integral for the neighbourhood of the col contains logarithmic terms, as is in general the case, either the first or higher limiting derivatives of $\log f'(s)$ at σ are $\pm\infty$. In the exceptional case the formal integral converges, and $f(s)$ is regular in $\sigma \leqslant s \leqslant \sigma + S$; the apparent inconsistency of this result with the discontinuity of $f(s)$ at σ is due to the fact that $f(s)$ coincides with different branches of the same analytic function in $(\sigma - S, \sigma)$ and $(\sigma, \sigma + S)$.

In Lemma II, as originally conceived, a singular point is not regarded as belonging to the trajectories which are asymptotic thereto. We now introduce a new convention whereby that trajectory which enters any col not from within the associated guide-curve loop is regarded as attaining the col and as being *irregularly* continued by *one* of the two trajectories which leave the col. The choice between these is arbitrary, but congruent choices are to be made at all cols. The other two trajectories which respectively enter and leave any col are not regarded as attaining it. With this convention Lemmas I, II are still valid, and from Theorem I we have:

LEMMA III. *Every proper trajectory can be continued, regularly or irregularly, to x-positive infinity, except for the* ∞^2 *trajectories which enter the cols from the associated foci.*

4.2. We now define the relations of *above* and *below* between points and trajectories, with the help of the *simple* guide-curves. For two points P, Q on such a curve we say that P is above Q, and Q below P, if P lies in the positive sense from Q. With this definition the axioms of linear order are satisfied.

We say that a point P on a simple guide-curve $\mathcal{G}_P$ is above (below) a trajectory $\mathfrak{T}$ (which, in accordance with the preceding convention, may contain one or more cols), and that $\mathfrak{T}$ passes below (above) P, if $\mathfrak{T}$ meets $\mathcal{G}_P$, at R say, and if P is above (below) R; the point R is, by Theorem I, unique if it exists. If P is not on a simple guide-curve (through being on a guide-curve loop or cycle), or if $\mathfrak{T}$ does not meet $\mathcal{G}_P$, the relation of P to $\mathfrak{T}$ is not defined.

The proof of the following two Lemmas follows from the definitions together with Lemmas I, II:

LEMMA IV. *If P lies above (below) a trajectory $\mathfrak{T}$, any point of the trajectory $\mathfrak{T}_P$ through P lies above (below) $\mathfrak{T}$ provided that its relation thereto is defined; and, with a corresponding proviso, any point of $\mathfrak{T}$ lies below (above) $\mathfrak{T}_P$.*

LEMMA V. *If P lies above (below) the trajectory through Q, and a trajectory $\mathfrak{T}$ passes above (below) P and has a defined relation to Q, then $\mathfrak{T}$ passes above (below) Q.*

4.3. THEOREM II. *Let m, m', n, n' be integers, of which m, m' are positive. Let $\mathfrak{T}$ be the trajectory through any point (x_0, y_0) on any simple guide-curve.*

(i) *If* $\mathfrak{T}$ *passes above or through* $(x_0+m,\ y_0+nc)$, *it passes above*

$$(x_0+m',\ y_0+n'c)$$

provided that $n/m > n'/m'$.

(ii) *If* $\mathfrak{T}$ *passes below or through* $(x_0+m,\ y_0+nc)$, *it passes below*

$$(x_0+m',\ y_0+n'c)$$

provided that $n/m < n'/m'$.

(iii) *If* $\mathfrak{T}$ *passes through (above, below)* $(x_0+m,\ y_0+nc)$, *then it passes through (above, below)* $(x_0+m',\ y_0+n'c)$ *provided that* $n/m = n'/m'$.

(iv) *If* $\mathfrak{T}$ *meets any other simple guide-curve at* $(x_1,\ y_1)$ *and passes above (through, below)* $(x_0+m,\ y_0+nc)$, *then it passes above (through, below)*

$$(x_1+m,\ y_1+nc).$$

By Lemma III, $\mathfrak{T}$ can be continued from $(x_0,\ y_0)$ to x-positive infinity, so that it meets every complete guide-curve for which $z \geqslant z(x_0, y_0)$; and by Theorem I the intersection is on the simple guide-curve. Hence the relation of $\mathfrak{T}$ to all points concerned in the enunciation is defined.

For shortness, write $(x_0+m,\ y_0+nc) = [m,\ n]$.

Assume† the contradictory of (i): that the trajectory $\mathfrak{T}$ through $[0,\ 0]$ passes above (or through) $[m,\ n]$ and below (or through) $[m',\ n']$, where

$$m > 0, \quad m' > 0, \quad n/m > n'/m'.$$

If $m = m'$ then $n > n'$, and the statement is inconsistent with the axioms of linear order as they concern the point of $\mathfrak{T}$ and $[m,\ n]$, $[m,\ n']$ on the simple guide-curve $z = z(x_0,\ y_0)+m$. If, for definiteness, we take $m' > m$, the trajectory $\mathfrak{T}'$ through $[m'-m,\ n'-n]$, $= [m'',\ n'']$, is congruent to $\mathfrak{T}$, and by Lemma I passes above (or through) $[m',\ n']$ since $\mathfrak{T}$ passes through $[0,\ 0]$ and above (or through) $[m,\ n]$. If $\mathfrak{T}$ passes through $[m',\ n']$ it follows from Lemma IV (or II) that $\mathfrak{T}$ passes below (or through) $[m'',\ n'']$; if $\mathfrak{T}$ passes below $[m',\ n']$, the same conclusion follows from Lemma V (or IV). Also

$$m'' = m'-m > 0,$$

$$\frac{n}{m} - \frac{n''}{m''} = \frac{n}{m} - \frac{n'-n}{m'-m} = \frac{nm'-n'm}{m(m'-m)} > 0.$$

† The argument is best followed with the aid of a figure.

Hence $\mathfrak{T}$ passes above or through $[m, n]$ and below or through $[m'', n'']$, where

$$m > 0, \quad m'' > 0, \quad n/m > n''/m'';$$

which is another statement of the type to be disproved with one of the abscissae smaller.

By a finite number of such steps we find that $\mathfrak{T}$ must pass above or through $[m, n]$ and below or through $[\overline{m}, \overline{n}]$, where

$$m > 0, \quad \overline{m} > 0, \quad n/m > \overline{n}/\overline{m}, \quad 0 < \overline{m} \leqslant m.$$

If $\overline{m} = m$ we have, as at first, a contradiction. If $\overline{m} < m$, we reason similarly, and get a new statement of the type to be disproved, with $\overline{m}$ as the greater abscissa; and so on. But if M is the highest common factor of m, m' (where possibly $M = 1$), it is also the highest common factor of m, m''; and so at each stage. Hence after the smaller abscissa is reduced to M the larger abscissa will finish at the value M, and at this stage we shall have, as at first, a contradiction. Hence the statement (i) is established.

The statements (ii), (iii) are proved similarly.

The statement (iv) follows if we remark that a trajectory congruent to $\mathfrak{T}$ passes through the two points $(x_0+m, y_0+nc), (x_1+m, y_1+nc)$, so that by Lemmas II, IV these points have the same relation to $\mathfrak{T}$.

4.4. *Definition of the type-number.* If any trajectory $\mathfrak{T}$ meets a simple guide-curve at (x_0, y_0), the congruent points (x_0+m, y_0+nc), where m, n are integers and $m > 0$, can be divided into the three classes of those above, those below, and those on $\mathfrak{T}$, the third class being in general empty. If we let the rational number n/m correspond to the point (x_0+m, y_0+nc), all rational numbers are hence divided into three classes, which by Theorem II, (i)–(iii) satisfy the conditions for a Dedekind section. Further, by Theorem II (iv), the same section of the rationals is given by the similar procedure starting with (x_1, y_1), the point where $\mathfrak{T}$ meets any other simple guide-curve.

The number α, rational or irrational, specified by the section is called *the type-number of* $\mathfrak{T}$; if the third class is not empty and n/m is a number in it, $\alpha = n/m$.

4.5. THEOREM III. *All trajectories which can be regularly or irregularly continued to x-positive infinity have the same type-number.*

For two congruent trajectories this follows directly from the definition. Also, if any trajectory $\mathfrak{T}$ which can be continued to x-positive infinity and any simple guide-curve $\mathfrak{C}_0$ are assigned, a trajectory congruent to $\mathfrak{T}$ meets

$\mathcal{G}_0$ since $\mathcal{T}$ meets curves congruent to $\mathcal{G}_0$. Hence it is sufficient to consider trajectories $\mathcal{T}_P$, $\mathcal{T}_Q$ which cut a simple guide-curve $\mathcal{G}_0$ at points P, Q, where Q is between P and the next point congruent to and above P on $\mathcal{G}_0$.

Write

$$(x_P+m,\ y_P+nc) = [m,\ n]_P, \quad (x_Q+m,\ y_Q+nc) = [m,\ n]_Q,$$

where m, n are integers, and call $\mathcal{G}_m$ the curve congruent to $\mathcal{G}_0$ through $[m, n]_P$, $[m, n]_Q$, these points lying on the same such curve for all integers n. The trajectories $\mathcal{T}_P$, $\mathcal{T}_Q$ cut $\mathcal{G}_m$ at P_m, Q_m say, for all positive integers m, so their relations with all the points with which we shall be concerned are defined.

Assume that, for certain integers $m > 0$ and n, $\mathcal{T}_P$ passes above $[m, n]_P$ and $\mathcal{T}_Q$ below $[m, n]_Q$. Then Q_m is above P_m since Q is above P (Lemma IV), and on $\mathcal{G}_m$ we have the four points

$$[m, n]_P, \quad P_m, \quad Q_m, \quad [m, n]_Q \tag{7}$$

in the order indicated. Let $P^{(m)} = [-m, -n]_{P_m}$, $Q^{(m)} = [-m, -n]_{Q_m}$, so that on $\mathcal{G}_0$ we have the four points

$$P, \quad P^{(m)}, \quad Q^{(m)}, \quad Q$$

in the order indicated (Lemma I). Then, if the trajectories $\mathcal{T}_{P^{(m)}}$, $\mathcal{T}_{Q^{(m)}}$ through $P^{(m)}$, $Q^{(m)}$ cut $\mathcal{G}_m$ at $P^{(2m)}$, $Q^{(2m)}$, we have on $\mathcal{G}_m$ the six points

$$[m, n]_P, \quad P_m, \quad P^{(2m)}, \quad Q^{(2m)}, \quad Q_m, \quad [m, n]_Q$$

in the order indicated (Lemma IV). Since the translation

$$(x, y) \to (x+m, y+nc)$$

converts $\mathcal{T}_{P^{(m)}}$, $\mathcal{T}_{Q^{(m)}}$ into $\mathcal{T}_{P_m} = \mathcal{T}_P$, $\mathcal{T}_{Q_m} = \mathcal{T}_Q$, we have (Lemma I) on $\mathcal{G}_{2m}$ the four points

$$[2m, 2n]_P, \quad P_{2m}, \quad Q_{2m}, \quad [2m, 2n]_Q$$

in the order indicated. This set is analogous to (7), with m, n multiplied by 2. Hence, by repetition of the argument, we conclude that, if i is a power of 2†, $\mathcal{G}_{im}$ contains the four points

$$[im, in]_P, \quad P_{im}, \quad Q_{im}, \quad [im, in]_Q$$

in the order indicated.

† The result is true actually for all positive integral i.

But, since Q is between $P \equiv [0, 0]_P$ and $[0, 1]_P$, $[im, in]_Q$ is between $[im, in]_P$ and $[im, in+1]_P$. Hence, when i is a power of 2, $\mathfrak{T}_P$ passes above $[im, in]_P$ and below $[im, in+1]_P$. In the corresponding section of the rationals, $(in)/(im)$ belongs to the lower class and $(in+1)/(im)$ to the upper, so the type-number of $\mathfrak{T}_P$ is n/m. Similarly the type-number of $\mathfrak{T}_Q$ is n/m.

By considering all the possible relationships of $[m, n]_P$, $[m, n]_Q$ to $\mathfrak{T}_P$, $\mathfrak{T}_Q$ in this way we deduce

LEMMA VI. *If $\mathfrak{T}_P$ passes above (through, below) $[m, n]_P$, then either $\mathfrak{T}_Q$ passes above (through, below) $[m, n]_Q$ or $\mathfrak{T}_P$, $\mathfrak{T}_Q$ both have the type-number n/m.*

Applying this for all integers $m > 0$ and n, we see that either $\mathfrak{T}_P$, $\mathfrak{T}_Q$ have the same rational type-number, or they specify the same separation of all rationals into two Dedekind classes; and in the latter case their type-numbers are again by definition equal, being either rational or irrational.

When we take P fixed and Q variable on $\mathfrak{J}_0$ the theorem is proved. In virtue of it we can speak of "the type-number" without reference to any specific trajectory.

COROLLARY. *If the two trajectories which leave the col (a, a') pass one above and the other below, or "enclose", the col $(a+m, a'+nc)$, $m > 0$, the type-number is n/m.*

This follows from Lemma VI and an argument from continuity, if we take for $\mathfrak{J}_0$ a simple guide-curve for which z is sufficiently little greater than $z(a, a')$, and for P, Q its points of intersection with the trajectories leaving (a, a'). (Compare Fig. 1 where, for the case $\beta = 0$, $\mathfrak{T}_P$ passes above $[0, 1]_P$ and $\mathfrak{T}_Q$ below $[0, 1]_Q$.)

5. *Dependence of the type-number on the parameter.*

5.1. When β has any value between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ the trajectories, which we may call *β-trajectories* so as to put in evidence the value of β, are determinate and a type number α, the same for all β-trajectories, is defined. α is thus a function of β defined and single-valued for $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$.

THEOREM IV. *The type-number α is a continuous function of β in $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, and $\alpha \to \infty$, $-\infty$ respectively as $\beta \to \frac{1}{2}\pi - 0$, $-\frac{1}{2}\pi + 0$.*

To prove the continuity of α, choose any value β_0 between $\pm\frac{1}{2}\pi$, and a point A, the β_0-trajectory $\mathfrak{T}_{\beta_0}$ through which can be continued regularly to

x-positive infinity [Theorem I (iv)]. We can, and shall, suppose that A is not on a guide-curve loop or cycle, so that the type-number α_0 of $\mathcal{T}_{\beta_0}$ can be defined (§ 4. 4) with reference to A. Let m be an arbitrarily large positive integer, and let $\mathcal{T}_{\beta_0}$ cut the guide-curve $\mathcal{G}_m$ through $[m, 0]_A$ at a point P_{β_0} between the points $[m, n]_A$, $[m, n+1]_A$ or at the lower point. Then, from the definition of α_0, $n/m \leqslant \alpha_0 \leqslant (n+1)/m$.

Since the right-hand sides of the differential equations (4) are regular and not simultaneously zero at x, y, β_0, when (x, y) is any point of a certain domain which contains the arc AP_{β_0} of $\mathcal{T}_{\beta_0}$ and its end-points, the β-trajectory through A cuts $\mathcal{G}_m$ at a point P_β which is arbitrarily near P_{β_0} when β is sufficiently near β_0. Thus, when $|\beta-\beta_0| < \delta$ say, P_β is between $[m, n+1]_A$ and $[m, n-1]_A$, and for such a value of β the type-number α satisfies

$$\frac{n-1}{m} \leqslant \alpha \leqslant \frac{n+1}{m}.$$

Hence $|\alpha-\alpha_0| \leqslant 2/m$ when $|\beta-\beta_0| < \delta$, and since m is arbitrarily large this proves the continuity of α at β_0.

The behaviour enunciated for α when β is near $\pm\frac{1}{2}\pi$ is intuitive from the fact that the guide-curves are the β-trajectories for $\beta = \frac{1}{2}\pi$ and $\beta = -\frac{1}{2}\pi$, the positive senses being coincident for $\beta = \frac{1}{2}\pi$ and opposite for $\beta = -\frac{1}{2}\pi$. For the proof, however, something more is needed than the continuity theorem just used, on account of the singularities at the cols.

Let $\mathcal{D}$ be the continuum consisting of all the points of the simple guide-curves on which z is between $z(a, a')$, $= z_1$ say, and $z(a+1, a') = z_1+1$, together with their limit-points. On the right $\mathcal{D}$ is bounded by the simple guide-curve $z = z_1+1$, $\mathcal{G}_2$ say, which contains the cols $(a+1, a'+nc)$, and on the left it is bounded by the *complete* guide-curve $z = z_1$, $\mathcal{G}_1$ say, which contains the cols $(a, a'+nc)$. [See Fig. 1.] The points of $\mathcal{G}_1$ are in linear order provided that each col is regarded as two distinct points, one above and one below the points of the associated loop.

We specify the arcs of the β-trajectories in $\mathcal{D}$ by variables z, s for which the coordinate curves are the guide-curves and their orthogonal trajectories, *i.e.* the 0-trajectories (Fig. 1). The value of s associated with any curve $\mathcal{C}$ of the latter family is taken to be the (algebraic) arc-length along the simple guide-curve $z = z_1+\frac{1}{2}$ to its unique point of intersection with $\mathcal{C}$, measured from a fixed point in the positive sense. From Theorem I a unique point (x, y) of $\mathcal{D}$ corresponds to any pair of values (s, z) for which $z_1 \leqslant z \leqslant z_1+1$; and the inverse transformation is unique provided that each col on $\mathcal{G}_1$ is (as above) regarded as two points of $\mathcal{D}$. The direct and inverse transformations are regular in $\mathcal{D}$ except at the cols on its boundaries, but here they are one-sidedly continuous from within $\mathcal{D}$. If S is the arc-length

along $z = z_1 + \frac{1}{2}$ from any point to the next congruent point, we have

$$\left.\begin{aligned} s(x, y+c) &= s(x, y)+S, \\ y(s+S, z) &= y(s, z)+c. \end{aligned}\right\} \tag{8}$$

Since the coordinate curves of s, z are orthogonal,

$$\frac{\partial z}{\partial x}\frac{\partial s}{\partial y} - \frac{\partial z}{\partial y}\frac{\partial s}{\partial x} = \pm\sqrt{\left\{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right\}}\sqrt{\left\{\left(\frac{\partial s}{\partial x}\right)^2+\left(\frac{\partial s}{\partial y}\right)^2\right\}},$$

where the sign is by continuity the same at all non-singular points of $\mathfrak{D}$; and is $+$ since, at a point where $y = \frac{1}{2}c$, the vector (z_x, z_y) points into the fourth or first quadrant, and (s_x, s_y) into the first or second. Hence the equations (4) give, on use of (5),

$$\begin{aligned} \frac{ds}{dz} &= \tan\beta\sqrt{\frac{s_x^2+s_y^2}{z_x^2+z_y^2}} \\ &= \tan\beta\, F(s, z), \quad \text{say,} \end{aligned} \tag{9}$$

where $F(s, z)$ is an analytic function, positive and regular for $z_1 \leqslant z \leqslant z_1+1$ except at those points (s, z_1) and (s, z_1+1) which correspond to cols on the boundaries of $\mathfrak{D}$. Since also, from (8), $F(s+S, z) = F(s, z)$, there is a positive lower bound L to the values of $F(s, z)$ for $z_1+\frac{1}{4} \leqslant z \leqslant z_1+\frac{3}{4}$.

Let P be any point of $\mathfrak{C}_1$ which is not on a loop of this curve, or at a col. The β-trajectory $\mathfrak{T}_\beta$ through P is specified by the solution $s = f(z)$ of (9) for which $f(z_1) = s_P$; $f(z)$ being regular for $z_1 \leqslant z < z_1+1$, and also for $z = z_1+1$ unless the point Q where $\mathfrak{T}_\beta$ meets $\mathfrak{C}_2$ is a col. If $0 < \beta < \frac{1}{2}\pi$, $f(z)$ increases with z, and for the ordinary or limiting value $s_Q = f(z_1+1)$ we have

$$s_Q - s_P > \tfrac{1}{2}L\tan\beta.$$

The type-number α of $\mathfrak{T}_\beta$ is not less than n if Q is above the point (x_P+1, y_P+nc); this is the case if

$$\begin{aligned} s_\alpha &> s(x_P+1,\ y_P+nc) \\ &= s(x_P+1,\ y_P)+nS, \end{aligned}$$

i.e. if $\qquad s_Q - s_P > nS+s(x_P+1,\ y_P)-s_P$;

hence $\alpha \geqslant n$ if $\qquad \tfrac{1}{2}L\tan\beta > nS+s(x_P+1,\ y_P)-s_P.$ (10)

Since L, S and $s(x_P+1, y_P)-s_P$ are independent of β, $\alpha \to \infty$ as $\beta \to \frac{1}{2}\pi - 0$. Similarly $\alpha \to -\infty$ as $\beta \to -\frac{1}{2}\pi + 0$.

COROLLARY. *By a suitable assignment of β the type-number can be given any desired value.*

5.2. *The trajectory field corresponding to a rational type-number.* The following facts are stated with brief indications only of their proof, since they are not essential to our main object.

5.21. The β_2-trajectories cut the β_1-trajectories everywhere at a counter-clockwise angle $\beta_2-\beta_1$. Hence it is intuitive that the point P, in which the β-trajectory through a fixed point A meets a simple guide-curve $\mathcal{G}$ on which $z>z_A$, moves in the positive sense along $\mathcal{G}$ as β increases from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$. The motion of P is continuous so long as $\mathcal{T}_\beta$ does not enter a col between A and P; and as β increases through a value β_0 for which this occurs P jumps from the lower β_0-trajectory leaving the col to the upper one.

Hence it is apparent that, *as β increases, α never decreases.*

5.22. If, for $\beta=\beta_0$, α is rational and equal to n/m ($m>0$ and n integers without common factor), all points of a simple guide-curve $\mathcal{G}_0(z=z_0)$ are in correspondence with those of $\mathcal{G}_m(z=z_0+m)$ *via* the β_0-trajectories which pass between $\mathcal{G}_0$, $\mathcal{G}_m$. We specify the points of $\mathcal{G}_0$, $\mathcal{G}_m$ by the arc-lengths s, s_m from the related origins A, $[m, n]_A$ on the respective curves, and have a relation

$$s_m=g(s),$$

where $g(s)+nS$ is the m-th iterate of the $f(s)$ of § 3.4. $g(s)$ is regular for all s except a set of isolated values where it has simple finite discontinuities, which correspond to trajectories that enter cols between $\mathcal{G}_0$, $\mathcal{G}_m$. The limiting gradients at the discontinuities depend on γ as in § 3.4.

From the fact that the type-number is n/m we prove that the upper and lower bounds of $g(s)-s$ cannot be both positive or both negative, and from congruence properties we prove that the limits of $g(s)-s$ on the right (left) of each discontinuity have the same relation to 0. Hence each continuous piece of the curve $s_m=g(s)$ meets the line $s_m=s$, and *there exist trajectories which join points P to congruent points $[m, n]_P$.* On the torus these are *periodic*, and *regularly* or *irregularly* periodic according as the continuation from P to $[m, n]_P$ does not or does lead through a col. Thus *a necessary and sufficient condition for the existence of periodic trajectories is that the type-number is rational.*

Since the bounds of $g(s)-s$ are not coincident and vary continuously with β, *the type-number keeps any rational value over a finite closed range of β.*

From the existence of pervasive trajectories when α is irrational (§6.6 below) it follows that *an irrational type-number occurs for one value only of* β.

5.23†. Since $g(s)-s$ is analytic and not identically zero, the β-trajectories sufficiently near to and on each side of a regular periodic β-trajectory asymptote thereto as $z\to\infty$ or as $z\to-\infty$; the sense of the approach ($z\to\infty$ or $z\to-\infty$) is in general the same from both sides (*cf.* Fig. 5), but exceptionally may be opposite (*cf.* Fig. 6).

Fig. 5. *Trajectory field (in the plane) when* $\gamma<\frac{1}{2}\pi$ *and there is an irregular periodic trajectory* p_1 *of type* 0/1.

If β now increases, p_2 moves upwards while p_1 becomes regular and moves downwards, the two ultimately coinciding in p, Fig. 6.

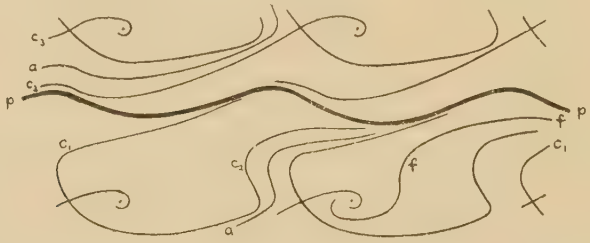


Fig. 6. *Trajectory field when* $\gamma<\frac{1}{2}\pi$ *and there is a limiting regular periodic trajectory* p *o type* 0/1.

Any increase in β will cause α to pass over infinitely many rational and irrational values.

Fig. 7. *Trajectory field when* $\gamma>\frac{1}{2}\pi$ *and there is a limiting irregular periodic trajectory* p *of type* 0/1.

† The results of this section are proved by consideration of the successive iterates of $g(s)$.

The trajectories near an irregular periodic trajectory on the side remote from the associated foci are asymptotic to it as $z \to \infty$ if $\gamma > \frac{1}{2}\pi$, and as $z \to -\infty$ if $\gamma < \frac{1}{2}\pi$. In the latter case there must exist also at least one regular periodic trajectory (Fig. 5); in the former case there may be none such (Fig. 7).

A regular periodic β-trajectory varies continuously with β, and disappears only by uniting with another regular periodic trajectory or by passing through a limiting irregular periodic form.

5.24. The value $\beta = \beta_0$, which is at the upper end of the range in which α has a rational value n/m, is of special significance for the deeper study of the trajectory-field for α near to and greater than n/m. We then have (on the torus) one, and in general only one, periodic β_0-trajectory p, which is always regular if $\gamma < \frac{1}{2}\pi$ (Fig. 6), but may be irregular if $\gamma > \frac{1}{2}\pi$ (Fig. 7). In the former case the trajectories near p on opposite sides are asymptotic thereto in opposite senses, and fall into the three classes: (1) c_1, c_2, which leave the col, and c_3 which enters the col, (2) trajectories f which leave the focus, (3) trajectories a which proceed regularly from x-negative to x-positive infinity and are doubly asymptotic to p. In the latter case there are asymptotic to p as $z \to \infty$ one trajectory c_1 which leaves the col and those f which leave the focus.

For $\beta > \beta_0$ there is no (real) periodic trajectory of type n/m, and every trajectory except the one which passes from the focus to the col can be analysed into a set of comparatively long arcs practically coincident with arcs of p joined by comparatively short arcs practically coincident with the "central portion" of an a or c_3+c_1 or c_3+c_2 (Fig. 6), or with c_1 (Fig. 7); this set continues indefinitely in both senses ($z \to \infty$ and $z \to -\infty$) unless the trajectory originates at the focus, in which case it begins with an arc practically coincident with an f. As $\beta \to \beta_0$ the ratio of the lengths of a short and a long arc tends to 0. Thus all trajectories lie more densely in the neighbourhood of p than elsewhere, and the relative density tends to infinity as $\beta \to \beta_0$.

These considerations apply also to the simpler case in which the trajectory-field has no singular points, so that *though all trajectories are everywhere dense on the torus when α is irrational*†, *for suitably chosen α the ratio of the densities at different points is as great as we please.*

5.3. *Case in which $z(x, y)$ and β are functions of a parameter λ.* Suppose that z depends analytically upon a (real) parameter λ as well as on x, y, and is regular for any (x, y, λ) for which x, y are real and λ lies in an open

† Denjoy, 4.

range (λ_1, λ_2); the family of guide-curves then varies continuously with λ. Suppose that for all λ in (λ_1, λ_2) the hypotheses of § 2 . 1 are valid, where we can without loss of generality suppose the values z_1, $z_1 \pm 1$, ... of z at the cols to be independent of λ. Suppose further that β is an analytic function of λ regular in (λ_1, λ_2).

The proof in § 5 . 1 that α varies continuously with β depends merely on the regularity in β of the right-hand sides of the equations (4). Hence we can assert that, in any sub-range of (λ_1, λ_2) in which $-\frac{1}{2}\pi < \beta < \frac{1}{2}\pi$, α is a continuous function of λ.

For each λ we have a continuum $\mathfrak{D}_\lambda$ defined as in § 5 . 1, and s is regular in λ as well as in x, y when (x, y) is any point of $\mathfrak{D}_\lambda$ other than a col; so $F(s, z, \lambda)$ is regular for $z_1 < z < z_1 + 1$, $\lambda_1 < \lambda < \lambda_2$ and all real s. S is regular in λ and positive for $\lambda_1 < \lambda < \lambda_2$, while L is continuous and positive in this range. If we define the variation of P with λ to be such that P is always on $\mathfrak{G}_1$ (which varies with λ) but never at a col, with x_P, y_P continuous in λ, then $s(x_P + 1, y_P) - s_P$ is continuous in λ. Hence, if for the moment we regard β, λ as independent parameters, (10) shows that $\alpha \to \infty$ as $\beta \to \frac{1}{2}\pi - 0$, uniformly for λ in any closed range interior to (λ_1, λ_2). Hence we can assert

THEOREM IVa. *If z is a regular function of x, y, λ for all real x, y and $\lambda_1 < \lambda < \lambda_2$, and satisfies the hypotheses of § 2 . 1 for each λ in this range; and if β is a regular function of λ for $\lambda_1 < \lambda_1' \leqslant \lambda \leqslant \lambda_2' < \lambda_2$, which lies between $\pm\frac{1}{2}\pi$ for λ between λ_1', λ_2' and takes the values $-\frac{1}{2}\pi$, $\frac{1}{2}\pi$ for $\lambda = \lambda_1'$, λ_2'; then the type-number α varies continuously with λ for $\lambda_1' < \lambda < \lambda_2'$ and tends to $-\infty$, ∞ as $\lambda \to \lambda_1' + 0$, $\lambda_2' - 0$.*

6. *Trajectories whose type-number is irrational.*

6 . 1. Let β, or λ, be fixed at a value for which the type-number α is irrational (corollary to Theorem IV or IVa). Then by the corollary to Theorem III and the definition of α we have:

When α is irrational the two trajectories $\mathfrak{T}_1$, $\mathfrak{T}_2$ which leave the col (a, a') enclose no cols; they pass above all cols $(a+m, a'+nc)$ for which $m > 0$ and $\alpha > n/m$, and below all cols $(a+m, a'+nc)$ for which $m > 0$ and $\alpha < n/m$. They can hence be continued regularly to x-positive infinity.

Choose the notation so that $\mathfrak{T}_1$ is above $\mathfrak{T}_2$. The trajectory $\mathfrak{T}$ through any point P of the guide-curve loop associated with the col (a, a') has all its points for which $z > z(a, a')$ above $\mathfrak{T}_2$ and below $\mathfrak{T}_1$, and all its points for

which $z < z(a, a')$ inside the loop, these points also being between $\mathfrak{T}_1$, $\mathfrak{T}_2$ in an intuitive sense but not in the technical sense of § 4.2. Hence, from the preceding statement and Theorem I, $\mathfrak{T}$ goes regularly to x-positive infinity as $t \to \infty$, and to the associated focus as $t \to -\infty$. From the continuity of the trajectory-field (one-sided in each compartment at the col), the aggregate of the points of the trajectories through all points of the loop, together with its limit-points, gives all points between and on $\mathfrak{T}_1$, $\mathfrak{T}_2$.

No two points of the continuum, $\mathfrak{B}^*(a, a')$ say, so defined are congruent; for the trajectories through such points would belong entirely to $\mathfrak{B}^*(a, a')$ and for $t \to -\infty$ would be asymptotic to two non-coincident foci or cols belonging to $\mathfrak{B}^*(a, a')$, which from the preceding is impossible. Similarly $\mathfrak{B}^*(a, a')$ contains no point congruent to a point of the trajectory $\mathfrak{T}_3$ which enters (a, a') not from within the associated loop, for the trajectory through for $t \to \infty$ such a point would be asymptotic to a col other than (a, a') in $\mathfrak{B}^*(a, a')$. Hence $\mathfrak{T}_3$ can be continued regularly to x-negative infinity.

It is convenient now to regard $\mathfrak{B}^*(a, a')$ together with $\mathfrak{T}_3$ as constituting one pseudo-trajectory, which we call a *black trajectory* and denote by $\mathfrak{B}(a, a')$; we call $\mathfrak{T}_3 + \mathfrak{T}_1$ its upper boundary and $\mathfrak{T}_3 + \mathfrak{T}_2$ its lower boundary. There is a congruent black trajectory associated with each col $(a+m, a'+nc)$, and by the preceding no two of these have a point in common. In Fig. 3, portions of two black trajectories are indicated by shading (which for clearness is shown only near the two boundaries $\mathfrak{T}_1$, $\mathfrak{T}_2$ and omitted from the interior portion between); the upper one of these black trajectories is congruent to the continuation of the lower one towards the right. Any simple guide-curve is cut by the black trajectories in an enumerable set of closed *black intervals*, some of which are *proper* intervals and some are *degenerate* point-intervals. Since no two of these overlap or abut there are on the guide-curve $2^{\aleph_0}$ non-black points, the trajectories through which can be continued regularly to x-positive and x-negative infinity, since no col or focus is available for their termination; we call such trajectories *non-singular*.

The legitimacy of the black-trajectory convention lies in the facts: (1) for the aggregate of all non-singular and black trajectories Lemmas I, II are valid; and (2) the aggregate possesses a continuity property (Lemma VII below) similar to that of an ordinary trajectory-field. The utility of the convention lies in the facts: (3) every trajectory of the aggregate can be continued to x-positive and to x-negative infinity; and (4) the aggregate falls into a linear order (§ 6.2 below).

On the torus there are one black trajectory and $2^{\aleph_0}$ non-singular trajectories.

LEMMA VII. *If P_1, P_2 are any points on the same trajectory $\mathfrak{T}$, through which pass simple guide-curves $\mathfrak{G}_1$, $\mathfrak{G}_2$, then all points on $\mathfrak{G}_1$ in a sufficiently small neighbourhood of P_1 furnish trajectories, whether non-singular or black, which cut $\mathfrak{G}_2$ in points or closed intervals lying in an arbitrary neighbourhood of P_1; provided that, if $\mathfrak{T}$ is a black trajectory, P_1, P_2 are to lie on the upper (lower) boundary thereof and the continuity may hold only on the upper (lower) side.*

The proof follows from the standard theorem of continuity of a trajectory-field in a domain free from singularities, together with the one-sided continuity in each compartment at a col, and the fact that there is a neighbourhood of the arc of $\mathfrak{T}$ between P_1, P_2 which contains no col other than the one which may lie on this arc when $\mathfrak{T}$ is black.

6.2. *Order relations between congruent points.* The system of non-singular and black trajectories falls into a linear order which is that of the points and black intervals in which they cut any simple guide-curve, all such curves furnishing the same order (Lemma IV). From Theorem II, with the definition of the type-number α and the non-intersection of black trajectories, we have:

If (x_0, y_0) is any point on a non-singular trajectory or on the boundary of a black trajectory $\mathfrak{T}(x_0, y_0)$, the congruent point (x_0+m, y_0+nc) and the trajectory $\mathfrak{T}(x_0+m, y_0+nc)$ through this point lie above $\mathfrak{T}(x_0, y_0)$ if

$$m > 0 \quad \text{and} \quad n/m > \alpha,$$

or if

$$m < 0 \quad \text{and} \quad n/m < \alpha,$$

or if

$$m = 0 \quad \text{and} \quad n > 0;$$

with corresponding criteria for below. The extension of Theorem II to $m \leqslant 0$ that is here implied is possible because every trajectory can be continued to x-negative infinity; the proof for $m < 0$ follows from the congruence of the point-pair (x_0-m, y_0-nc), (x_0, y_0) with the pair (x_0, y_0), (x_0+m, y_0+nc), while the case $m = 0$ is trivial.

The fundamental application of this criterion is as follows. Let the irrational number α be developed as a simple continued fraction of which p_i/q_i is the i-th convergent, q_i being positive and p_i positive or negative according to the sign of α. For definiteness take the case in which i is of such parity that

$$\frac{p_i}{q_i} < \alpha < \frac{p_i+p_{i-1}}{q_i+q_{i-1}} < \frac{p_{i-1}}{q_{i-1}}, \tag{11}$$

$$p_{i-1}q_i - p_i q_{i-1} = 1, \tag{12}$$

The condition (12) implies that all points $(x_0+m,\ y_0+nc)$ fall into a lattice† of which $(x_0+q_i,\ y_0+p_ic)$, $(x_0+q_{i-1},\ y_0+p_{i-1}c)$ are the unit points with respect to $(x_0,\ y_0)$. Hence it is evident that, if $0<m<q_i+q_{i-1}$ and $n/m>p_i/q_i$, then $n/m\geqq p_{i-1}/q_{i-1}$; the equality sign can occur here only if $q_{i-1}\leqq m<q_i+q_{i-1}$. Hence, from the order-criterion italicized above, with the help of (11), we get:

$\mathfrak{T}(x_0+q_{i-1},\ y_0+p_{i-1}c)$ is above $\mathfrak{T}(x_0,\ y_0)$. All those points $(x_0+m,\ y_0+nc)$, $0\leqq m\leqq q_i+q_{i-1}$, which lie above $\mathfrak{T}(x_0,\ y_0)$ lie also above $\mathfrak{T}(x_0+q_{i-1},\ y_0+p_{i-1}c)$, except that $(x_0+q_{i-1},\ y_0+p_{i-1}c)$ lies on the latter trajectory and

$$\left(x_0+q_i+q_{i-1},\ y_0+(p_i+p_{i-1})c\right)$$

lies between the two trajectories.

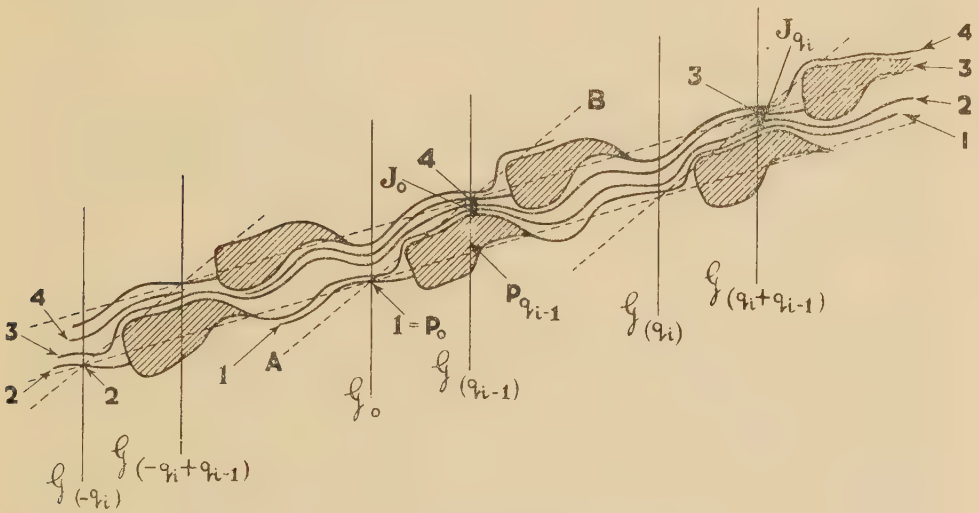


Fig. 8. *Order relations between congruent trajectories.*

The case where $\mathfrak{T}(x_0,\ y_0)$ is a black trajectory is illustrated. The diagram is schematic only: the trajectories should be more convoluted, the portions near and to the right of the cols should be broader, and the bands between them narrower.

1: $(x_0,\ y_0)$ and $\mathfrak{T}(x_0,\ y_0)$.

2: $(x_0-q_i,\ y_0-p_ic)$ and $\mathfrak{T}(x_0-q_i,\ y_0-p_ic)$.

3: $(x_0+q_i+q_{i-1},\ y_0+p_ic+p_{i-1}c)$ and $\mathfrak{T}(x_0+q_i+q_{i-1},\ y_0+p_ic+p_{i-1}c)$.

4: $(x_0+q_{i-1},\ y_0+p_{i-1}c)$ and $\mathfrak{T}(x_0+q_{i-1},\ y_0+p_{i-1}c)$.

From this we derive by congruence properties (intuitive from the topological anti-symmetry of Fig. 8 about the line AB): all those points

† Indicated by dotted lines in Fig. 8.

$\left(x_0+q_{i-1}-m,\ y_0+(p_{i-1}-n)c\right)$, $0 \leqslant m \leqslant q_i+q_{i+1}$, which lie below

$$\mathfrak{T}(x_0+q_{i-1},\ y_0+p_{i-1}c)$$

lie also below $\mathfrak{T}(x_0, y_0)$, except that (x_0, y_0) lies on the latter trajectory and $(x_0-q_i,\ y_0-p_ic)$ lies between the two trajectories.

If we denote by $\mathcal{G}_n$ the simple guide-curve $z = z(x_0, y_0)+n$, these results combine to give: *There are no points congruent to* (x_0, y_0) *inside the open domain bounded by* $\mathfrak{T}(x_0, y_0)$, $\mathfrak{T}(x_0+q_{i-1}, y_0+p_{i-1}c)$, $\mathcal{G}_{-q_i}$, $\mathcal{G}_{q_{i-1}+q_i}$, *and there is one such point on each of its four boundaries.* See Fig. 8.

If $\mathfrak{T}(x_0, y_0)$ cuts $\mathcal{G}_n$ in a point or closed black interval P_n it follows that there is no point† congruent to P_n inside the domain bounded by the two trajectories and $\mathcal{G}_{-q_i+n}$, $\mathcal{G}_{q_i+q_{i-1}+n}$, and one such point on each of the four boundaries.

Combining the results for $n = 0, 1, \ldots, q_i$, we get

There is no point congruent to any of $P_0, P_1, \ldots, P_{q_i}$ *inside the domain bounded by the two trajectories and* $\mathcal{G}_0$, $\mathcal{G}_{q_{i-1}+q_i}$; *while for its boundaries we have a point congruent to* P_{q_i} *on* $\mathcal{G}_0$, *a point congruent to* P_0 *on* $\mathcal{G}_{q_{i-1}+q_i}$, *and one point congruent to each* P *on each trajectory.*

6.3. Now on the torus let any trajectory $\mathfrak{T}$ have successive intersections $P_0, P_1, \ldots$ in the sense of z increasing with a definite simple guide curve $\mathcal{G}$; if $\mathfrak{T}$ is the black trajectory, all the P from a certain one onwards are closed black intervals. In the plane representation, Fig. 8, we take P_0 to be the intersection of $\mathfrak{T}(x_0, y_0)$ with $\mathcal{G}_0$, or any congruent point, so that P_n is the intersection of $\mathfrak{T}(x_0, y_0)$ with $\mathcal{G}_n$, or any congruent point. Hence from the preceding result, one of the two open arcs into which the closed curve $\mathcal{G}$ on the torus is divided by P_0, $P_{q_{i-1}}$ contains none of the points $P_0, P_1, \ldots, P_{q_i}$, namely that which is represented by J_0 in Fig. 8. The trajectories which traverse this arc J_0 form a strip bounded on each side by $\mathfrak{T}$. If this strip is followed in the sense of z-increasing, let $J_1, J_2, \ldots$ be the open arcs in which it successively cuts $\mathcal{G}$; so that the plane representation of J_n is the arc of $\mathcal{G}_{n+q_{i-1}}$ between $\mathfrak{T}(x_0, y_0)$, $\mathfrak{T}(x_0+q_{i-1}, y_0+p_{i-1}c)$. Then, as in Theorem III, if J_n completely contained J_0 or was completely contained by it the type-number would be rational; so, if J_n has points common with J_0, each must contain an end-point of the other. Hence, from the preceding result, the first of $J_1, J_2, \ldots$ which has points common

† For shortness we write "point" for "point or closed black interval."

with J_0 is J_{q_i}, which contains the end-point P_0 of J_0. From this it follows further that no two of $J_1, J_2, \ldots, J_{q_i}$ have a point in common.

This same result is found in the case $p_i/q_i > \alpha > p_{i-1}/q_{i-1}$, and we have proved

LEMMA VIII. *Let $\mathcal{G}$ be a simple guide-curve on the torus, one of the senses of description of which is designated as positive; let successive intersections, for z increasing, of the black trajectory or any non-singular trajectory with $\mathcal{G}$ be the black intervals or points $P_0, P_1, P_2, \ldots$. Let the i-th convergent to the simple continued fraction for α be p_i/q_i $(q_i > 0)$. Then on either the positively or negatively sensed open arc $J_0^{(i)}$ from P_0 to $P_{q_{i-1}}$ there lies none of $P_0, P_1, \ldots, P_{q_i}$; no two of the correspondingly sensed open arcs*

$$J_0^{(i)}, J_1^{(i)}, \ldots, J_{q_i-1}^{(i)} \quad (J_n^{(i)} = P_n P_{n+q_{i-1}}),$$

have a point in common; and $J_{q_i}^{(i)}$ overlaps $J_0^{(i)}$ with P_0 in common. The sense of $J_0^{(i)}$ changes with the parity of i. See Fig. 9.

6.4. LEMMA IX. *Let $\ldots, P_{-1}, P_0, P_1, \ldots, \ldots, Q_{-1}, Q_0, Q_1, \ldots$ be successive intersections with $\mathcal{G}$ of any two trajectories, non-singular or black and possibly coincident, and r, s any integers. Then both the open arcs $P_r P_s$ of $\mathcal{G}$ contain points or black intervals $Q_{-n}, Q_{n'}$, for which n, n' are arbitrarily large.*

If an arc $P_r P_s$ contains Q_i, the correspondingly sensed arc $P_{r+m} P_{s+m}$ contains Q_{i+m}; hence it is sufficient to prove the case in which $r = 0$ and s has any positive fixed value. Choose one of the arcs $P_0 P_s$ (in applications it is the one which is "small" that is required) and consider the sequence of correspondingly sensed open arcs $P_0 P_s$, $P_s P_{2s}$, $P_{2s} P_{3s}$, ... Let $P_{(k-1)s} P_{ks}$ be the first one which contains P_0; there must be such a one since, for a suitably chosen i, $P_0 P_{q_{i-1}}$ is a part of $P_0 P_s$ and is overlapped by $P_{q_i} P_{q_i+q_{i-1}}$ with P_0 in common (Lemma VIII). If Q_0 is not on the trajectory through P_0, then Q_m belongs to one of $P_0 P_s, P_s P_{2s}, \ldots, P_{(k-1)s} P_{ks}$ for every integer m; in the contrary case the same is true of all the Q_m except the finite number that coincide with $P_0, P_s, \ldots, P_{ks}$. Hence, for all m for which $|m|$ is sufficiently large, one of $Q_m, Q_{m-s}, \ldots, Q_{m-(k-1)s}$ belongs to $P_0 P_s$; this proves the Lemma.

6.5. THEOREM V. *When the type-number is irrational, the α- and ω-limit sets of the black trajectory and of every non-singular trajectory on the torus coincide, and the section of this limit set by any simple guide-curve is perfect and nowhere dense.*

Let $\ldots, P_{-1}, P_0, P_1, \ldots, \ldots, Q_{-1}, Q_0, Q_1, \ldots$ be successive intersections of any two trajectories, non-singular or black and possibly coincident, with

a simple guide-curve $\mathfrak{C}_1$. Write $D\{P_n\}$, $D\{P_{-n}\}$ for the derived sets of the sequences

$$P_0, \quad P_1, \quad P_2, \quad \ldots,$$

$$P_0, \quad P_{-1}, \quad P_{-2}, \quad \ldots;$$

if the P_n are proper black intervals, $D\{P_n\}$ is the derived set of their end-points. Since no two of the P's coincide, $D\{P_n\}$ and $D\{P_{-n}\}$ are not empty.

Any neighbourhood of any point A of $D\{P_n\}$ contains an open interval $P_r P_s$, and hence by Lemma IX contains Q_{-n} and $Q_{n'}$ for which n, n' are arbitrarily large. Hence A belongs to $D\{Q_n\}$ and $D\{Q_{-n}\}$; and, in by particular, taking $Q = P$, to $D\{P_{-n}\}$. Similarly every point of each of $D\{P_n\}$, $D\{P_{-n}\}$, $D\{Q_n\}$, $D\{Q_{-n}\}$ belongs to all, so these sets are coincident. Calling this set $\mathfrak{S}$, and regarding Q as fixed and P variable over $\mathfrak{C}_1$, we have: *for the black trajectory and for every non-singular trajectory, the α- and ω-limit sets coincide and consist of the set of trajectories through the points of* $\mathfrak{S}$.

If the set of all black intervals (proper and degenerate) is everywhere dense on $\mathfrak{C}_1$, then, by Lemma IX, the proper black intervals form a set everywhere dense on $\mathfrak{C}_1$. Since no two of these overlap or abut, their end-points form a set $\mathfrak{E}$ which is dense in itself and nowhere dense. Since $\mathfrak{S}$ is the derived set of $\mathfrak{E}$ it is perfect and nowhere dense, consisting of $\mathfrak{E}$ together with $2^{\aleph_0}$ other points. This completes the proof for this case.

If the black intervals are not everywhere dense on $\mathfrak{C}_1$ there are on $\mathfrak{C}_1$ open intervals, which we call *grey*†, which contain no black intervals, and whose end-points are either black or are limit-points of black intervals (or possibly have both properties). If an open interval contains no *proper* black interval or portion thereof, it can by Lemma IX contain at most one degenerate black interval, so it consists of either one or two grey intervals or portions thereof. Hence the grey intervals and the proper black intervals are together everywhere dense on $\mathfrak{C}_1$. Since no interior point of a grey or proper black interval can belong to $\mathfrak{S}$, $\mathfrak{S}$ is nowhere dense on $\mathfrak{C}_1$.

To complete the proof for this case we must show that $\mathfrak{S}$ is perfect, establishing for this and later purposes a property of homogeneity in the structure of the system of black and grey intervals, as follows:

LEMMA X. (i) *The upper (lower) end-points of all black intervals, proper and degenerate, are either all limit-points of black intervals, or all end-points of grey intervals.*

† For convenience of representation in a figure.

(ii) *If there is a grey interval on any simple guide-curve, the set of trajectories through its points forms a "grey domain" which cuts any simple guide-curve in a grey interval; the upper (lower) end-points of all the intervals so associated with the same grey domain are either all black, or all non-black limit-points of black intervals.*

(iii) *If a grey domain cuts a simple guide-curve in successive intervals* $\ldots, G_{-1}, G_0, G_1, \ldots$, *the G's are all distinct and their lengths tend to* 0 *as* $n \to \infty$ *and as* $n \to -\infty$.

(iv) *No grey interval can have both its end-points black.*

The proof of (i) and (ii) follows directly from Lemma VII, § 6.1. As regards (iii), the only relations of G_m, G_n which are consistent with the definition of a grey interval are that they should be coincident or without common points; and the first possibility is inconsistent with the irrationality of α since then the trajectories through their end-points would be periodic. The second part of (iii) follows immediately. Finally, if (iv) were not true, we should have two black intervals bounding an open arc containing no black intervals, in contradiction with Lemma IX. The possibility that a black interval is bordered on both sides by grey intervals is not excluded, but in this case Lemma IX shows that the latter cannot be sections of the same grey domain.

We now complete the proof of the theorem. If the black intervals on $\mathcal{G}$ do not form a set dense in itself, then each of them is by Lemma X (i) bordered on both sides by a grey interval. Suppose that A_r is the upper end-point of a grey interval whose lower end-point belongs to a black interval B_r. Then, by Lemma X (iv) and the definition of a grey interval, a set of black intervals $B_{n_1}, B_{n_2}, \ldots$ converges to A_r, necessarily from above. Above each B_n there abuts by (i), (ii) a grey interval whose upper end A_n is on the trajectory through A_r, and A_n must be below those of the B_{n_i} that are above B_n. Hence the A_{n_i} converge to A_r, and the set $\{A_n\}$, $n = 0, \pm 1, \pm 2, \ldots$, is dense in itself.

Hence there is always a trajectory, black or non-singular, which cuts $\mathcal{G}$ in a set dense in itself, and consequently $\mathcal{S}$ is perfect.

6.6. If we now resolve the black trajectory into its constituent proper trajectories, we can easily specify the α- and ω-limit sets of all trajectories on the torus. The said constituents are the focus and col, the *focal* trajectories (asymptotic to the focus as $t \to -\infty$), and the trajectories $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ asymptotic to the col, which bound the black trajectory.

In all cases, we have a perfect nowhere-dense limit set $\mathcal{S}$ on the guide-curve $\mathcal{C}$, the $2^{\aleph_0}$ trajectories through which form a perfect nowhere-dense set, $\mathfrak{L}$ say, on the torus.

Case I. *The black intervals are everywhere dense on* $\mathcal{C}$. Here the col and $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ belong to $\mathfrak{L}$. $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are asymptotic-pervasive, *e.g.* $\mathfrak{T}_3$ has $\mathfrak{L}$ for α-limit set and the col for ω-limit set. The remaining trajectories of $\mathfrak{L}$ have $\mathfrak{L}$ as α- and ω-limit set and are *pervasive*, the pervasiveness being non-uniform because $\mathfrak{L}$ is not a minimal set. The focal trajectories are asymptotic, having the focus as α-limit set; the ω-limit set is for one of them the col, and for the remainder $\mathfrak{L}$.

Sufficient criteria for the occurrence of this case will be given in §§ 7.1, 7.2.

Case II. *Grey intervals exist on* $\mathcal{C}$, *and no black interval has a grey interval abutting on either side.* As in Case I, the col and $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ belong to $\mathfrak{L}$. The circumstances are as in Case I, except for the existence of grey trajectories. These are not pervasive, but of asymptotic type, with $\mathfrak{L}$ as α- and ω-limit set. It will be proved in § 7.2 that this case cannot occur when $\gamma = \frac{1}{2}\pi$.

Case III. *Every black interval has a grey interval abutting on one side only.* The limit set of each species of trajectory is as in Case II, except that one of $\mathfrak{T}_1, \mathfrak{T}_2$ does not belong to $\mathfrak{L}$, and is of asymptotic type. It will be proved in § 7.2 that this case cannot occur when $\gamma \geqslant \frac{1}{2}\pi$.

Case IV. *Every black interval has a grey interval abutting on both sides.* The col and the associated trajectories $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ do not now belong to $\mathfrak{L}$, which is a minimal set of uniformly pervasive trajectories; while $\mathfrak{T}_1, \mathfrak{T}_2, \mathfrak{T}_3$ are of asymptotic type. It will be proved in § 7.2 that (with our hypothesis that z is an analytic function) this case can never occur.

Anticipating the results of § 7 we can say:

The common limit-set referred to in the enunciation of Theorem V always includes the col and two or three of the trajectories that are asymptotic thereto, and it includes $2^{\aleph_0}$ *other trajectories which are non-uniformly pervasive and of discontinuous type.*

7. *On the existence of grey intervals and their abuttal on black intervals.*

7.1. The results under this heading which have been stated in § 6.6 are proved by the method of Denjoy, 4, and rest upon Lemmas VIII and X.

Take any simple guide-curve $\mathcal{G}$ on the torus other than that through the col. On this there is a positive sense corresponding to that of its plane representation, and an angular variable s which is the arc-length in the plane representation. It is convenient to take the zero of this at the point B_0, where the trajectory $\mathcal{T}_3$ first cuts $\mathcal{G}$ on its passage from the col back to x-negative infinity.

The successive positive transforms $s_1, s_2, \ldots$ of any point s_0 of $\mathcal{G}$, *i.e.* the points at which the (proper) trajectory through s_0 cuts $\mathcal{G}$ after successive unit increases in z, are given by

$$s_{n+1} = f(s_n) \quad (n = 0, 1, \ldots),$$

where the properties of $f(s)$ have been given in § 3.4. $f(s)$ is determinate unless $s = mS$, with m an integer. If P, Q are any points of $\mathcal{G}$, such that the open arc I_0 in the positive sense from P to Q does not contain B_0, we can attach to these points values s_P, $s_Q > s_P$, such that no multiple of S lies between them; the first transform of I_0 is an open arc I_1, and the lengths of the plane representations of I_1, I_0 are related by

$$I_1 = I_0 f'(\bar{s}_0),$$

where $s_P < \bar{s}_0 < s_Q$. This relation is still valid if either or both of P, Q is at B_0. By repetition of this we get a relation between the lengths of I_n, the n-th positive transform of I_0, and I_0 in the plane representation, provided that B_0 lies within none of $I_0, I_1, \ldots, I_{n-1}$. If I_0 is a grey or proper black interval, this condition for B_0 is satisfied for all $n \geqslant 0$ (Lemma X), and we have

$$I_{n+1} = I_n f'(\bar{s}_n), \quad (n = 0, 1, 2, \ldots), \tag{13}$$

where $\bar{s}_n$ is a point inside I_n. Moreover, if I_0 is a grey interval, its successive negative transforms $I_{-1}, I_{2-}, \ldots$ are all grey intervals, and choosing any negative integer m we can apply (13) for $n = m, m+1, m+2, \ldots$.

Since the transformation from the torus to the plane is regular the number I_n in (13) tends to 0 as $n \to \infty$ if I_0 is a black or grey interval; and, if I_0 is a grey interval, $I_n \to 0$ as $n \to -\infty$. There follows immediately

THEOREM VI. *If $f'(s) \leqslant 1$ for all values of s, no grey intervals exist, and the circumstances are those of* § 6.6, *Case I. This case can occur only if $\gamma \geqslant \frac{1}{2}\pi$, where γ is the angle (in the plane) at a col between the two guide-curve branches which unite to form the associated loop.*

7.2. Let $B_1, B_2, \ldots$ be the black intervals in which the black trajectory successively cuts $\mathcal{G}$ as z increases from its value at the col; in the conven-

tional sense introduced in §6.1, but not in the analytic sense, B_1 is the first positive transform of B_0, and B_2, B_3, ... are in both senses the successive transforms of B_1. Let $J_0^{(i)}$, ..., $J_{q_i}^{(i)}$ be the open intervals defined in §6.3, Lemma VIII, with $P_n = B_n$ (Fig. 9). Then $J_0^{(i)}$, ..., $J_{q_i-1}^{(i)}$ have no common

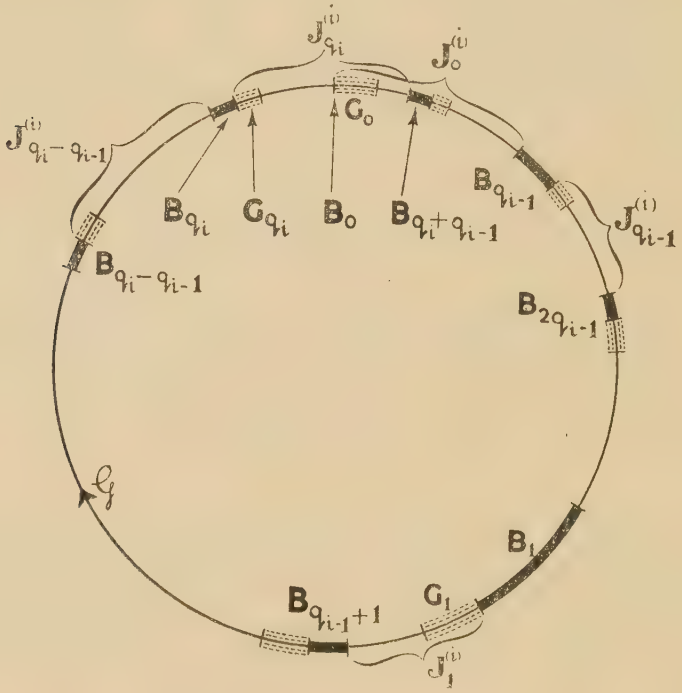


Fig. 9. *Order relations between black intervals on torus.*

The arrow head shows the positive sense on $\mathcal{C}_f$.

points, and $J_{q_i}^{(i)}$ overlaps $J_0^{(i)}$ with B_0 in common. For definiteness, suppose that the parity of i is such that $J_0^{(i)}$ extends in the positive sense from B_0 to $B_{q_{i-1}}$, so that we can describe $B_{q_{i-1}}$ as *above* B_0.

Let G_0, H_0 be any open sub-intervals of $J_0^{(i)}$, with all points of H_0 above G_0. Their successive positive transforms G_n, H_n lie then in the $J_n^{(i)}$ with H_n above G_n, for $n = 0, 1, \ldots, q_i$, and from (13) we have

$$G_{n+1} = G_n f'(g_n), \quad H_{n+1} = H_n f'(h_n) \quad (n = 0, 1, \ldots, q_i - 1),$$

where g_n, h_n $(0 < g_n < S,\ 0 < h_n < S)$ are suitably chosen points of G_n, H_n. Hence

$$\log \frac{H_{q_i} G_0}{H_0 G_{q_i}} = \sum_{n=0}^{q_i-1} \{\log f'(h_n) - \log f'(g_n)\}. \tag{14}$$

If we remove the curly brackets and arrange the terms on the right of (14) in the order of increasing magnitude of the arguments g_n, h_n, this becomes

$$-\log f'(g_0)+\log f'(h_0)-\ldots-\log f'(g_{q_i-q_{i-1}})+\log f'(h_{q_i-q_{i-1}}), \qquad (15)$$

where the prefixed signs are alternately $-$ and $+$; for h_n is above g_n in $J_n^{(i)}$, and $J_0^{(i)}$, ..., $J_{q_i-1}^{(i)}$ do not overlap.

At the two ends of the range $(0, S)$ which contains the g_n, h_n, we have (§ 3 . 4)

$$\left.\begin{array}{lll} \text{if} \quad \gamma<\frac{1}{2}\pi, \quad \log f'(s)\to\infty & \text{as} \quad s\to 0,\ S; \\ \text{if} \quad \gamma=\frac{1}{2}\pi, \quad \log f'(s)\to\text{finite limits} & \text{as} \quad s\to 0,\ S; \\ \text{if} \quad \gamma>\frac{1}{2}\pi, \quad \log f'(s)\to-\infty & \text{as} \quad s\to 0,\ S. \end{array}\right\} \qquad (16)$$

THEOREM VII. (i) *Grey intervals cannot abut on a black interval on both sides.*

(ii) *If $\gamma>\frac{1}{2}\pi$, a grey interval cannot abut on a black interval on either side.*

(iii) *If $\gamma=\frac{1}{2}\pi$, there are no grey intervals.*

Proof of (i), (ii). Suppose that B_0 has a grey interval G_0 abutting on it from above (Fig. 9). Then G_0 belongs to $J_0^{(i)}$, and any H_0 in $J_0^{(i)}$ which has no points common with G_0 must be above G_0. Since $J_{q_i}^{(i)}$ contains B_0 and is bounded by black intervals, it contains G_0. Hence we can choose H_0 as that grey component of $J_0^{(i)}$ for which

$$H_{q_i}=G_0;$$

so that

$$H_0=G_{-q_i},$$

and the left-hand side of (14) becomes

$$\log\frac{G_0^{\,2}}{G_{-q_i}G_{q_i}}.$$

This choice of H_{q_i} is possible for all i of that parity for which $J_0^{(i)}$ lies above B_0, say for $i=i_0$, i_0+2, i_0+4, ...

Now the modulus of the sum (15) does not exceed the total variation of $\log f'(s)$ in $g_0\leqslant s\leqslant h_{q_i-q_{i-1}}$. g_0 is here a number independent of i, and *if a grey interval abuts on B_0 from below*, with its lower end at $s=k$, B_{q_i} must lie below this, and since $h_{q_i-q_{i-1}}$ is below B_{q_i} we have

$$h_{q_i-q_{i-1}}<k<S.$$

Hence $$\left|\log \frac{G_0^2}{G_{-q_i} G_{q_i}}\right| \leqslant A \quad (i = i_0, i_0+2, i_0+4, \ldots), \quad (17)$$

where A is the total variation of $\log f'(s)$ in $g_0 \leqslant s \leqslant k$, and is finite since $f(s)$ is regular in this interval. The relation (17) is inconsistent with $G_n \to 0$ as $n \to \infty$ and as $n \to -\infty$, and we have proved (i).

Suppose now that B_0 has the grey interval G_0 abutting from above, but no grey interval abutting from below. Then the arguments in the end-terms of (15) may be arbitrarily near to S. By § 3.4, $\log f'(s)$ has a finite number only of turning points in $0 < s < S$; let $s = l$ be the greatest of them. Then the terms in (15) whose arguments do not exceed l give a sum whose modulus does not exceed $A+B$, where A is the total variation of $\log f'(s)$ in $g_0 \leqslant s \leqslant l$ and B is the upper bound of $|\log f'(s)|$ in this interval.

The remaining terms in (15), whose arguments exceed l, may be grouped in pairs from the end, where there may be one term, to which the sign $+$ is prefixed, left over. If $\gamma < \frac{1}{2}\pi$, (16) shows that $\log f'(s)$ increases with s from a finite value at $s = l$ to $+\infty$ at $s = S$, so that the groups are all positive, and the odd term, if it exists, exceeds $\log f'(l)$. Hence

$$\log \frac{G_0^2}{G_{-q_i} G_{q_i}} \geqslant -B-(A+B). \quad (18)$$

If $\gamma > \frac{1}{2}\pi$, (16) shows that $\log f'(s)$ decreases from a finite value at $s = l$ to $-\infty$ at $s = S$, so that the groups are all negative, and the odd term, if it exists, is less than $\log f'(l)$. Hence

$$\log \frac{G_0^2}{G_{-q_i} G_{q_i}} \leqslant B+(A+B). \quad (19)$$

The relations (18), (19) hold for all i of the same parity as i_0; and (19), but not (18), is inconsistent with $G_n \to 0$ as $n \to \infty$ and $n \to -\infty$. Hence, if $\gamma > \frac{1}{2}\pi$, and no grey interval abuts on B_0 from below, a grey interval cannot abut thereon from above.

The corresponding statement with the words "above" and "below" interchanged is proved by taking i to be of the opposite parity to i_0. Hence (ii) is proved.

The proof of (iii) rests on the facts that $\log f'(s)$ is of bounded variation in $0 < s < S$ when $\gamma = \frac{1}{2}\pi$, and that, by the definition of a grey interval and Lemma X, none of the positive or negative transforms of such an interval contains B_0. If G_0 is a definite grey interval, we employ the preceding reasoning with $J_0^{(i)}; \ldots; J_{q_i}^{(i)}$ bounded by $G_0, G_{q_{i-1}}; \ldots; G_{q_i}, G_{q_i+q_{i-1}}$ instead of by the B's, and the situation of B_0 among the G's is not relevant.

7.3. The deeper examination of the cases $\gamma \neq \frac{1}{2}\pi$ must be based on some knowledge of the relative magnitudes of those g_n, h_n that lie near $s = 0$, S, and not merely on their relative order. One line of attack is to allow an irrational type-number α to be approached through a sequence of rational α by variation of the parameter β or λ (§ 5, and in particular § 5.24). Hence we may prove that: *if when λ lies at the upper (lower) end λ_0 of a range in which α is rational the only periodic trajectory is irregularly periodic (cf. Fig. 7), then for those values of λ greater (less) than and sufficiently near to λ_0 for which α is irrational, the circumstances are those of Case I, i.e. no grey domain exists.* This circumstance can occur only when $\gamma \geqslant \frac{1}{2}\pi$; and I think it probable that, for any given function $z(x, y, \lambda)$ for which $\gamma \geqslant \frac{1}{2}\pi$, it always occurs for those rational type-numbers whose (terminating) simple continued fraction developments proceed to a sufficient number of terms.

8. *Example of the preceding theory.*

8.1. Let u, v, w be rectangular coordinates in space. With a view to finishing with differential equations

$$\frac{du}{dt} = U, \qquad \frac{dv}{dt} = V, \qquad \frac{dw}{dt} = W$$

in which U, V, W are polynomials in u, v, w, we start with the family of algebraic tori specified in terms of parameters x, y by

$$u = \frac{b\lambda \cos x}{1+\lambda' \cos(\lambda y/\lambda')}, \quad v = \frac{b\lambda \sin x}{1+\lambda' \cos(\lambda y/\lambda')}, \quad w = \frac{b\lambda' \sin(\lambda y/\lambda')}{1+\lambda' \cos(\lambda y/\lambda')}, \tag{20}$$

where $$b > 0, \quad 0 \leqslant \lambda \leqslant 1, \quad \lambda' = +\sqrt{(1-\lambda^2)}.$$

λ is the parameter of the family. For $0 < \lambda < 1$ the surface is a torus enclosing the core-circle $u^2+v^2 = b^2$, $w = 0$ $(\lambda = 1)$; while for $\lambda = 0$ we get the axis $u = v = 0$. The cartesian equation of the family is

$$\lambda^2(u^2+v^2+w^2+b^2)^2 = 4b^2(u^2+v^2). \tag{21}$$

Since, for λ constant, $du^2+dv^2+dw^2 = (u^2+v^2)(dx^2+dy^2)$, each torus is in conformal correspondence with the Cartesian xy-plane.

The guide-curves are to be the sections of the tori by the family of equiangular spiral cylinders given in terms of a parameter θ by

$$u = a+be^{z-k\theta}\cos\theta, \quad v = be^{z-k\theta}\sin\theta \quad (a > 0,\ k > 0). \tag{22}$$

z is the parameter of the family. The cylinders have the algebraic differential equation

$$\frac{du}{dv} = \frac{k(u-a)+v}{kv-(u-a)}. \tag{23}$$

The singular points of the guide-curves (col and centre) are to arise by tangencies of two of the cylinders with a torus. Contacts occur only in the plane $w = 0$, and their locus is the circle

$$u^2+v^2-au+av/k = 0.$$

We write $a/b = \mu,$

and choose $$\mu < \frac{k}{\sqrt{(1+k^2)}}, \tag{24}$$

so that this circle shall lie inside the core-circle. It is then found that:

I. The tori for which

$$\lambda_2 = \frac{2k\mu\sqrt{(1+k^2)}}{k^2+\mu^2(1+k^2)} < \lambda < 1,$$

i.e. those nearest the core, have no contacts with the spirals.

II. The tori for which

$$\lambda_1 = \frac{2\mu}{1+\mu^2} < \lambda < \lambda_2 = \frac{2k\mu\sqrt{(1+k^2)}}{k^2+\mu^2(1+k^2)} \tag{25}$$

have two contacts with the spirals and are not cut by their axis (Fig. 2).

III. The tori for which

$$0 < \lambda < \lambda_1$$

have two contacts with the spirals and are cut by their axis, so that there are four singularities of the corresponding families of guide-curves, two of them being foci.

It is with tori of the set II that we are concerned. It is apparent from Fig. 2 that the topological properties of the guide curves on any such torus are as demanded in § 2.1. This may, of course, be verified from their

xy-equation, obtained by eliminating u, v, θ from (20), (22). We find

$$\cot\theta = \frac{\lambda\cos x - \mu\left(1+\lambda'\cos(\lambda y/\lambda')\right)}{\lambda\sin x}, \tag{26}$$

$$\begin{aligned}\operatorname{cosec}^2\theta &= \frac{\lambda^2 - 2\lambda\mu\cos x\left(1+\lambda'\cos(\lambda y/\lambda')\right)+\mu^2\left(1+\lambda'\cos(\lambda y/\lambda')\right)^2}{\lambda^2\sin^2 x} \\ &= \frac{E}{\lambda^2\sin^2 x}, \text{ say};\end{aligned}$$

giving

$$z = k\theta + \tfrac{1}{2}\log E - \log\left(1+\lambda'\cos(\lambda y/\lambda')\right). \tag{27}$$

If we start with the determination $\theta = 0$ when $x = 0$, θ is, by (26), a single-valued regular function of x, y which takes the value $m\pi$ when $x = m\pi$ (m integral). The lower bounds of E, $\left(1+\lambda'\cos(\lambda y/\lambda')\right)$ are positive in virtue of (24), (25), and so z is single-valued and regular for real x, y, with

$$\left.\begin{aligned} z(x+2\pi,\, y) &= z(x,\, y)+2\pi k, \\ z(x,\, y+2\pi\lambda'/\lambda) &= z(x,\, y). \end{aligned}\right\}$$

The properties initially demanded in § 2 . 1 are satisfied if we replace x, y, z by $x/(2\pi)$, $y/(2\pi)$, $z/(2\pi k)$, the constant c being λ'/λ.

8 . 2. We have

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\lambda\left\{k\lambda+\mu(\sin x - k\cos x)\left(1+\lambda'\cos(\lambda y/\lambda')\right)\right\}}{E}, \\ \frac{\partial z}{\partial y} &= -\frac{\lambda^2\sin(\lambda y/\lambda')\left\{\mu(k\sin x+\cos x)\left(1+\lambda'\cos(\lambda y/\lambda')\right)-\lambda\right\}}{E\left(1+\lambda'\cos(\lambda y/\lambda')\right)}. \end{aligned}\right\} \tag{28}$$

Under the conditions (24) and (25), we have $\partial z/\partial x > 0$ when $y = \lambda'\pi/\lambda$, and the only real points where $\partial z/\partial x$, $\partial z/\partial y$ simultaneously vanish are given by

$$y \equiv 0 \pmod{2\pi\lambda'/\lambda}, \quad \sin x - k\cos x = -\frac{k\lambda}{\mu(1+\lambda')}.$$

If we put

$$\cos\phi = \frac{1}{\sqrt{(1+k^2)}}, \quad \sin\phi = \frac{k}{\sqrt{(1+k^2)}}, \quad \sin\psi = \frac{\lambda\sin\phi}{\mu(1+\lambda')},$$

so that

$$0 < \phi < \psi < \tfrac{1}{2}\pi, \tag{29}$$

the solutions are

$$(A) \quad y = 0, \quad x = -\pi+\phi+\psi,$$

$$(B) \quad y = 0, \quad x = \phi-\psi.$$

In the neighbourhood of the point A, we put $x = -\pi+\phi+\psi+\xi$, and the expansion of z is, as far as the quadratic terms,

$$z = -k\pi+k\psi+\log\frac{\mu\sin(\phi+\psi)}{\sin\phi}+\frac{\lambda\mu y^2\sin^2\psi}{\lambda'\sin 2\phi\sin(\phi+\psi)}-\frac{\xi^2\sin 2\psi\sin\phi}{4\cos\phi\sin^2(\phi+\psi)};$$

so at this point z is a minimum for variation of y alone, and a maximum for variation of x alone. The point is a col, the tangents whereat make with the x-axis the equal angles $\frac{1}{2}\gamma$ given by

$$\tan^2\tfrac{1}{2}\gamma = \frac{\lambda'}{(1-\lambda')(1+\cot\phi\tan\psi)}.$$

As λ decreases from λ_2 to λ_1, γ increases from 0 to the value γ_0 given by

$$\tan^2\tfrac{1}{2}\gamma_0 = \frac{1-\mu^2}{4\mu^2},$$

and $\gamma_0 > \frac{1}{2}\pi$ if $\mu^2 < \frac{1}{5}$.

In the neighbourhood of the point B we put $x = \phi-\psi+\xi$, and the expansion of z is, as far as the quadratic terms,

$$z = -k\psi+\log\frac{\mu\sin(\psi-\phi)}{\sin\phi}+\frac{\lambda\mu y^2\sin^2\psi}{\lambda'\sin 2\phi\sin(\psi-\phi)}+\frac{\xi^2\sin 2\psi\sin\phi}{4\cos\phi\sin^2(\psi-\phi)};$$

so z is here a minimum. The analytical verification of the conditions demanded in § 2.1 is now complete.

From (29) we have $\phi-\psi > -\pi+\phi+\psi$, so that the minimum point B lies on the side of x-increasing from the minimax A; and it is easy to verify that $z_A > z_B$, and thence that the curve through A must have a loop surrounding B. Thus B is the centre associated with the col A.

8.3. We now regard the aggregate of the trajectories on all the tori for $0 \leqslant \lambda \leqslant 1$ as one family, properly to be specified by differential equations in u, v, w. For the trajectories on the torus λ the differential equations are

$$\frac{dx}{dt} = \frac{\partial z}{\partial x}\cos\beta-\frac{\partial z}{\partial y}\sin\beta, \quad \frac{dy}{dt} = \frac{\partial z}{\partial y}\cos\beta+\frac{\partial z}{\partial x}\sin\beta, \tag{30}$$

where $\partial z/\partial x$, $\partial z/\partial y$ are given by (28), and β is an arbitrary function of λ. The system in u, v, w is to have (21) as an integral, λ being the arbitrary

constant, and is to be reduced to (30) under the transformation (20) [which implies (21)]. The system is hence

$$\left.\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial x}\left(\frac{\partial z}{\partial x}\cos\beta - \frac{\partial z}{\partial y}\sin\beta\right) + \frac{\partial u}{\partial y}\left(\frac{\partial z}{\partial y}\cos\beta + \frac{\partial z}{\partial x}\sin\beta\right),\\
\frac{dv}{dt} &= \frac{\partial v}{\partial x}\left(\frac{\partial z}{\partial x}\cos\beta - \frac{\partial z}{\partial y}\sin\beta\right) + \frac{\partial v}{\partial y}\left(\frac{\partial z}{\partial y}\cos\beta + \frac{\partial z}{\partial x}\sin\beta\right),\\
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\left(\frac{\partial z}{\partial x}\cos\beta - \frac{\partial z}{\partial y}\sin\beta\right) + \frac{\partial w}{\partial y}\left(\frac{\partial z}{\partial y}\cos\beta + \frac{\partial z}{\partial x}\sin\beta\right),
\end{aligned}\right\} \qquad (31)$$

where we are to replace β by the chosen function of λ and express the right-hand sides in terms of u, v, w from (20).

The simplest and most illuminating method of calculating these equations is to use the geometrical definitions from which we started in § 8.1. Since the torus λ is represented conformally by (20) on the Cartesian xy-plane, and (30) specify a direction inclined at each point at an angle $\frac{1}{2}\pi - \beta$ to the (plane) guide-curve through the point, the corresponding direction on the torus is inclined at an angle $\frac{1}{2}\pi - \beta$ to the guide-curve thereon. Hence we proceed as follows:

From (21) the direction cosines (l, m, n) of the normal at (u, v, w) to the torus through this point are proportional to polynomials in u, v, w; we find that

$$\frac{l}{u(u^2+v^2-w^2-b^2)} = \frac{m}{v(u^2+v^2-w^2-b^2)} = \frac{n}{2w(u^2+v^2)} = \frac{\lambda}{2b\lambda'(u^2+v^2)}.$$

The guide-curve through this point has that perpendicular direction (l_1, m_1, n_1) to (l, m, n) which satisfies (23); we find that

$$\begin{aligned}
\frac{l_1}{2w(u^2+v^2)(ku-ka+v)} &= \frac{m_1}{2w(u^2+v^2)(kv-u+a)}\\
&= \frac{n_1}{(w^2+b^2-u^2-v^2)(ku^2+kv^2-kau+av)},\\
&= \frac{1}{R} \quad \text{say.}
\end{aligned}$$

The direction cosines (l_2, m_2, n_2) of the perpendicular to (l_1, m_1, n_1) in the surface of the torus through (u, v, w) are given by $l_2 = m_1 n - m n_1, \ldots$; and the required differential equations are

$$\frac{du}{l_1\sin\beta - l_2\cos\beta} = \frac{dv}{m_1\sin\beta - m_2\cos\beta} = \frac{dw}{n_1\sin\beta - n_2\cos\beta}.$$

When we remove some common factors, and in particular the irrational function R, these become $du/U = dv/V = dw/W$, where

$$U = 4bw(u^2+v^2)^2\,(v+ku-ka)(\lambda'/\lambda)\tan\beta - 4w^2(u^2+v^2)^2(kv-u+a)$$
$$-v(u^2+v^2-w^2-b^2)^2\,(ku^2+kv^2-kau+av),$$

$$V = 4bw(u^2+v^2)^2\,(kv-u+a)(\lambda'/\lambda)\tan\beta + 4w^2(u^2+v^2)^2(ku-ka+v)$$
$$+u(u^2+v^2-w^2-b^2)^2\,(ku^2+kv^2-kau+av),$$

$$W = -2(u^2+v^2)(u^2+v^2-w^2-b^2)\{b(ku^2+kv^2-kau+av)(\lambda'/\lambda)\tan\beta$$
$$+w(u^2+v^2-au-kav)\}.$$

These values check with the ratios $du : dv : dw$ given by (31) with use of (20) and (28).

The parameter β is at our disposal as a function of λ. We choose it so as to remove the irrational function λ'/λ and at the same time to ensure the appearance of irrational type-numbers as λ varies over the range (25). For the latter purpose we make β take all values in $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ as λ varies over a closed range interior to (25); for the former purpose we make $(\lambda'/\lambda)\tan\beta$ a rational function of λ^2, and hence by (21) of u, v, w. For example, with λ_1, λ_2 defined as in (25), put

$$\lambda_3 = \frac{2\lambda_1+\lambda_2}{3}, \qquad \lambda_4 = \frac{\lambda_1+2\lambda_2}{3}$$

and

$$\frac{\lambda'}{\lambda}\tan\beta = \frac{1}{\lambda^2-\lambda_3{}^2} + \frac{1}{\lambda^2-\lambda_4{}^2}.$$

Substituting for λ^2 from (21), we have

$$U : V : W = U_1 : V_1 : W_1,$$

where U_1, V_1, W_1 are polynomials in u, v, w, actually of the fifteenth degree. The remaining constants a, b, k are absolute, and are subject only to the inequality (24), viz.

$$0 < a/b < k/\sqrt{(1+k^2)}.$$

Hence our final conclusion:

THEOREM VIII. *On a family of algebraic tori in space in which u, v, w are rectangular coordinates it is possible to specify curves, some of which are non-uniformly pervasive with a limit set which is perfect and nowhere dense,*

by means of differential equations

$$\frac{du}{U} = \frac{dv}{V} = \frac{dw}{W},$$

in which U, V, W are polynomials in u, v, w.

References.

1. G. D. Birkhoff, "Quelques théorèmes sur le mouvement des systèmes dynamiques", *Bull. de la Soc. Math. de France*, 40 (1912), 305.
2. ———, "Divergente Reihen und singuläre Punkte gewöhnlicher Differentialgleichungen", *Sitz. Preuss. Akad. Wiss.* (Phys. Math. Klasse) (1929), 171.
3. T. M. Cherry, "Topological properties of solutions of ordinary differential equations", *Amer. J. of Math.*, 59 (1937), 957.
4. A. Denjoy, "Sur les caractéristiques à la surface du tore", *Comptes rendus*, 194 (1932), 830.
5. H. Dulac, "Recherches sur les points singuliers des équations différentielles", *Journ. de l'Ecole Polytechnique* (2), 9 (1904), 1.
6. ———, "Sur les cycles limites", *Bull. de la Soc. Math. de France*, 51 (1923), 45.
7. E. E. Levi, "Sur les équations différentielles périodiques", *Comptes rendus*, 153 (1911), 799.
8. H. Poincaré, "Mémoire sur les courbes définies par une équation différentielle", *Journal de Math.* (3), 7 (1881), 375.
9. ———, "Sur les courbes définies par les équations différentielles", *Journal de Math.* (4), 1 (1885), 167.

The University,
Melbourne, Victoria.

AN EXTENSION OF POINCARÉ'S THEOREM ON DEFECTIVE INTEGRALS

By W. V. D. HODGE.

[Received 10 August, 1937.—Read 18 November, 1937.]

Poincaré's theorem* on defective integrals of the first kind on a curve of genus p establishes the fact that, if there are n integrals with only $2n$, instead of $2p$, columns of periods, there is a complementary set of $p-n$ integrals with only $2(p-n)$ columns of periods. The extension of this theorem to the algebraic simple integrals of the first kind which exist on an algebraic variety of any number of dimensions is immediate. But on an algebraic variety V_d of d dimensions we have to consider, not only simple integrals, but algebraic integrals of multiplicity m, where $1 < m \leqslant d$, and it is possible to extend Poincaré's theorem so as to give a result which is of importance (for example, in the theory of correspondences) for algebraic multiple integrals which are finite everywhere on V_d.

Just as it is a simple matter to extend Poincaré's original theorem for curves to simple integrals on a variety of d dimensions, by considering the curve sections of the variety by a linear space, so, when we have made the extension to m-fold integrals on a variety V_m, we can immediately deduce it for m-fold integrals on a V_d, where $d > m$. Therefore, it is only necessary to prove the extension in the case of m-fold integrals on a V_m. The case $m = 2$ illustrates the nature and the method of the extension adequately, so we shall confine ourselves to this case, adding at the end a note on some points which arise in the proof of the extension for a general value of m.

1. We begin by expressing Poincaré's theorem for curves in the form which is most suitable for our extension. Let $u_1, \ldots, u_p$ be the integrals of the first kind on a curve of genus p, and let the period matrix of the

* H. Poincaré, *American J. of Math.*, 8 (1886), 289.

integrals $u_1, \ldots, u_n$, for a canonical set of cycles $\gamma_1, \ldots, \gamma_{2p}$, be $\boldsymbol{\omega}\mathbf{a}$, where $\boldsymbol{\omega}$ is a $(n, 2n)$ Riemann matrix, and $\mathbf{a}$ is a $(2n, 2p)$ matrix of integers. The matrix $\mathbf{a}$ is of rank $2n$. It is therefore possible to find a $(2n-2p, 2p)$ matrix $\mathbf{b}$ of integers such that

$$\mathbf{k} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$$

is not singular (and has determinant k). Let us take a new base (not now canonical) $\delta_1, \ldots, \delta_{2p}$ for the cycles on the Riemann surface, given by

$$k\gamma_i \sim \sum_{j=1}^{2p} k_{ji}\delta_j.$$

Then the period matrix of the integrals $u_1, \ldots, u_n$ for the cycles δ_i is $(k\boldsymbol{\omega}, 0)$, where 0 is a $(n, 2p-2n)$ matrix whose elements are zero. The n integrals have, therefore, zero periods on the cycles $\delta_{2n+1}, \ldots, \delta_{2p}$. The $2p-2n$ cycles are determined uniquely (save that they can be replaced by linear combinations of themselves), while the $2n$ cycles $\delta_1, \ldots, \delta_{2n}$ are not determined uniquely, owing to the fact that $\mathbf{b}$ is not unique. Poincaré's theorem amounts to saying that we can choose (a) the cycles $\delta_1, \ldots, \delta_{2n}$ and (b) the integrals $u_{n+1}, \ldots, u_p$, so that the p independent integrals $u_1, \ldots, u_p$ have period matrix of the form

$$\begin{pmatrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_2 \end{pmatrix},$$

where $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are Riemann matrices of type $(n, 2n)$ and $(p-n, 2p-2n)$ respectively*, with regard to the cycles $\delta_1, \ldots, \delta_{2p}$. Since

$$k\gamma_{2n+i} \sim \sum_{j=1}^{2p} b_{ji}\delta_j \quad (i = 1, \ldots, 2p-2n),$$

the period matrix of the integrals $u_{n+1}, \ldots, u_p$ so chosen with regard to the canonical cycles $\gamma_1, \ldots, \gamma_{2p}$ is $k^{-1}\boldsymbol{\omega}_2\mathbf{b}$. The proof of these statements is an immedate consequence of Poincaré's theorem. It is not necessary to give it here, since the proof given below in §§ 4, 5 for double integrals is exactly similar.

2. Now consider an algebraic surface F, and consider its Riemannian. This Riemannian can be constructed in the way described in my paper,

* Here, as in the remainder of the paper, " 0 " in the representation of a matrix will denote a rectangular array of zeros, the number of rows and columns being such as to complete the matrix.

"Harmonic integrals associated with algebraic varieties"*, and this we shall suppose done. Among the 2-cycles on this are those known as algebraic cycles, which are characterised by the fact that the algebraic double integrals of the first kind have zero periods on them. Then, if there are ρ independent algebraic cycles, it is known† that $r_2 = R_2 - \rho$ independent 2-cycles can be chosen, each of which has zero intersection number with each algebraic cycle. Such cycles may be called the transcendental 2-cycles. When the surface F is subject to a birational transformation, algebraic 2-cycles on the Riemannian may be created or destroyed, but the transcendental 2-cycles are invariant. On the Riemannian M of F we can introduce a metric, and by means of this metric harmonic integrals, in the manner described in the paper just quoted. I have shown‡ that there are r_2 independent harmonic integrals, consisting of the real and imaginary parts of the p_g algebraic integrals of the first kind

$$\int R_i dx\, dy \quad (i = 1, \ldots, p_g),$$

and $r_2 - 2p_g$ integrals which can be written in the form

$$\int (A\, dx\, d\bar{x} + B\, dx\, d\bar{y} + \bar{B}\, d\bar{x}\, dy + C\, dy\, d\bar{y}),$$

where x, y are complex parameters on F and $\bar{x}$, $\bar{y}$, $\bar{B}$ are the conjugate imaginaries of x, y, B, while A and C are pure imaginaries, which have periods only on the transcendental cycles. If

$$\int P, \quad \int Q$$

are any two of these r_2 integrals, which have periods

$$\omega_1, \ldots, \omega_{r_2},$$

$$\nu_1, \ldots, \nu_{r_2}$$

on the r_2 independent transcendental 2-cycles

$$\Gamma_1, \ldots, \Gamma_{r_2},$$

then

$$\int_M P \times Q = \sum_{i,j=1}^{r_2} \omega_i A_{ij} \nu_j, \tag{1}$$

where the matrix $\mathbf{A} = (A_{ij})$ is the inverse of the intersection matrix of the cycles Γ_i.

* W. V. D. Hodge, *Proc. London Math. Soc.* (2), 39 (1935), 249.

† S. Lefschetz, *L'analyse situs et la géométrie algébrique* (Paris, 1924), 130.

‡ W. V. D. Hodge, *Journal London Math. Soc.*, 12 (1937), 58.

Since the only integrals which we shall have to consider in this paper are linear combinations of these r_2 integrals, the only period cycles to be taken into account are the transcendental 2-cycles, and we may therefore make the convention in the remainder of this paper that the term "cycle" will mean "transcendental 2-cycle". In this sense cycles are absolute birational invariants.

3. Let us write the integrals under consideration as

$$\int P_i \qquad (i = 1, \ldots, r_2), \tag{2}$$

where

$$P_i = R_i dx\, dy \qquad (i = 1, \ldots, p_g), \tag{2.1}$$

$$P_{p_g+i} = \bar{R}_i d\bar{x}\, d\bar{y} \qquad (i = 1, \ldots, p_g), \tag{2.2}$$

$$P_{2p_g+i} = A_i dx\, d\bar{x} + B_i dx\, d\bar{y} + \bar{B}_i d\bar{x}\, dy + C_i dy\, d\bar{y} \quad (i = 1, \ldots, r_2 - 2p_g). \tag{2.3}$$

Write the period matrix of these integrals with respect to the cycles Γ_i in the form

$$\begin{pmatrix} \omega \\ \bar{\omega} \\ \lambda \end{pmatrix},$$

where ω is the (p_g, r_2) matrix of periods of the algebraic integrals, $\bar{\omega}$ as usual denotes its conjugate imaginary, and λ is a real $(r_2 - 2p_g, r_2)$ matrix. If we apply equation (1) to each pair of integrals (2) and observe that the product of a form (2.1) by a form (2.1) or (2.3) is zero, and similarly that the product of a form (2.2) by a form (2.2) or (2.3) is zero, we obtain the matrix equation

$$\begin{pmatrix} \omega \\ \bar{\omega} \\ \lambda \end{pmatrix} \mathbf{A} (\omega' \bar{\omega}' \lambda') = \begin{pmatrix} 0 & \alpha & 0 \\ \bar{\alpha} & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}, \tag{3}$$

where α is a *definite* hermitian matrix of type (p_g, p_g) and β is a *definite* real symmetric matrix of type $(r_2 - 2p_g, r_2 - 2p_g)$. In particular, we have the equations

$$\omega \mathbf{A} \lambda' = 0,$$

$$\bar{\omega} \mathbf{A} \lambda' = 0.$$

The matrix $\begin{pmatrix} \omega \\ \bar{\omega} \end{pmatrix}$ is of rank $2p_g$, and λ is of rank $r_2 - 2p_g$; it follows that, if $\mathbf{x} = (x_1, \ldots, x_{r_2})$ is any solution of

$$\omega \mathbf{A} \mathbf{x}' = 0, \quad \bar{\omega} \mathbf{A} \mathbf{x}' = 0,$$

then

$$\mathbf{x} = \mathbf{b} \lambda,$$

where $\mathbf{b}$ is a $(1, r_2-2p_g)$ matrix. In other words, $\boldsymbol{\lambda}$ is a matrix whose rows are r_2-2p_g independent *real* solutions of

$$\boldsymbol{\omega}\mathbf{A}\mathbf{x}' = 0,$$

and is defined by this property, uniquely save for a forward multiplier. This forward multiplier corresponds to replacing the integrals (2.3) by linear combinations of themselves, an operation which can be performed at any time without making any essential alteration of the argument. The matrix $\boldsymbol{\lambda}$ can therefore be defined in this way, as an absolute birational invariant of F, without any reference to the theory of harmonic integrals. This definition is sufficient for the extension of Poincaré's theorem which follows. But, on account of the application of this extension to the theory of correspondences, it is important to see that $\boldsymbol{\lambda}$ is the period matrix of a set of harmonic integrals of the form (2.3). For convenience, we shall retain the terminology of harmonic integrals.

4. Let us suppose that, if a suitable arrangement of the integrals (2) is made, the period matrix of the integrals

$$\int P_i \quad (i = 1, \ldots, n;\ i = p_g+1, \ldots, p_g+n;\ i = 2p_g+1, \ldots, 2p_g+\nu) \tag{4}$$

can be written in the form

$$\begin{pmatrix} \boldsymbol{\omega}_1 \\ \overline{\boldsymbol{\omega}}_1 \\ \boldsymbol{\lambda}_1 \end{pmatrix} \mathbf{a},$$

where $\boldsymbol{\omega}_1$ is a $(n, 2n+\nu)$ matrix, $\boldsymbol{\lambda}_1$ is a $(\nu, 2n+\nu)$ matrix, and $\mathbf{a}$ is a $(2n+\nu, r_2)$ matrix of integers. The matrix $\mathbf{a}$ is of rank $2n+\nu$, for otherwise the period matrix of the integrals (4) would be of rank less than $2n+\nu$, and it would follow that there would exist a linear combination of these integrals with all its periods zero, which is impossible since the integrals are harmonic. Hence we can find a matrix $\mathbf{b}$ of integers of type $(r_2-2n-\nu, r_2)$ such that

$$\mathbf{k} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$$

is a non-singular matrix, of determinant k. We now take a new set of period cycles $\Gamma_1', \ldots, \Gamma_{r_2}'$, where

$$k\,\Gamma_i \sim \sum_{j=1}^{r_2} k_{ji}\Gamma_j'.$$

Then the period matrix of the integrals (4) with regard to the new cycles is the $(2n+\nu, r_2)$ matrix

$$\begin{pmatrix} k\boldsymbol{\omega}_1 & 0 \\ k\bar{\boldsymbol{\omega}}_1 & 0 \\ k\boldsymbol{\lambda}_1 & 0 \end{pmatrix}.$$

Let the symmetrical intersection matrix of the cycles Γ_i' be written as

$$\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 \\ \mathbf{a}_3 & \mathbf{a}_4 \end{pmatrix},$$

where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are matrices of types $(2n+\nu, 2n+\nu)$, $(2n+\nu, r_2-2n-\nu)$, $(r_2-2n-\nu, 2n+\nu)$, $(r_2-2n-\nu, r_2-2n-\nu)$, and similarly write the inverse of this matrix as

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}.$$

The matrix $\mathbf{a}_3$ is the transpose of the matrix $\mathbf{a}_2$, and similarly $\mathbf{A}_3$ is the transpose of $\mathbf{A}_2$.

From (3) we deduce that

$$\begin{pmatrix} k\boldsymbol{\omega}_1 \\ k\bar{\boldsymbol{\omega}}_1 \\ \boldsymbol{\lambda}_1 \end{pmatrix} \mathbf{A}_1 (k\boldsymbol{\omega}_1', k\bar{\boldsymbol{\omega}}_1', \boldsymbol{\lambda}_1')$$

is non-singular; hence $\mathbf{A}_1$ is non-singular. Therefore $\mathbf{a}_4$ is non-singular by Jacobi's theorem on inverse determinants. Let its determinant be d. Now make a further change of base for the cycles to a new base $\Delta_1, \ldots, \Delta_{r_2}$, where

$$\Delta_i \sim \sum_{j=1}^{r_2} c_{ij} \Gamma_j',$$

and

$$\mathbf{c} = \begin{pmatrix} dE_{2n+\nu} & -d\mathbf{a}_2\mathbf{a}_4^{-1} \\ 0 & E_{r_2-2n-\nu} \end{pmatrix},$$

E_r being the unit matrix of order r.

Such a change of base replaces the period matrix of the integrals (4) by the matrix

$$\begin{pmatrix} dk\boldsymbol{\omega}_1 & 0 \\ dk\bar{\boldsymbol{\omega}}_1 & 0 \\ dk\boldsymbol{\lambda}_1 & 0 \end{pmatrix},$$

and the matrix $\mathbf{A}$ by one of the form

$$\begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_4 \end{pmatrix}.$$

5. Absorbing the unimportant multiplier dk in ω_1 and λ_1, we now write the period matrix of the integrals (2) in the form

$$\begin{pmatrix} \omega_1 & 0 \\ \omega_2 & \omega_3 \\ \bar{\omega}_1 & 0 \\ \bar{\omega}_2 & \bar{\omega}_3 \\ \lambda_1 & 0 \\ \lambda_2 & \lambda_3 \end{pmatrix},$$

where ω_2, ω_3 are matrices of types $(p_g-n,\ 2n+\nu)$ and $(p_g-n,\ r_2-2n-\nu)$, and λ_2, λ_3 are real matrices of types

$$(r_2-2p_g-\nu,\ 2n+\nu), \quad (r_2-2p_g-\nu,\ r_2-2n-\nu).$$

From equation (3) we have

$$\omega_1 \mathbf{A}_1 \omega_1' = 0, \quad \omega_2 \mathbf{A}_1 \omega_1' = 0,$$

$$\lambda_1 \mathbf{A}_1 \omega_1' = 0, \quad \lambda_2 \mathbf{A}_1 \omega_1' = 0.$$

Since ω_1' is of rank n_1 and $\mathbf{A}_1$ is non-singular, the matrix

$$\begin{pmatrix} \omega_1 \\ \lambda_1 \\ \omega_2 \\ \lambda_2 \end{pmatrix}$$

is of rank $n+\nu$ at most. But $\begin{pmatrix} \omega_1 \\ \lambda_1 \end{pmatrix}$ is of rank $n+\nu$. Hence, there exist matrices $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$, of types $(p_g-n,\ n)$, $(p_g-n,\ \nu)$, $(r_2-2p_g-\nu,\ n)$, $(r_2-2p_g-\nu,\ \nu)$, respectively, such that

$$\omega_2 = \mathbf{k}_1 \omega_1 + \mathbf{k}_2 \lambda_1,$$

$$\lambda_2 = \mathbf{k}_3 \omega_1 + \mathbf{k}_4 \lambda_1.$$

Now apply the relations (3). We have

$$0 = \omega_2 \mathbf{A}_1 \lambda_1' = \mathbf{k}_1 \omega_1 \mathbf{A}_1 \lambda_1' + \mathbf{k}_2 \lambda_1 \mathbf{A}_1 \lambda_1'.$$

But $$\omega_1 \mathbf{A}_1 \lambda_1' = 0, \quad |\lambda_1 \mathbf{A}_1 \lambda_1'| \neq 0;$$

hence $$\mathbf{k}_2 = 0.$$

Similarly,

$$0 = \lambda_2 \mathbf{A}_1 \bar{\omega}_1' = \mathbf{k}_3 \omega_1 \mathbf{A}_1 \bar{\omega}_1' + \mathbf{k}_4 \lambda_1 \mathbf{A}_1 \bar{\omega}_1'.$$

But $$|\boldsymbol{\omega}_1 \mathbf{A}_1 \bar{\boldsymbol{\omega}}_1'| \neq 0, \quad \boldsymbol{\lambda}_1 \mathbf{A}_1 \bar{\boldsymbol{\omega}}_1' = 0;$$

hence $$\mathbf{k}_3 = 0.$$

Therefore

$$\begin{pmatrix} E_n & 0 & 0 & 0 & 0 & 0 \\ -\mathbf{k}_1 & E_{p_g-n} & 0 & 0 & 0 & 0 \\ 0 & 0 & E_n & 0 & 0 & 0 \\ 0 & 0 & -\bar{\mathbf{k}}_1 & E_{p_g-n} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_\nu & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{k}_4 & E_{r_2-2p_g-\nu} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega}_1 & 0 \\ \boldsymbol{\omega}_2 & \boldsymbol{\omega}_3 \\ \bar{\boldsymbol{\omega}}_1 & 0 \\ \bar{\boldsymbol{\omega}}_2 & \bar{\boldsymbol{\omega}}_3 \\ \boldsymbol{\lambda}_1 & 0 \\ \boldsymbol{\lambda}_2 & \boldsymbol{\lambda}_3 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega}_1 & 0 \\ 0 & \boldsymbol{\omega}_3 \\ \bar{\boldsymbol{\omega}}_1 & 0 \\ 0 & \bar{\boldsymbol{\omega}}_3 \\ \boldsymbol{\lambda}_1 & 0 \\ 0 & \boldsymbol{\lambda}_3 \end{pmatrix}. \tag{5}$$

Hence the integrals which, with (4), make up the integrals (2), can be chosen so that the integrals (2) have, with respect to the cycles Δ_i, the period matrix of the form (5).

This generalization of Poincaré's theorem can now be stated in the following form which makes no reference to harmonic integrals:

If it is possible to choose the r_2 period cycles $\Delta_1, \ldots, \Delta_{r_2}$ of the double integrals of the first kind on a surface F so that

(*a*) *n of the integrals have zero periods on the cycles $\Delta_{2n+\nu+1}, \ldots, \Delta_{r_2}$,*

and (*b*) *the equations*

$$\boldsymbol{\omega} \mathbf{A} \mathbf{x}' = 0$$

have ν independent real solutions of the form $\mathbf{x} = (x_1, \ldots, x_{r_2})$ with

$$x_{2n+\nu+1} = \ldots = x_{r_2} = 0,$$

where $\boldsymbol{\omega}$ is the period matrix of the p_g double integrals on F, and $\mathbf{A}$ is the inverse of the intersection matrix of the r_2 cycles Δ_i, then we can choose the cycles $\Delta_1, \ldots, \Delta_{2n+\nu}$ and the remaining $p_g - n$ double integrals so that these integrals have zero periods on the cycles $\Delta_1, \ldots, \Delta_{2n+\nu}$.

When the property implied by our extension of Poincaré's theorem holds, the period matrix of the integrals (2) will be said to be *impure*, otherwise it is *pure*.

6. Let, now,

$$\sum_{j=1}^{n} a_{ij} P_j + \sum_{j=n+1}^{p_g} \beta_{ij} P_j \quad (i = 1, \ldots, n_1),$$

$$\sum_{j=2p_g+1}^{2p_g+\nu} \gamma_{ij} P_j + \sum_{j=2p_g+\nu+1}^{r_2} \delta_{ij} P_j \quad (i = 1, \ldots, \nu_1)$$

be a set of forms whose integrals have the period matrix

$$\begin{pmatrix} \Omega \\ \Lambda \end{pmatrix} (\mathbf{A}_1, \mathbf{A}_2),$$

where Ω, Λ are $(n_1, 2n_1+\nu_1)$, $(\nu_1, 2n_1+\nu_1)$ period matrices, and $\mathbf{A}_1$, $\mathbf{A}_2$ are matrices of integers of types $(2n_1+\nu_1, 2n+\nu)$, $(2n_1+\nu_1, r_2-2n-\nu)$. Then

$$\alpha\omega_1 = \Omega\mathbf{A}_1, \quad \gamma\lambda_1 = \Lambda\mathbf{A}_1,$$

$$\beta\omega_2 = \Omega\mathbf{A}_2, \quad \delta\lambda_2 = \Lambda\mathbf{A}_2.$$

If we apply the results of §§ 6, 7 of my paper on correspondences* we conclude that

$$\begin{pmatrix} \omega_1 \\ \bar{\omega}_1 \\ \lambda_1 \end{pmatrix}$$

is itself impure unless $n = n_1$, $\nu = \nu_1$, and α is of rank n, and a similar result holds for

$$\begin{pmatrix} \omega_2 \\ \bar{\omega}_2 \\ \lambda_2 \end{pmatrix},$$

Applying this result repeatedly, we show that the period matrix of the integrals (2) for any surface F can be resolved uniquely into the form

$$\begin{pmatrix} \omega_1 & \cdot & \cdot & \cdot \\ \cdot & \omega_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \omega_r \\ \bar{\omega}_1 & \cdot & \cdot & \cdot \\ \cdot & \bar{\omega}_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \bar{\omega}_r \\ \lambda_1 & \cdot & \cdot & \cdot \\ \cdot & \lambda_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \lambda_r \end{pmatrix},$$

where the sub-matrices

$$\begin{pmatrix} \omega_i \\ \bar{\omega}_i \\ \lambda_i \end{pmatrix}$$

are all pure.

* W. V. D. Hodge, *Proc. London Math. Soc.* (2), 44 (1938), 226–242.

7. There are one or two points to be noticed in connection with the extension of this result to integrals of multiplicity greater than *two*. In the first place, for $m > 2$ the resolution of the m-cycles on a V_m into algebraic cycles and transcendental cycles cannot be effected satisfactorily, since we know practically nothing of the cycles characterized by the fact that algebraic integrals of the first kind have zero periods on them. Moreover, the behaviour of m-cycles under birational transformations of the variety is not yet clearly understood. Hence we must, at present, take into account *all* the m-cycles on the variety, and the theorem becomes one of relative invariance, and not absolute invariance.

The remainder of the argument necessary in the general case is similar to that given in §§ 4, 5. We have to make use of the fact* that the harmonic m-fold integrals on a V_m can be resolved into $m+1$ sets, with the $(i+1)$-th set composed of integrals of the form

$$\int \sum_{j,\,k} A_{j_1 \dots j_{m-i} k_1 \dots k_i} dz^{j_1} \dots dz^{j_{m-i}} d\bar{z}^{k_1} \dots d\bar{z}^{k_i}.$$

Then, in place of a single matrix $\boldsymbol{\lambda}$, we have $m-1$ matrices (corresponding to the period matrices of the 2nd, ..., m-th sets) which are conjugate imaginaries in pairs. When m is even one of them may therefore be taken as real.

Pembroke College,
Cambridge.

* W. V. D. Hodge, *Proc. Cambridge Phil. Soc.*, 31 (1935), 18.

ALGEBRAIC CORRESPONDENCES BETWEEN SURFACES

By W. V. D. HODGE.

[Received 10 August, 1937.—Read 18 November, 1937.]

The foundations of the geometrical theory of correspondences between two algebraic surfaces have been laid by Severi in a series of papers†. The transcendental theory is not, however, so well advanced, and it is the purpose of this paper to consider certain aspects of such a theory. The general conclusion reached is that, while there is a general resemblance between this theory and the corresponding theory for curves, there are in some respects marked differences, and that until a certain problem, proposed in § 3, is solved, we cannot hope to make great progress along these lines. In this paper we assume a knowledge of the representation of correspondences by means of product varieties, and of the topological theory of correspondences‡.

The matter of notation causes a little difficulty: The two surfaces whose correspondence is to be examined will be denoted by F_1 and F_2. Any symbol, whether it denotes a number, matrix, cycle, or differential form, with a suffix 1 refers to the surface F_1, and the same symbol with the suffix 2 will denote the corresponding object associated with F_2. Upper indices will serve to distinguish different objects of the same class, *e.g.* the 1-cycles on F_1 will be denoted by $b_1{}^1$, $b_1{}^2$, ... Symbols without a suffix will, in general, denote objects symmetrically related to F_1 and F_2 arising out of the correspondence. Matrices will be printed in Clarendon type, and upper indices will sometimes be used to distinguish different matrices. The elements of a matrix will be written with upper indices; thus α^{ij} or $\alpha^{(1)ij}$ will denote elements of the matrices $\boldsymbol{\alpha}$ or $\boldsymbol{\alpha}^1$. The transpose of the matrix $\boldsymbol{\alpha}$ will be denoted by $\boldsymbol{\alpha}'$. The conjugate imaginary of a number x

† F. Severi, *Rend. Lincei* (6), 17 (1933) (seven notes); *Mem. Reale Acc. d'Italia*, 5 (1934), 239–283. The latter paper contains references to further papers, by Severi and others, on this subject.

‡ *Cf.* S. Lefschetz, *Topology* (American Math. Soc. Colloquium Lectures, 1930), 245.

will be denoted by $\bar{x}$, and the matrix whose elements are the conjugate imaginaries of the elements of $\boldsymbol{\alpha}$ will be denoted by $\bar{\boldsymbol{\alpha}}$.

1. Let q_1 denote the irregularity of F_1, p_1 its geometric genus, r_1 the number of independent transcendental 2-cycles† on F_1, ρ_1 the base-number for curves on F_1, *i.e.* the number of independent algebraic 2-cycles on F_1. We shall use F_1 to denote either the surface or its Riemannian manifold. Similarly, q_2, p_2, r_2, ρ_2 are defined on F_2. We shall denote by x_1 a point of F_1, and by b_1^i $(i = 1, \ldots, 2q_1)$ the 1-cycles of F_1, by c_1^i $(i = 1, \ldots, r_2)$ the transcendental 2-cycles, by C_1^i $(i = 1, \ldots, \rho_1)$ the algebraic 2-cycles, by d_1^i $(i = 1, \ldots, 2q_1)$ the 3-cycles of F_1. Similar definitions, with suffix 2, hold on F_2. On the product manifold $F_1 \times F_2$ there are

$$(r_1+\rho_1)(r_2+\rho_2)+8q_1q_2+2$$

independent 4-cycles, a base for which is given by

$$x_1 \times F_2, \quad b_1^i \times d_2^j, \quad c_1^i \times C_2^j, \quad C_1^i \times c_2^j, \quad c_1^i \times c_2^j, \quad C_1^i \times C_2^j,$$

$$d_1^i \times b_2^j, \quad F_1 \times x_2.$$

Of these, the $\rho_1\rho_2+2$ cycles

$$x_1 \times F_2, \quad C_1^i \times C_2^j, \quad F_1 \times x_2$$

are algebraic‡. In general, any algebraic 4-cycle on $F_1 \times F_2$ is related to these $\rho_1\rho_2+2$ cycles by a homology, but in special cases there may be algebraic 4-cycles which are independent of these. It is with this possibility that we are concerned in this paper.

Consider an algebraic (ϵ_1, ϵ_2) correspondence T between F_1 and F_2. Any r-cycle Γ of F_1 is transformed by T into an r-cycle of F_2 which we denote by $T(\Gamma)$. Similarly, any r-cycle of F_2 is transformed into an r-cycle of F_1 which we denote by $T^{-1}(\Gamma)$. The transformations of the cycles of

† These are 2-cycles which have zero intersection number with every algebraic 2-cycle, *cf.* S. Lefschetz, *L'analyse situs et la géométrie algébrique* (Paris, 1924), 130.

‡ An algebraic 4-cycle is a cycle homologous on $F_1 \times F_2$ to the image of a (possibly virtual) algebraic surface.

F can then be represented by the equations

$$T(x_1) \simeq \epsilon_2 x_2,$$

$$T(b_1) \simeq \mathbf{P}_1 b_2,$$

$$T(c_1) \simeq \mathbf{Q}_1{}^1 c_2 + \mathbf{Q}_1{}^2 C_2,$$

$$T(C_1) \simeq \mathbf{Q}_1{}^3 c_2 + \mathbf{Q}_1{}^4 C_2,$$

$$T(d_1) \simeq \mathbf{S}_1 d_2,$$

$$T(F_1) = \epsilon_1 F_2,$$

where, for instance, the matrix equation

$$T(b_1) \simeq \mathbf{P}_1 b_2$$

stands for $$T(b_1{}^i) \simeq \sum_{j=1}^{2q_2} \mathbf{P}_1{}^{ij} b_2{}^j \quad (i = 1, \ldots, 2q_1).$$

It is not desirable to restrict ourselves to fundamental bases for the cycles on F_1 and F_2. Moreover, we are only concerned with homology with division. Hence we may allow fractional coefficients to appear in the expressions for cycles, and in the above homologies the elements of the matrices $\mathbf{P}_1, \mathbf{Q}_1{}^i, \mathbf{S}_1$ may be rational fractions. Some of the cycles which appear in this paper may not be cycles in the ordinary sense, but may be fractional multiples of such cycles. Cycles of the ordinary type are then obtained by multiplying by suitable integers, and this can always be done without making any essential alteration to the form of the equations which we obtain.

In a similar manner we obtain the following equations for the cycles of F_1 which correspond to cycles of F_2:

$$T^{-1}(x_2) \simeq \epsilon_1 x_1,$$

$$T^{-1}(b_2) \simeq \mathbf{P}_2 b_1,$$

$$T^{-1}(c_2) \simeq \mathbf{Q}_2{}^1 c_1 + \mathbf{Q}_2{}^2 C_1,$$

$$T^{-1}(C_2) \simeq \mathbf{Q}_2{}^3 c_1 + \mathbf{Q}_2{}^4 C_1,$$

$$T^{-1}(d_2) \simeq \mathbf{S}_2 d_1,$$

$$T^{-1}(F_2) = \epsilon_2 F_1.$$

The correspondence is represented on $F_1 \times F_2$ by the 4-cycle T which is given by

$$T \simeq \epsilon_1 x_1 \times F_2 + \sum_{r,s} f^{rs} b_1^r \times d_2^s + \sum_{r,s} j^{rs} c_1^r \times C_2^s + \sum_{r,s} k^{rs} C_1^r \times c_2^s + \sum_{r,s} l^{rs} c_1^r \times c_2^s + \sum_{r,s} m^{rs} C_1^r \times C_2^s + \sum_{r,s} g^{rs} d_1^r \times b_2^s + \epsilon_2 F_1 \times x_2.$$

The matrices $\mathbf{f}$, $\mathbf{j}$, $\mathbf{k}$, $\mathbf{l}$, $\mathbf{m}$, $\mathbf{g}$ are given by the equations†

$$\left.\begin{aligned} \mathbf{f} &= (\boldsymbol{\xi}_1')^{-1}\mathbf{S}_1 = -\mathbf{P}_2'\boldsymbol{\zeta}_2^{-1}, \\ \mathbf{g} &= (\boldsymbol{\zeta}_1')^{-1}\mathbf{P}_1 = -\mathbf{S}_2'\boldsymbol{\xi}_2^{-1}, \\ \begin{pmatrix} \mathbf{l} & \mathbf{j} \\ \mathbf{k} & \mathbf{m} \end{pmatrix} &= \begin{pmatrix} (\eta_1^1)' & . \\ . & (\eta_1^2)' \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{Q}_1^1 & \mathbf{Q}_1^2 \\ \mathbf{Q}_1^3 & \mathbf{Q}_1^4 \end{pmatrix} = \begin{pmatrix} (\mathbf{Q}_2^1)' & (\mathbf{Q}_2^3)' \\ (\mathbf{Q}_2^2)' & (\mathbf{Q}_2^4)' \end{pmatrix} \begin{pmatrix} \eta_2^1 & . \\ . & \eta_2^2 \end{pmatrix}^{-1}, \end{aligned}\right\} \quad (1)$$

where $\boldsymbol{\xi}_1$ is the intersection matrix $\langle (b_1^i . d_1^j) \rangle$, η_1^1 is the intersection matrix of the cycles c_1^i, η_1^2 is the intersection matrix of the cycles C_1^i, and $\boldsymbol{\zeta}_1$ is the intersection matrix $\langle (d_1^i . b_1^j) \rangle$, while $\boldsymbol{\xi}_2$, η_2^1, η_2^2, $\boldsymbol{\zeta}_2$ are similarly defined on F_2.

Now, since the correspondence is algebraic, algebraic cycles must correspond to algebraic cycles. Hence

$$\mathbf{Q}_1^3 = 0, \quad \mathbf{Q}_2^3 = 0.$$

It follows immediately from the third equation (1) that

$$\mathbf{Q}_1^2 = 0, \quad \mathbf{Q}_2^2 = 0, \quad \mathbf{j} = 0, \quad \mathbf{k} = 0.$$

We shall therefore write

$$\mathbf{Q}_1^1 = \mathbf{Q}_1, \quad \mathbf{Q}_1^4 = \mathbf{R}_1, \quad \mathbf{Q}_2^1 = \mathbf{Q}_2, \quad \mathbf{Q}_2^4 = \mathbf{R}_2.$$

Then the equations of transformation of the cycles can be written in the form

$$\left.\begin{aligned} T(x_1) &\simeq \epsilon_2 x_2, & T^{-1}(x_2) &\simeq \epsilon_1 x_1, \\ T(b_1) &\simeq \mathbf{P}_1 b_2, & T^{-1}(b_2) &\simeq \mathbf{P}_2 b_1, \\ T(c_1) &\simeq \mathbf{Q}_1 c_2, & T^{-1}(c_2) &\simeq \mathbf{Q}_2 c_1, \\ T(C_1) &\simeq \mathbf{R}_1 C_2, & T^{-1}(C_2) &\simeq \mathbf{R}_2 C_1, \\ T(d_1) &\simeq \mathbf{S}_1 d_2, & T^{-1}(d_2) &\simeq \mathbf{S}_2 d_1, \\ T(F_1) &= \epsilon_1 F_2, & T^{-1}(F_2) &= \epsilon_2 F_1. \end{aligned}\right\} \quad (2)$$

† Lefschetz, *Topology* (*loc. cit*).

The 4-cycle T which represents the correspondence on $F_1 \times F_2$ is given by the homology

$$T \sim \epsilon_1 x_1 \times F_2 + \sum_{r,s} f^{rs} b_1^r \times d_2^s + \sum_{r,s} l^{rs} c_1^r \times c_2^s + \sum_{r,s} m^{rs} C_1^r \times C_2^s + \sum_{r,s} g^{rs} d_1^r \times b_2^s + \epsilon_2 F_1 \times x_2. \quad (3)$$

If the matrices $\mathbf{f}, \mathbf{g}, \mathbf{l}$ are zero, the correspondence is a correspondence of valency zero†. Any cycle homologous to

$$\epsilon_1 x_1 \times F_2 + \sum_{r,s} m^{rs} C_1^r \times C_2^s + \epsilon_2 F_1 \times x_2$$

is algebraic for all ϵ_1, ϵ_2, $\mathbf{m}$; hence the possibility of the existence of a non-valency correspondence reduces to the possibility of the existence of an algebraic cycle homologous to

$$T^* \sim \sum_{r,s} f^{rs} b_1^r \times d_2^s + \sum_{r,s} g^{rs} d_1^r \times b_2^s + \sum_{r,s} l^{rs} c_1^r \times c_2^s. \quad (4)$$

If this cycle is algebraic, any cycle of the form (3) with the same matrices $\mathbf{f}, \mathbf{g}, \mathbf{l}$ is algebraic. We cannot say immediately that ϵ_1, ϵ_2, $\mathbf{m}$ can be chosen so that an effective surface on $F_1 \times F_2$ exists satisfying the homology (3). This problem involves geometrical questions which are not yet solved, and we do not enter into it here. If (4) is algebraic, then any cycle given by (3) is algebraic and defines a virtual surface, which in turn defines a correspondence which we may speak of as a virtual correspondence.

2. We devote the rest of this paper to the problem of finding conditions to be satisfied in order that (4) may be algebraic. We first show that a necessary and sufficient condition for this is that the three cycles represented by the three terms of this expression should each be algebraic. Without loss of generality we may suppose that the curves C_1^1, C_2^1 define linear systems of curves on F_1, F_2, respectively, of positive grade, and that

$$C_1^1 . d_1^i \sim b_1^i \quad (i = 1, \ldots, 2q_1),$$

on F_1, and that

$$C_2^1 . d_2^i \sim b_2^i \quad (i = 1, \ldots, 2q_2),$$

on F_2. Then, in order that T^* may be algebraic, we require

$$T^* . F_1 \times C_2^1 = \sum_{r,s} f^{rs} b_1^r \times b_2^s$$

† Severi, *loc. cit.*

to be algebraic. Conversely, if this 2-cycle is algebraic, the 4-cycle

$$\sum_{r,s} f^{rs} b_1^r \times d_2^s$$

in $C_1^1 \times F_2$ is algebraic†, and therefore this cycle is algebraic in $F_1 \times F_2$. A similar argument applies to the cycle

$$\sum_{r,s} g^{rs} d_1^r \times b_2^s.$$

Thus we deduce that a necessary and sufficient condition for T^* to be algebraic is that the three cycles

$$\sum_{r,s} f^{rs} b_1^r \times d_2^s, \quad \sum_{r,s} g^{rs} d_1^r \times b_2^s, \quad \sum_{r,s} l^{rs} c_1^r \times c_2^s$$

are algebraic. The conditions to be satisfied in order that the first two cycles are algebraic are that the 2-cycles

$$\sum_{r,s} f^{rs} b_1^r \times b_2^s, \quad \sum_{r,s} g^{rs} b_1^r \times b_2^s$$

are algebraic. These last conditions can be expressed transcendentally. A necessary condition for a 2-cycle on an algebraic variety to be algebraic is that the double integrals of the first kind have zero periods on it. In the present case, if $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are the period matrices with respect to our base cycles of the Picard integrals of the first kind on F_1, F_2, these conditions reduce to

$$\boldsymbol{\omega}_1 \mathbf{f} \boldsymbol{\omega}_2' = 0, \quad \boldsymbol{\omega}_1 \mathbf{g} \boldsymbol{\omega}_2' = 0. \tag{5}$$

Conversely‡, if the equations (5) are satisfied, some integral multiple of the cycles

$$\sum_{r,s} f^{rs} b_1^r \times b_2^s, \quad \sum_{r,s} g^{rs} b_1^r \times b_2^s$$

is algebraic.

3. We have now reduced our problem to that of determining conditions to be fulfilled in order that

$$\Gamma \simeq \sum_{r,s} l^{rs} c_1^r \times c_2^s \tag{6}$$

may be algebraic, and we now proceed to an examination of this problem. At present, no sufficient condition for Γ to be algebraic is known, but we can give some necessary conditions, which may, eventually, prove sufficient.

† S. Lefschetz, *L'analyse situs et la géométrie algébrique* (Paris, 1924), 103.

‡ W. V. D. Hodge, *Proc. Cambridge Phil. Soc.*, 31 (1935), 18. In the present case it is easy to prove that, provided that $\sum_{r,s} f^{rs} b_1^r \times b_2$, $\sum_{r,s} g^{rs} b_1^r \times b_2^s$ are proper cycles (and not fractional multiples of proper cycles), the multiple may be replaced by unity.

We suppose that the Riemannian manifolds of F_1 and F_2 are constructed so that harmonic integrals may be introduced. The integrals which we have to consider are the same as those which I considered in my paper† "An extension of Poincaré's theorem on defective integrals", and those on F_k will now be written

$$\int \pi_k^i(x_k) \quad (i = 1, \ldots, r_k),$$

where $\pi_k^i(x_k)$ $(i = 1, \ldots, p_k)$ is the differential of an algebraic double integral of the first kind on F_k, $\pi_k^{p_k+i}(x_k)$ $(i = 1, \ldots, p_k)$ is the conjugate imaginary of $\pi_k^i(x_k)$, and $\pi_k^{2p_k+i}(x_k)$ is a total differential of the form

$$\pi_k^{2p_k+i}(x_k) = A^i\,dx\,d\bar{x} + B^i\,dx\,d\bar{y} + \bar{B}^i\,d\bar{x}\,dy + C^i\,dy\,d\bar{y},$$

when expressed in terms of local parameters (x, y), which is real, and the integral

$$\int \pi_k^{2p_k+i}(x_k)$$

has periods only on the cycles c_k^j. The period matrix of these integrals with respect to the cycles c_k^j will be written as

$$\begin{pmatrix} \boldsymbol{\theta}_k \\ \bar{\boldsymbol{\theta}}_k \\ \boldsymbol{\lambda}_k \end{pmatrix},$$

where $\boldsymbol{\theta}_k$ is the period matrix of the algebraic integrals, $\bar{\boldsymbol{\theta}}_k$ is the conjugate imaginary of this, and λ_k is real. An invariant definition of λ_k, independent of harmonic integrals, is given in the paper quoted.

We consider the integrals

$$\int \pi_1^i(x_1) \times \pi_2^j(x_2)$$

on $F_1 \times F_2$, and evaluate over Γ. To evaluate over any algebraic 4-cycle we evaluate over the component parts of a virtual surface homologous to it. On any component we express $\pi_1^i(x_1) \times \pi_2^j(x_2)$ in terms of the local parameters on the surface. When we do this it follows at once that

$$\pi_1^i(x_1) \times \pi_2^j(x_2) = 0$$

for (i) $(i = 1, \ldots, p_1;\ j = 1, \ldots, p_2)$, (ii) $(i = 1, \ldots, p_1;\ j = 2p_2+1, \ldots, r_2)$, (iii) $(i = 2p_1+1, \ldots, r_1;\ j = 1, \ldots, p_2)$. But in virtue of the homology

† W. V. D. Hodge, *Proc. London Math. Soc.*, 44 (1938), 216–225.

(6) for Γ, these equations give the relations

$$\left.\begin{aligned} \theta_1 1 \theta_2' &= 0, \\ \theta_1 1 \lambda_2' &= 0, \\ \lambda_1 1 \theta_2' &= 0. \end{aligned}\right\} \tag{7}$$

We thus have $p_1 r_2 + p_2 r_1 - p_1 p_2$ conditions which are necessary in order that Γ be algebraic. Are these sufficient? We do not know at present. In order to answer this question we shall presumably have to answer the more general question: *Is it a sufficient condition for a 4-cycle Γ on any variety of four complex dimensions to be algebraic, that every everywhere finite integral of a total differential form which can be expressed in terms of the local parameters z^1, z^2, z^3, z^4 on the variety in the form*

$$\int A dz^1 dz^2 dz^3 dz^4,$$

or in the form

$$\int \sum_{i,j,k,l=1}^{4} A_{ijkl} dz^i dz^j dz^k d\bar{z}^l,$$

has zero periods on Γ? The condition is clearly necessary. If it is sufficient it can be proved that conditions (7) are sufficient to tell us that the Γ given by (6) is algebraic. For it can be deduced from the theory of harmonic integrals that every integral which satisfies the conditions laid down can be expressed in terms of the $p_1 r_2 + p_2 r_1 - p_1 p_2$ integrals

$$\int \pi_1^i(x_1) \times \pi_2^j(x_2)$$

$(i = 1, \ldots, p_1;\ j = 1, \ldots, p_2)$, $(i = 1, \ldots, p_1;\ j = 2p_2+1, \ldots, r_2)$,

$(i = 2p_1+1, \ldots, r_1;\ j = 1, \ldots, p_2)$,

together with certain integrals which have zero periods on the cycles $c_1^i \times c_2^j$, and an integral having no periods on any of the 4-cycles of $F_1 \times F_2$.

4. We now proceed to obtain and examine certain equations which are a generalization of the equations of Hurwitz which are classical in the theory of correspondences between curves. Let x_1 be a general point of F_1 and let $x_2^1, \ldots, x_2^{e_2}$ be the points of F_2 which correspond to it in the correspondence T defined in § 1. Similarly, if x_2 is a general point of F_2, denote the corresponding points of F_1 by $x_1^1, \ldots, x_1^{e_1}$. Then, if

$$u_1^1(x_1), \quad \ldots, \quad u_1^{q_1}(x_1)$$

are q_1 independent Picard integrals of the first kind on F_1, having period matrix $\boldsymbol{\omega}_1$, and

$$u_2^1(x_2), \quad \ldots, \quad u_2^{q_2}(x_2)$$

are similarly defined on F_2, we have, just as in the theory of correspondences between curves,

$$\sum_{j=1}^{e_2} u_2^i(x_2^j) = \sum_{k=1}^{q_1} a_2^{ik} u_1^k(x_1) + a_2^i \quad (i = 1, \ldots, q_2),$$

where a_2^{ik}, a_2^i are constants. As x_1 describes the cycle b_1^a, the points $x_2^1, \ldots, x_2^{e_2}$ describe the cycle $T(b_1^a)$ in the aggregate. Hence, equating the increases on the two sides of this equation as x_1 describes this cycle, we obtain the matrix equation

$$\boldsymbol{\omega}_2 \mathbf{P}_1' = \boldsymbol{a}_2 \boldsymbol{\omega}_1.$$

Similarly, from the equations

$$\sum_{j=1}^{e_1} u_1^i(x_1^j) = \sum_{k=1}^{q_2} a_1^{ik} u_2^k(x_2) + a_1^i \quad (i = 1, \ldots, q_1),$$

where a_1^{ik}, a_1^i are constants, we obtain the equation

$$\boldsymbol{\omega}_1 \mathbf{P}_2' = \alpha_1 \omega_2.$$

These equations, which are similar to the equations obtained in the theory of curves, have already been studied†, and need not detain us here.

5. The same methods can be applied to obtain relations between the double integrals on F_1 and F_2, and between their period matrices. We shall suppose the harmonic integrals involved arranged as in § 3. Then we have, as in the case of simple integrals, equations of the form

$$\sum_{j=1}^{e_2} \pi_2^i(x_2^j) = \sum_{k=1}^{p_1} a_2^{ik} \pi_1^k(x_1) \qquad (i = 1, \ldots, p_2),$$

and

$$\sum_{j=1}^{e_2} \pi_2^{p_2+i}(x_2^j) = \sum_{k=1}^{p_1} \bar{a}_2^{ik} \pi_1^{p_1+k}(x_1) \quad (i = 1, \ldots, p_2),$$

where $a_2^{ik}, \bar{a}_2^{ik}$ are conjugate constants. The equations which arise from the remaining harmonic integrals require a little more examination.

$$\sum_{j=1}^{e_2} \pi_2^{2p_2+i}(x_2^j)$$

† For a summary, with references, see O. Zariski, *Algebraic surfaces* (Berlin, 1935), Appendix B.

is expressible as a differential form $\Re_1{}^i(x_1)$, which must be real and expressible in terms of the local parameters (x, y) on F_1 in the form

$$\Re_1{}^i(x_1) = A^i\,dx\,d\bar{x} + B^i\,dx\,d\bar{y} + \bar{B}^i\,d\bar{x}\,dy + C^i\,dy\,d\bar{y}.$$

Further, if C_1 is any 2-cycle of F_1,

$$\int_{C_1} \Re_1{}^i(x_1) = \int_{T(C_1)} \pi_2^{2p_2+i}(x_2).$$

Hence, since $$C_1 \simeq 0$$

implies $$T(C_1) \simeq 0,$$

$$\int \Re_1{}^i(x_1)$$

vanishes when evaluated over any bounding cycle of F_1. Hence $\Re_1{}^i(x_1)$ is a total differential. Again, since $T(C_1)$ is algebraic whenever C_1 is algebraic,

$$\int \Re_1{}^i(x_1)$$

has no periods on the algebraic cycles of F_1. There must, therefore, exist constants γ_1^{ij} such that

$$\Re_1{}^i(x_1) = \sum_{j=1}^{r_2} \gamma_1^{ij}\,\pi_1{}^j(x_1) + \mathfrak{L}_1^{i1}(x_1) + \mathfrak{L}_1^{i2}(x_1) + \mathfrak{L}_1^{i3}(x_1),$$

where $$\mathfrak{L}_1^{i1}(x_1) = L_1^{i1}\,dx\,dy,$$

$$\mathfrak{L}_1^{i2}(x_1) = L_1^{i2}\,d\bar{x}\,d\bar{y},$$

$$\mathfrak{L}_1^{i3}(x_1) = U_1{}^i\,dx\,d\bar{x} + V_1{}^i\,dx\,d\bar{y} + W_1{}^i\,d\bar{x}\,dy + T_1{}^i\,dy\,d\bar{y},$$

and $$\mathfrak{L}_1^{i1}(x_1) + \mathfrak{L}_1^{i2}(x_1) + \mathfrak{L}_1^{i3}(x_1)$$

is a total differential whose integral is without periods. But the expression for $\Re_1{}^i(x_1)$ is an identity, hence

$$\mathfrak{L}_1^{i1}(x_1) = -\sum_{j=1}^{p_1} \gamma_1^{ij}\,\pi_1{}^j(x_1),$$

$$\mathfrak{L}_1^{i2}(x_1) = -\sum_{j=1}^{p_1} \gamma_1^{ij+p_1}\,\pi_1^{j+p_1}(x_1).$$

Now, consider the integrals

$$\int_{F_1} \pi_1{}^k(x_1) \times [\mathfrak{L}_1^{i1}(x_1) + \mathfrak{L}_1^{i2}(x_1) + \mathfrak{L}_1^{i3}(x_1)] \quad (k = p_1+1, \ldots, 2p_1).$$

By the classical bilinear relations†, these integrals are all zero. On the other hand, by considering the forms of $L_1^{ij}(x_1)$, we reduce the integrals to

$$\int_{F_1} \pi_1{}^k(x_1) \times \mathfrak{L}_1^{i1}(x_1).$$

Again, using the bilinear relations and writing the values of this integral for $k = p_1+1, \ldots, 2p_1$ as a column matrix, we obtain the equation

$$\bar{\theta}_1[(\eta_1{}^1)']^{-1}\theta_1'\gamma_1' = 0,$$

where $$\gamma_1 = (\gamma_1^{i1}, \ldots, \gamma_1^{ip_1}).$$

Now the matrix $$\bar{\theta}_1[(\eta_1{}^1)']^{-1}\theta_1'$$

is non-singular, hence $$\gamma_1 = 0.$$

If we repeat the analysis with $k = 1, \ldots, p_1$, we obtain a similar result, the two together giving

$$\gamma_1^{ij} = 0 \quad (j = 1, \ldots, 2p_1).$$

We now write

$$\gamma_1^{i\,j+2p_1} = b_2^{ij}, \quad \mathfrak{L}_1^{i3}(x_1) = \mathfrak{L}_1{}^i(x_1),$$

and obtain the equations

$$\sum_{j=1}^{e_2} \pi_2^{2p_2+i}(x_2{}^j) = \sum_{k=1}^{r_1-2p_1} b_2^{ik}\pi_1^{2p_1+k}(x_1) + \mathfrak{L}_1{}^i(x_1) \quad (i = 1, \ldots, r_2-2p_2),$$

where $\mathfrak{L}_1{}^i(x_1)$ is a total differential whose integral is without periods.

Now integrate the expressions for $\sum_{j=1}^{e_2} \pi_2{}^i(x_2{}^j)$ $(i = 1, \ldots, r_2)$ with respect to x_1 over the transcendental 2-cycles of F_1. We obtain the equations

$$\left.\begin{aligned} \boldsymbol{\theta}_2\mathbf{Q}_1' &= \mathbf{a}_2\boldsymbol{\theta}_1, \\ \bar{\boldsymbol{\theta}}_2\mathbf{Q}_1' &= \bar{\mathbf{a}}_2\bar{\boldsymbol{\theta}}_1, \\ \boldsymbol{\lambda}_2\mathbf{Q}_1' &= \mathbf{b}_2\boldsymbol{\lambda}_1. \end{aligned}\right\} \tag{8}$$

Since $\boldsymbol{\lambda}_1$ is of rank r_1-2p_1 and $\mathbf{Q}_1$, $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ are real, it follows that $\mathbf{b}_2$ is real. Similarly, from the reverse transformation we obtain equations of the form

$$\left.\begin{aligned} \boldsymbol{\theta}_1\mathbf{Q}_2' &= \mathbf{a}_1\boldsymbol{\theta}_2, \\ \bar{\boldsymbol{\theta}}_1\mathbf{Q}_2' &= \bar{\mathbf{a}}_1\bar{\boldsymbol{\theta}}_2, \\ \boldsymbol{\lambda}_1\mathbf{Q}_2' &= \mathbf{b}_1\boldsymbol{\lambda}_2. \end{aligned}\right\} \tag{9}$$

† *Cf.* my paper in these *Proceedings, loc. cit.*, and references there given.

6. Let us suppose that $\mathbf{a}_2$ is of rank n, and that $\mathbf{b}_2$ is of rank ν. Write

$$\Lambda_k = \begin{pmatrix} \theta_k \\ \bar{\theta}_k \\ \lambda_k \end{pmatrix}, \qquad \mathbf{M}_k = \begin{pmatrix} \mathbf{a}_k & . & . \\ . & \bar{\mathbf{a}}_k & . \\ . & . & \mathbf{b}_k \end{pmatrix}.$$

Then from

$$\Lambda_2 \mathbf{Q}_1' = \mathbf{M}_2 \Lambda_1,$$

$$\mathbf{Q}_1' = \Lambda_2^{-1} \mathbf{M}_2 \Lambda_1$$

is of rank $2n+\nu$. Also

$$\begin{aligned} \mathbf{M}_1 &= \Lambda_1 \mathbf{Q}_2' \Lambda_2^{-1} \\ &= \Lambda_1 [(\eta_1^1)']^{-1} \mathbf{Q}_1 \eta_2^1 \Lambda_2^{-1} \\ &= \{\Lambda_1 [(\eta_1^1)']^{-1} \Lambda_1'\} \mathbf{M}_2' \{\Lambda_2 [(\eta_2^1)']^{-1} \Lambda_2'\}^{-1}. \end{aligned}$$

In these equations we have used (1), the expression for $\mathbf{Q}_1$ obtained above, and the fact that η_2^1 is symmetrical.

But, by the bilinear relations, we know that

$$\Lambda_k [(\eta_k^1)']^{-1} \Lambda_k' = \begin{pmatrix} . & \phi_k & . \\ \bar{\phi}_k & . & . \\ . & . & \psi_k \end{pmatrix},$$

where ϕ_k is a non-singular matrix of type (p_k, p_k), $\bar{\phi}_k$ is its conjugate imaginary, and ψ_k is a non-singular real square matrix. Hence

$$\mathbf{a}_1 = \phi_1 \bar{\mathbf{a}}_2 \phi_2^{-1}, \quad \mathbf{b}_1 = \psi_1 \mathbf{b}_2 \psi_2^{-1}.$$

In particular, $\mathbf{a}_1$ is of rank n, and $\mathbf{b}_1$ is of rank ν. It follows that $\mathbf{Q}_2$ is of rank $2n+\nu$.

7. Now, since $\mathbf{a}_2$ is of rank n, and $\mathbf{b}_2$ is of rank ν, we can find non-singular square matrices $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ such that

$$\mathbf{a}_2 = (\mathbf{u}_2)^{-1} \begin{pmatrix} \mathbf{E}^n & . \\ . & . \end{pmatrix} \mathbf{u}_1,$$

$$\mathbf{b}_2 = (\mathbf{v}_2)^{-1} \begin{pmatrix} \mathbf{E}^\nu & . \\ . & . \end{pmatrix} \mathbf{v}_1,$$

where $\mathbf{E}^r$ denotes the unit matrix of r rows and columns. The substitution

$$[\pi_k^i(x_k)]^* = \sum_{j=1}^{r_k} \delta_k^{ij} \pi_k^j(x_k),$$

where $$\delta_k = \begin{pmatrix} \mathbf{u}_k & . & . \\ . & \bar{\mathbf{u}}_k & . \\ . & . & \mathbf{v}_k \end{pmatrix},$$

replaces the differential forms $\pi_k^i(x_k)$ by a new set arranged according to the scheme of § 3.

Again, since $\mathbf{Q}_1$ is of rank $2n+\nu$, we can find rational non-singular matrices $\mathbf{d}_1$, $\mathbf{d}_2$ such that

$$\mathbf{Q}_1 = \mathbf{d}_1^{-1} \begin{pmatrix} \mathbf{E}^{2n+\nu} & . \\ . & . \end{pmatrix} \mathbf{d}_2.$$

The substitutions $$(c_k^i)^* \simeq \sum_{j=1}^{r_k} d_k^{ij} c_k^j$$

on the transcendental 2-cycles replace the period cycles by new period cycles.

If these substitutions for $k = 1, 2$ are made on the differential forms and cycles of F_1, F_2, the matrices $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{Q}_1'$ in (8) then assume forms

$$\begin{pmatrix} \mathbf{E}^n & . \\ . & . \end{pmatrix}, \quad \begin{pmatrix} \mathbf{E}^\nu & . \\ . & . \end{pmatrix}, \quad \begin{pmatrix} \mathbf{E}^{2n+\nu} & . \\ . & . \end{pmatrix},$$

respectively. We may, therefore, without loss of generality, suppose that the differential forms and period cycles have been arranged so that $\mathbf{a}_2, \mathbf{b}_2, \mathbf{Q}_1'$ in (8) have these forms, and this will be assumed for the remainder of the paper.

Equations (8), in this form, tell us immediately that the integrals

$$\int \pi_1^i(x_1) \quad (i = 1, \ldots, n;\ i = 2p_1+1, \ldots, 2p_1+\nu)$$

have zero periods on the cycles $c_1^{2n+\nu+1}, \ldots, c_1^{r_1}$. These integrals therefore satisfy the conditions for the generalization of Poincaré's theorem on defective integrals†, and hence, by that generalization, we can choose the remaining integrals and period cycles so that $\boldsymbol{\theta}_1$, $\boldsymbol{\lambda}_1$ are of the forms

$$\begin{pmatrix} \boldsymbol{\theta}_1{}^1 & . \\ . & \boldsymbol{\theta}_1{}^2 \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{\lambda}_1{}^1 & . \\ . & \boldsymbol{\lambda}_1{}^2 \end{pmatrix},$$

respectively, where $\boldsymbol{\theta}_1{}^1$, $\boldsymbol{\theta}_1{}^2$, $\boldsymbol{\lambda}_1{}^1$, $\boldsymbol{\lambda}_1{}^2$ are matrices of types $(n, 2n+\nu)$, $(p_1-n, r_1-2n-\nu)$, $(\nu, 2n+\nu)$, $(r_1-2p_1-\nu, r_1-2n-\nu)$ respectively.

† W. V. D. Hodge, *loc. cit.*

The specialized form of equations (8), however, also shows that the integrals

$$\int \pi_2^{i}(x_2) \quad (i = n+1, \ldots, p_2;\ i = 2p_2+\nu+1, \ldots, r_2)$$

have zero periods on the cycles $c_2^1, \ldots, c_2^{2n+\nu}$. Hence, by the extension of Poincaré's theorem, the remaining integrals and period cycles can be chosen so that $\boldsymbol{\theta}_2$, $\boldsymbol{\lambda}_2$ are of the forms

$$\begin{pmatrix} \theta_2^{1} & . \\ . & \theta_2^{2} \end{pmatrix}, \qquad \begin{pmatrix} \lambda_2^{1} & . \\ . & \lambda_2^{2} \end{pmatrix},$$

where $\boldsymbol{\theta}_2^1$, $\boldsymbol{\theta}_2^2$, $\boldsymbol{\lambda}_2^1$, $\boldsymbol{\lambda}_2^2$ are matrices of types $(n, 2n+\nu)$, $(p_2-n, r_2-2n-\nu)$, $(\nu, 2n+\nu)$, $(r_2-2p_2-\nu, r_2-2n-\nu)$, respectively. Further, these changes in the integrals and cycles are such that the forms obtained above for $\mathbf{a}_2$, $\mathbf{b}_2$, $\mathbf{Q}_1'$ in (8) are not affected. Equations (8) may therefore now be written as

$$\left.\begin{aligned} \begin{pmatrix} \theta_2^{1} & . \\ . & \theta_2^{2} \end{pmatrix} \begin{pmatrix} E^{2n+\nu} & . \\ . & . \end{pmatrix} &= \begin{pmatrix} E^{n} & . \\ . & . \end{pmatrix} \begin{pmatrix} \theta_1^{1} & . \\ . & \theta_1^{2} \end{pmatrix}, \\ \begin{pmatrix} \lambda_2^{1} & . \\ . & \lambda_2^{2} \end{pmatrix} \begin{pmatrix} E^{2n+\nu} & . \\ . & . \end{pmatrix} &= \begin{pmatrix} E^{\nu} & . \\ . & . \end{pmatrix} \begin{pmatrix} \lambda_1^{1} & . \\ . & \lambda_1^{2} \end{pmatrix}, \end{aligned}\right\} \tag{10}$$

where the second equation (8) has been omitted, since it is merely the conjugate imaginary of the first. In the new base set of cycles, the matrices $\boldsymbol{\eta}_1^1$, $\boldsymbol{\eta}_2^1$ are now of the form

$$\begin{pmatrix} \eta_1^{11} & . \\ . & \eta_1^{12} \end{pmatrix}, \qquad \begin{pmatrix} \eta_2^{11} & . \\ . & \eta_2^{12} \end{pmatrix},$$

where $\boldsymbol{\eta}_1^{11}$, $\boldsymbol{\eta}_1^{12}$, $\boldsymbol{\eta}_2^{11}$, $\boldsymbol{\eta}_2^{12}$ are non-singular rational square matrices of ranks $2n+\nu$, $r_1-2n-\nu$, $2n+\nu$, $r_2-2n-\nu$, respectively. This is a consequence of the extended Poincaré theorem.

8. We now seek to determine the form of equations (9) when the integrals and cycles have been arranged according to the scheme of § 7. We have, from (1),

$$\begin{aligned} \mathbf{Q}_2' &= \begin{pmatrix} (\boldsymbol{\eta}_1^{11})' & . \\ . & (\boldsymbol{\eta}_1^{12})' \end{pmatrix}^{-1} \begin{pmatrix} E^{2n+\nu} & . \\ . & . \end{pmatrix} \begin{pmatrix} \boldsymbol{\eta}_2^{11} & . \\ . & \boldsymbol{\eta}_2^{12} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{e}_2 & . \\ . & . \end{pmatrix}, \end{aligned}$$

where $\mathbf{e}_2$ is a non-singular rational square matrix of type $(2n+\nu, 2n+\nu)$. Equations (9) then give

$$\begin{pmatrix} \theta_1^{\,1} & \cdot \\ \cdot & \theta_1^{\,2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_2 & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^{11} & \mathbf{a}_1^{12} \\ \mathbf{a}_1^{21} & \mathbf{a}_1^{22} \end{pmatrix} \begin{pmatrix} \theta_2^{\,1} & \cdot \\ \cdot & \theta_2^{\,2} \end{pmatrix},$$

$$\begin{pmatrix} \lambda_1^{\,1} & \cdot \\ \cdot & \lambda_1^{\,2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_2 & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1^{11} & \mathbf{b}_1^{12} \\ \mathbf{b}_1^{21} & \mathbf{b}_1^{22} \end{pmatrix} \begin{pmatrix} \lambda_2^{\,1} & \cdot \\ \cdot & \lambda_2^{\,2} \end{pmatrix},$$

in a notation whose meaning will be clear. From these equations it follows that

$$\mathbf{a}_1^{12} = 0, \quad \mathbf{a}_1^{21} = 0, \quad \mathbf{a}_1^{22} = 0,$$

$$\mathbf{b}_1^{12} = 0, \quad \mathbf{b}_1^{21} = 0, \quad \mathbf{b}_1^{22} = 0.$$

Thus equations (9) take the form

$$\left.\begin{aligned} \begin{pmatrix} \theta_1^{\,1} & \cdot \\ \cdot & \theta_1^{\,2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_2 & \cdot \\ \cdot & \cdot \end{pmatrix} &= \begin{pmatrix} \mathbf{a}_1^{11} & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \theta_2^{\,1} & \cdot \\ \cdot & \theta_2^{\,2} \end{pmatrix}, \\ \begin{pmatrix} \lambda_1^{\,1} & \cdot \\ \cdot & \lambda_1^{\,2} \end{pmatrix} \begin{pmatrix} \mathbf{e}_2 & \cdot \\ \cdot & \cdot \end{pmatrix} &= \begin{pmatrix} \mathbf{b}_1^{11} & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \theta_2^{\,1} & \cdot \\ \cdot & \theta_2^{\,2} \end{pmatrix}, \end{aligned}\right\} \tag{11}$$

where $\mathbf{a}_1^{11}$ and $\mathbf{b}_1^{11}$ are non-singular square matrices of ranks n and ν respectively†.

9. From this point on, the analysis of equations (8) and (9) is similar to that of the corresponding equations for simple integrals. Without difficulty we obtain the final result which follows. If the matrices Λ_k are analysed into the forms‡

$$\begin{pmatrix} \theta_k^{\,1} & & & & \\ & \ddots & & & \\ & & \theta_k^{\varepsilon_k} & & \\ & & & \bar{\theta}_k^{\,1} & \\ & & & & \ddots \\ & & & & & \bar{\theta}_k^{\varepsilon_k} \\ & & & & & & \lambda_k^{\,1} \\ & & & & & & & \ddots \\ & & & & & & & & \lambda_k^{\varepsilon_k} \end{pmatrix},$$

† This is a generalization of a known result in the theory of curves, *cf.* W. V. D. Hodge, *Proc. Cambridge Phil. Soc.*, 32 (1936), 337.

‡ *Cf.* my paper on the generalization of Poincaré's theorem, *loc. cit.*

where
$$\Lambda_k{}^i = \begin{pmatrix} \theta_k{}^i \\ \bar{\theta}_k{}^i \\ \lambda_k{}^i \end{pmatrix}$$

is not further reducible, equations (8) and (9) consist of equations of the form

$$\left.\begin{aligned} \Lambda_2{}^i\, \mathbf{Q}_1^{ij} &= \begin{pmatrix} \mathbf{a}_2^{ij} & . & . \\ . & \bar{\mathbf{a}}_2^{ij} & . \\ . & . & \mathbf{b}_2^{ij} \end{pmatrix} \Lambda_1{}^j, \\ \Lambda_1{}^j\, \mathbf{Q}_2^{ji} &= \begin{pmatrix} \mathbf{a}_1^{ji} & . & . \\ . & \bar{\mathbf{a}}_1^{ji} & . \\ . & . & \mathbf{b}_1^{ji} \end{pmatrix} \Lambda_2{}^i, \end{aligned}\right\} \tag{12}$$

in which the matrices $\mathbf{Q}_1^{ij}$, $\mathbf{Q}_2^{ij}$ are either zero or non-singular square matrices. If there are κ^{ij} such relations connecting $\Lambda_1{}^i$, $\Lambda_2{}^j$ and

$$\sigma = \sum_{i,\,j} \kappa^{ij},$$

there are altogether σ relations of the forms (8) and (9) connecting Λ_1 and Λ_2. But, until we have obtained an answer to the problem proposed in § 3, we cannot say that there is a correspondence between F_1 and F_2 arising from each such relation. If the answer to this question is in the affirmative, we have, from (8),

$$\begin{aligned} \theta_2\, \mathbf{Q}_1{}'(\eta_1{}^1)^{-1}\, \theta_1{}' &= \mathbf{a}_2\, \theta_1 (\eta_1{}^1)^{-1} \theta_2{}' \\ &= 0 \end{aligned}$$

by the bilinear relations, and by (1) and the fact that $\eta_1{}^1$ is symmetrical, this is

$$\theta_2 \mathbf{l}'\, \theta_1{}' = 0,$$

i.e.
$$\theta_1 \mathbf{l}\, \theta_2{}' = 0,$$

and there corresponds an algebraic cycle on $F_1 \times F_2$,

$$\Gamma \sim \sum_{i,\,j} l^{ij}\, c_1{}^i \times c_2{}^j.$$

Using this surface to define a correspondence we deduce that this correspondence gives rise to the equations (8) and (9). For $\mathbf{l}$ defines $\mathbf{Q}_1$, $\mathbf{Q}_2$ uniquely by (1) and we have

$$\begin{pmatrix} \mathbf{a}_1 & . & . \\ . & \bar{\mathbf{a}}_1 & . \\ . & . & \mathbf{b}_1 \end{pmatrix} = \Lambda_1 \mathbf{Q}_2{}'\, \Lambda_2^{-1}.$$

10. If there are τ independent algebraic 2-cycles of the type

$$\sum_{r,s} f^{rs} b_1^r \times b_2^s,$$

there are τ algebraic 4-cycles of the type

$$\sum_{r,s} f^{rs} b_1^r \times d_2^s,$$

and τ of the type $\sum_{r,s} f^{rs} d_1^r \times b_2^s$.

τ is the number of independent relations (5). The number μ of topologically independent algebraic 4-cycles on $F_1 \times F_2$ therefore satisfies the inequalities

$$\rho_1\rho_2+2+2\tau \leqslant \mu \leqslant \rho_1\rho_2+2+2\tau+\sigma.$$

If (7) is a sufficient condition for the cycle (6) to be algebraic, we shall have

$$\mu = \rho_1\rho_2+2+2\tau+\sigma.$$

Pembroke College,
Cambridge.

GENERAL TAUBERIAN THEOREMS

By H. R. PITT.

[Received 16 June, 1937.—Read 17 June, 1937.]

1. *Introduction.*

The theory of Tauberian theorems has been made much more systematic and comprehensive by the work of Norbert Wiener, but there are parts of the subject which are still imperfectly connected with the general theory. For example, the simplest proof of the "high indices theorem" of Hardy and Littlewood, given recently by Ingham, makes little use of Wiener's ideas. Another proof of the theorem, on quite different lines, has been given by Wiener himself, but here again there is no direct use of his general theorems.

Our object here is to prove a series of general† theorems which include and extend all those which are most familiar, such as, for example, Littlewood's theorem and its extensions by Hardy and Littlewood, Schmidt and Wiener, the high indices theorem, and the Hardy-Littlewood Tauberian theorem for Borel summation. There are also applications to other methods of summation, for example, those of Hausdorff and Valiron. The first of these is discussed in § 4.

We show in § 3.6 that Wiener's theorems can be deduced from the following result.

THEOREM *a*. *Suppose that*

(i) $$k(x) \in L(-\infty, \infty),$$

(ii) $$\int_{-\infty}^{\infty} k(x)\,dx = 1,$$

† A *general* Tauberian theorem is one which involves an arbitrary kernel $k(x)$; a *special* Tauberian theorem applies only to some particular kernel.

(iii) $$\int_{-\infty}^{\infty} k(x)\, e^{-itx}\, dx \neq 0 \quad \textit{for real } t,$$

(iv) $$|s(x)| \leqslant M,$$

(v) $$s(y)-s(x) \to 0 \quad \textit{when} \quad x \to \infty, \quad 0 \leqslant y-x \to \infty,$$

(vi) $$g(x) = \int_{-\infty}^{\infty} k(x-y)\, s(y)\, dy \to C \quad \textit{as} \quad x \to \infty.$$

Then $$\lim_{y \to \infty} s(y) = C.$$

This is a special case of Theorem 8, and it is plain that, if we were concerned only with this form of the theorem, the details of the proof would be rather simpler. In particular, we could define $h(x)$ to be $2(\sin \frac{1}{2}\pi t/\pi t)^2$ and dispense with the function Φ. Such a proof of Theorem a would be somewhat shorter than Wiener's proof of Theorem A, and it would seem natural to deduce the latter theorem from it. On the other hand, however, it is important to observe that Theorem a can also be deduced from Theorem A by a fairly simple argument.

Our main theorems are generalizations of Theorem a. For example, we show that all the familiar Tauberian conditions, including the "high-indices" condition, can be regarded as special cases of the more general condition defining the class T of Section 2, and that Theorem a remains true with this condition instead of (v).

My debts to other writers are acknowledged as they arise, with one exception. The whole paper is based on the general ideas introduced into the subject by Wiener, and it is difficult to say where I am indebted to him most.

Finally, I should like to acknowledge my debt to Prof. Hardy for his invaluable criticism and advice during the preparation of this paper, and to Mr. A. E. Ingham for pointing out a large number of errors and omissions in the manuscript.

2. *Functions which satisfy Tauberian conditions*

2.1. We suppose that $s(x)$ is any real or complex function defined for real values of x. We use the notation $s(x) \in C$ to mean that $s(x)$ belongs to a class of functions C, and write $C_1 \subset C_2$ if a class C_1 is contained in another class C_2.

2.2. *Definition.* We say that $s(x) \in T$ if it is measurable and if functions $\mu(\epsilon, x)$, $\delta(\epsilon, x)$, $\theta(\epsilon, x)$, $\xi(\epsilon, x)$ can be defined for real x and positive ϵ so

that

(2.2.1) $\mu(\epsilon, x)$ is an increasing function of ϵ for any value of x,

$$0 \leqslant \mu(\epsilon) = \overline{\lim_{x\to\infty}}\, \mu(\epsilon, x) < 1, \tag{2.2.2}$$

$$\delta(\epsilon) = \underline{\lim_{x\to\infty}}\, \delta(\epsilon, x) > 0, \tag{2.2.3}$$

$$|\xi - x| \leqslant \delta, \tag{2.2.4}$$

$$\Re[e^{i\theta} s(y)] \geqslant (1-\mu)|s(x)| - \epsilon \quad (|\xi - y| < \delta). \tag{2.2.5}$$

Here, and in what follows, we write μ, δ, θ, ξ for $\mu(\epsilon, x)$, $\delta(\epsilon, x)$, $\theta(\epsilon, x)$, $\xi(\epsilon, x)$, respectively.

Definition. We say that $s(x) \in T'$ if functions μ, δ, θ and ξ can be defined so that (2.2.1), (2.2.4), and (2.2.5) are satisfied, and

$$0 \leqslant \mu'(\epsilon) = \overline{\operatorname{bd}}_{-\infty < x < \infty}\, \mu(\epsilon, x) < 1, \tag{2.2.6}$$

$$\delta'(\epsilon) = \underline{\operatorname{bd}}_{-\infty < x < \infty}\, \delta(\epsilon, x) > 0. \tag{2.2.7}$$

It is plain that, for any choice of functions δ, μ, we have $\mu(\epsilon) \leqslant \mu'(\epsilon)$, $\delta'(\epsilon) \leqslant \delta(\epsilon)$, so that $T' \subset T$. In particular,

$$\mu(+0) \leqslant \mu'(+0) < 1. \tag{2.2.8}$$

2.3. It is convenient at this stage to introduce two subclasses T_α, T_β of T. We are not concerned with these in the later part of this paper, but they occur naturally in the theorems of this section. They are also of some interest in connection with certain elementary Tauberian theorems which I hope to discuss in a later paper.

Definition. We say that $s(x) \in T_\alpha$ if $\mu(\epsilon, x)$ and $\delta(\epsilon, x)$ satisfy (2.2.1), (2.2.2), and (2.2.3), and if, for every x and positive ϵ, and every ϕ of $(-\pi, \pi)$, there is a number $\xi = \xi(\epsilon, x, \phi)$ such that

$$|\xi - x| \leqslant \delta, \tag{2.3.1}$$

$$\Re\left[e^{i\phi}\big(s(y) - s(x)\big)\right] \geqslant -\epsilon - \mu|s(x)| \quad (|\xi - y| < \delta). \tag{2.3.2}$$

Definition. We say that $s(x) \in T_\beta$ if ξ is independent of ϕ. In this case, (2.3.2) may be written in the form

$$|s(y) - s(x)| \leqslant \epsilon + \mu|s(x)| \quad (|\xi - y| < \delta). \tag{2.3.3}$$

It is plain that $T_\beta \subset T_\alpha \subset T$. The classes T_α' and T_β' can be defined by replacing limits by bounds in the definitions of T_α and T_β.

2.4. Two important classes of functions which satisfy Tauberian conditions have been defined by R. Schmidt (**13**). We now extend the wider of these classes to include functions which take complex values, and show that this class is contained in T.

Definition. We say that $s(x) \epsilon S$ if

$$\left|\arg\left(s(y)-s(x)+\eta(x, y)\right)\right| \leqslant A < \tfrac{1}{2}\pi,$$

where $\eta(x, y) \to 0$ as $x \to \infty$, $0 \leqslant y-x \to 0$.

Definition. We say that $s(x) \epsilon S_\alpha$ if $s(x)$ is real and

$$\underline{\lim}\left(s(y)-s(x)\right) \geqslant 0$$

when $x \to \infty$, $0 \leqslant y-x \to 0$.

Definition. We say that $s(x) \epsilon S_\beta$ if

$$\lim\left(s(y)-s(x)\right) = 0$$

when $x \to \infty$, $0 \leqslant y-x \to 0$.

The class S_α consists of Schmidt's *slowly decreasing functions*; S_β is the class of *slowly oscillating functions*. It is plain that $S_\beta \subset S_\alpha \subset S$.

2.5. THEOREM 1. (i) $S_\beta \subset T_\beta$, (ii) $S_\alpha \subset T_\alpha$, (iii) $S \subset T$.

(i) If $\epsilon > 0$, we can define $X(\epsilon)$ and $\delta(\epsilon) > 0$ so that $|s(y)-s(x)| \leqslant \epsilon$ for $x \geqslant X(\epsilon)$, $|y-x| \leqslant \delta(\epsilon)$. Then, if we define $\delta(\epsilon, x) = \delta(\epsilon)$ for $x \geqslant X(\epsilon)$, $\delta(\epsilon, x) = 0$ for $x < X(\epsilon)$, $\mu(\epsilon, x) = 0$, $\xi(\epsilon, x) = x$, the conclusion is evident.

(ii) If $\epsilon > 0$ we can define $X(\epsilon)$ and $\delta(\epsilon) > 0$ so that

$$s(y)-s(x) \geqslant -\epsilon \quad [x \geqslant X(\epsilon),\ 0 \leqslant y-x \leqslant 2\delta],$$

$$s(x)-s(y) \geqslant -\epsilon \quad [x \geqslant X(\epsilon),\ 0 \leqslant x-y \leqslant 2\delta].$$

The conclusion follows if we define

$$\delta(\epsilon, x) = \delta(\epsilon) \ \ [x \geqslant X(\epsilon)], \qquad \delta(\epsilon, x) = 0 \ \ [x < X(\epsilon)], \qquad \mu(\epsilon, x) = 0,$$

$$\xi(\epsilon, x, \phi) = x+\delta(\epsilon, x) \quad (|\phi| < \tfrac{1}{2}\pi),$$

$$\xi(\epsilon, x, \phi) = x-\delta(\epsilon, x) \quad (\tfrac{1}{2}\pi \leqslant |\phi| \leqslant \pi).$$

(iii) Let $\epsilon > 0$, and define $X(\epsilon)$ and $\delta(\epsilon) > 0$ so that

$$\left|\arg\left(s(y)-s(x)+\epsilon\right)\right| \leqslant A \quad (x \geqslant X,\ 0 \leqslant y-x \leqslant 2\delta), \tag{2.5.1}$$

$$\left|\arg\left(s(x)-s(y)+\epsilon\right)\right| \leqslant A \quad (x \geqslant X,\ 0 \leqslant x-y \leqslant 2\delta). \tag{2.5.2}$$

Let $\operatorname{sign} s(x) = e^{i\gamma}$. If $|\gamma| \leqslant \frac{1}{2}\pi$, we can choose $\theta = \theta(\epsilon, x)$ so that

$$|\theta| \leqslant \tfrac{1}{2}\pi - A, \quad |\gamma+\theta| \leqslant A. \tag{2.5.3}$$

Then

$$\begin{aligned}\mathfrak{R}[s(y)\,e^{i\theta}] &= \mathfrak{R}[|s(x)|e^{i(\theta+\gamma)}]+\mathfrak{R}[(s(y)-s(x)+\epsilon)\,e^{i\theta}]-\mathfrak{R}[\epsilon e^{i\theta}]\\ &\geqslant |s(x)|\cos A-\epsilon \quad (0 \leqslant y-x \leqslant 2\delta),\end{aligned}$$

by (2.5.1) and (2.5.3). Similarly, if $|\pi-\gamma| \leqslant \frac{1}{2}\pi$, we use (2.5.2) to show that, for a suitable value of θ,

$$\mathfrak{R}[s(y)\,e^{i\theta}] \geqslant |s(x)|\cos A-\epsilon \quad (0 \leqslant x-y \leqslant 2\delta).$$

The conclusion follows if we define

$$\delta(\epsilon, x) = \delta(\epsilon) \ \ [x \geqslant X(\epsilon)], \qquad \delta(\epsilon, x) = 0 \ \ [x < X(\epsilon)],$$

$$\mu(\epsilon, x) = 1-\cos A,$$

$$\xi(\epsilon, x) = x+\delta(\epsilon, x) \ \ \{\mathfrak{R}[s(x)] \geqslant 0\}, \qquad \xi(\epsilon, x) = x-\delta(\epsilon, x) \ \ \{\mathfrak{R}[s(x)] < 0\}.$$

2.6. We shall now consider various conditions under which a step function belongs to one of the classes defined above.

THEOREM 2. *Let* $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_\nu \to \infty$. *Let* $s(x) = \sum_{\lambda_\nu \leqslant x} a_\nu$. *Suppose that, for a sequence* ν_k *(which may be nul), we have*

$$\lambda_{\nu_k} - \lambda_{\nu_k-1} \geqslant 2\delta_0 > 0. \tag{2.6.1}$$

Then $s(x) \in T_\beta'$ *if one of the following conditions is satisfied.*

$$(a) \qquad |a_\nu| \leqslant K(\lambda_\nu - \lambda_{\nu-1}) \quad (\nu \neq \nu_k),$$

$$(b) \qquad \sum_{\nu=1}^{n}{}' |a_\nu|^p e^{\lambda_\nu}(\lambda_\nu-\lambda_{\nu-1})^{1-p} \leqslant Ke^{\lambda_n} \quad (p > 1),$$

$$(c) \qquad \sum_{\nu=1}^{\infty}{}' |a_\nu|^p (\lambda_\nu-\lambda_{\nu-1})^{1-p} = K < \infty,$$

where Σ' means that the terms in which $\nu = \nu_k$ are omitted. In particular, $s(x) \in T_\beta'$ if

$$(d) \qquad \lambda_\nu - \lambda_{\nu-1} \geqslant 2\delta_0 > 0$$

for every ν. Moreover, $s(x) \in S_\beta$ if one of the above conditions is satisfied and if also $a_\nu \to 0$.

We define the function† $\Lambda(t)$ to be Kt, $Kt^{1/p'}e^{t/p}$ or $K^{1/p}t^{1/p'}$ according as $\{a_\nu\}$ satisfies (a), (b), or (c). Let $n = n(x)$ be defined by the relation

$$\lambda_n \leqslant x < \lambda_{n+1},$$

and let $m = m(\epsilon, x) \geqslant n$ be the greatest integer for which $\Lambda(\lambda_m - \lambda_n) \leqslant \epsilon$. We suppose, in the first instance, that there is no ν_k which satisfies $n+1 \leqslant \nu_k \leqslant m$. We prove first that

$$(2.6.3) \qquad |s(y) - s(x)| \leqslant \Lambda(\lambda_m - \lambda_n) \leqslant \epsilon \quad (\lambda_n \leqslant y < \lambda_{m+1}).$$

Case (a).
$$|s(y) - s(x)| \leqslant \sum_{n+1}^{m} |a_\nu| \leqslant K \sum_{n+1}^{m} (\lambda_\nu - \lambda_{\nu-1}).$$
$$= K(\lambda_m - \lambda_n) = \Lambda(\lambda_m - \lambda_n).$$

Case (b).

$$|s(y) - s(x)| \leqslant \sum_{n+1}^{m} |a_\nu|$$
$$= \sum_{n+1}^{m} e^{\lambda_\nu/p} |a_\nu| (\lambda_\nu - \lambda_{\nu-1})^{-1/p'} e^{-\lambda_\nu/p} (\lambda_\nu - \lambda_{\nu-1})^{1/p'}$$
$$\leqslant \left[\sum_{n+1}^{m} |a_\nu|^p e^{\lambda_\nu}(\lambda_\nu - \lambda_{\nu-1})^{1-p}\right]^{1/p} \left[\sum_{n+1}^{m} (\lambda_\nu - \lambda_{\nu-1}) e^{-\lambda_\nu p'/p}\right]^{1/p'}$$
$$\leqslant K^{1/p} e^{\lambda_m/p} e^{-\lambda_n/p} (\lambda_m - \lambda_n)^{1/p'} = \Lambda(\lambda_m - \lambda_n).$$

Case (c).

$$|s(y) - s(x)| \leqslant \sum_{n+1}^{m} |a_\nu|$$
$$= \sum_{n+1}^{\infty} |a_\nu| (\lambda_\nu - \lambda_{\nu-1})^{-1/p'} (\lambda_\nu - \lambda_{\nu-1})^{1/p'}$$
$$\leqslant \left[\sum_{n+1}^{m} |a_\nu|^p (\lambda_\nu - \lambda_{\nu-1})^{1-p}\right]^{1/p} \left[\sum_{n+1}^{m} (\lambda_\nu - \lambda_{\nu-1})\right]^{1/p'}$$
$$\leqslant K^{1/p} (\lambda_m - \lambda_n)^{1/p'} = \Lambda(\lambda_m - \lambda_n).$$

---

† We define p', as usual, by the formulae

$$\frac{1}{p} + \frac{1}{p'} = 1, \quad p' = \frac{p}{p-1}.$$

Let

$$\mu(\epsilon, x) = 0, \quad \xi(\epsilon, x) = \tfrac{1}{2}(\lambda_{m+1}+\lambda_n), \quad \delta(\epsilon, x) = \tfrac{1}{2}(\lambda_{m+1}-\lambda_n).$$

The inverse $\Lambda^*(u)$ of $\Lambda(t)$ is plainly positive and increasing for $u > 0$. Since, by our definition of m,

$$\Lambda[2\delta(\epsilon, x)] = \Lambda[\lambda_{m+1}-\lambda_n] > \epsilon,$$

it follows that

$$2\delta(\epsilon, x) \geqslant \Lambda^*(\epsilon) > 0. \tag{2.6.4}$$

On the other hand, if there are values of ν_k satisfying $n+1 \leqslant \nu_k \leqslant m$, we suppose that ν^* is the least, and define

$$\xi(\epsilon, x) = \tfrac{1}{2}(\lambda_{\nu^*}+\lambda_n), \quad \delta(\epsilon, x) = \tfrac{1}{2}(\lambda_{\nu^*}-\lambda_n).$$

Then

$$\delta(\epsilon, x) \geqslant \tfrac{1}{2}(\lambda_{\nu^*}-\lambda_{\nu^*-1}) \geqslant \delta_0 \tag{2.6.5}$$

and

$$\begin{aligned} |s(y)-s(x)| &\leqslant \sum_{n+1}^{\nu^*-1} |a_\nu| \leqslant \Lambda(\lambda_{\nu^*-1}-\lambda_n) \\ &\leqslant \Lambda(\lambda_m-\lambda_n) \leqslant \epsilon \quad (\lambda_n \leqslant y < \lambda_{\nu^*}). \end{aligned} \tag{2.6.6}$$

It follows from (2.6.3), (2.6.4), (2.6.5), and (2.6.6) that

$$\delta = \delta(\epsilon, x) \geqslant \min[\tfrac{1}{2}\Lambda^*(\epsilon), \delta_0] > 0 \quad \text{and} \quad |\xi-x| \leqslant \delta$$

for all values of x. Therefore $s(x) \in T_\beta'$.

If $a_n \to 0$, we have, from (2.6.3),

$$\begin{aligned} |s(y)-s(x)| &\leqslant |a_{n+1}| + \sum_{n+2}^{m} |a_\nu| \\ &\leqslant o(1)+\Lambda(\lambda_m-\lambda_{n+1}) \\ &\leqslant o(1)+\Lambda(y-x) \\ &= o(1), \quad \text{as} \quad x \to \infty, \quad 0 \leqslant y-x \to 0, \end{aligned}$$

and therefore $s(x) \in S_\beta$.

2.7 THEOREM 3. *Let* $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_\nu \to \infty$. *Let*

$$s(x) = \sum_{\lambda_\nu \leqslant x} a_\nu, \quad a_\nu = c_\nu + p_\nu,$$

where

$$|\arg p_\nu| \leqslant A < \tfrac{1}{2}\pi. \tag{2.7.1}$$

Let

$$\lambda_{\nu_k}-\lambda_{\nu_k-1} \geqslant 2\delta_0 > 0, \tag{2.7.2}$$

$$|c_{\nu-1}| \leqslant K(\lambda_\nu-\lambda_{\nu-1})+\eta_\nu \quad (\nu \neq \nu_k), \tag{2.7.3}$$

where $\lim_{\nu\to\infty} \eta_\nu = 0$. *Then* (i) $s(x) \epsilon T$ *if one of the following conditions is satisfied.*

$$|c_\nu| \leqslant K(\lambda_\nu-\lambda_{\nu-1}) \quad (\nu \neq \nu_k), \tag{a}$$

$$\sum_1^n{}' |c_\nu|^p e^{\lambda_\nu}(\lambda_\nu-\lambda_{\nu-1})^{1-p} \leqslant Ke^{\lambda_n} \quad (p > 1), \tag{b}$$

$$\sum_1^\infty{}' |c_\nu|^p(\lambda_\nu-\lambda_{\nu-1})^{1-p} = K < \infty, \tag{c}$$

where Σ' *means that the terms in which* $\nu = \nu_k$ *are omitted. Moreover,* (ii) $s(x) \epsilon T_a$ *if* $A = 0$, (iii) $s(x) \epsilon S$ *if* $c_\nu \to 0$.

Note. It is easy to show that some condition similar to (2.7.3) is required. That given here is rather more general than the simpler condition $c_n \to 0$. It is clear that if $\lambda_\nu-\lambda_{\nu-1} \to 0$, the latter condition is a consequence of (*a*), (*b*), or (*c*), so that in this case condition (2.7.3) may be omitted.

(i) Let $\epsilon > 0$. Since we are concerned only with large values of x, there is no loss of generality in supposing that

$$|c_{\nu-1}| \leqslant K(\lambda_\nu-\lambda_{\nu-1})+\tfrac{1}{3}\epsilon \quad (\nu \neq \nu_k). \tag{2.7.4}$$

We define $n = n(x)$ by the relation $\lambda_n \leqslant x < \lambda_{n+1}$. Let $t(x) = \sum_{\lambda_\nu \leqslant x} c_\nu$. We proved in Theroem 2 that

$$|t(y)-t(x)| \leqslant \tfrac{1}{3}\epsilon \quad [\lambda_n \leqslant y < \lambda_n+2\delta_1(\epsilon, x)], \tag{2.7.5}$$

where

$$2\delta_1(\epsilon, x) \geqslant \lambda_{n+1}-\lambda_n, \quad \delta_1(\epsilon, x) \geqslant \delta(\epsilon) > 0. \tag{2.7.6}$$

Since (2.7.2) and (2.7.6) hold *a fortiori*, for smaller values of δ_0 and $\delta(\epsilon)$, we may suppose that

$$0 < \delta(\epsilon) \leqslant \delta_0 \leqslant \epsilon/12K. \tag{2.7.7}$$

Now suppose that

$$\lambda_{n+1}-\lambda_n < 2\delta(\epsilon), \tag{2.7.8}$$

so that $n+1 \neq \nu_k$, on account of (2.7.2) and (2.7.7), and define

$$m = m(\epsilon, x) < n$$

by

$$\lambda_m \leqslant \lambda_n - 2\delta(\epsilon) < \lambda_{m+1}. \tag{2.7.9}$$

Then it is plain, by (2.7.2) and (2.7.7), that there is no ν_k satisfying $m+2 \leqslant \nu_k \leqslant n$. If $\lambda_m \leqslant y < \lambda_{n+1}$, then

$$\begin{aligned} |t(y)-t(x)| &\leqslant |c_{m+1}| + \Big| \sum_{\nu=m+2}^{n} c_\nu \Big| \\ &\leqslant K(\lambda_{m+2}-\lambda_{m+1}) + \tfrac{1}{3}\epsilon + \tfrac{1}{3}\epsilon, \end{aligned}$$

by (2.7.4) and (2.7.5) (with x, y and n, $m+1$ interchanged). Hence

$$\begin{aligned} |t(y)-t(x)| &\leqslant K(\lambda_{n+1}-\lambda_n) + K(\lambda_n-\lambda_{m+1}) + \tfrac{2}{3}\epsilon \\ &\leqslant 4K\delta(\epsilon) + \tfrac{2}{3}\epsilon \leqslant \epsilon, \end{aligned}$$

by (2.7.7), (2.7.8), and (2.7.9). Since

$$\lambda_{n+1} - \lambda_m \geqslant 2\delta(\epsilon),$$

we can define $\delta_2(\epsilon, x)$ so that

$$|t(y)-t(x)| \leqslant \epsilon \quad (\lambda_{n+1} - 2\delta_2(\epsilon, x) \leqslant y < \lambda_{n+1}), \tag{2.7.10}$$

$$2\delta_2(\epsilon, x) \geqslant \lambda_{n+1} - \lambda_n, \quad \delta_2(\epsilon, x) \geqslant \delta(\epsilon). \tag{2.7.11}$$

If we write $2\delta_2(\epsilon, x) = \lambda_{n+1} - \lambda_n$ when $\lambda_{n+1} - \lambda_n \geqslant 2\delta(\epsilon)$, then (2.7.10) and (2.7.11) are true for all values of x. Now let $r(x) = \sum_{\lambda_\nu \leqslant x} p_\nu$, so that $s(x) = r(x) + t(x)$. Let sign $s(x) = e^{i\gamma}$ and let θ be real. Then we can write

$$s(y)e^{i\theta} = |s(x)|\, e^{i(\theta+\gamma)} + e^{i\theta}[t(y)-t(x)] + e^{i\theta}[r(y)-r(x)]. \tag{2.7.12}$$

If $|\gamma| \leqslant \frac{1}{2}\pi$, we can choose $\theta = \theta(x)$ so that

$$|\theta| \leqslant \tfrac{1}{2}\pi - A, \quad |\gamma+\theta| \leqslant A. \tag{2.7.13}$$

Let

$$\delta(\epsilon, x) = \delta_1(\epsilon, x), \quad \xi(\epsilon, x) = \lambda_n + \delta_1(\epsilon, x), \quad \mu(\epsilon, x) = 1 - \cos A.$$

Since $|\arg [r(y)-r(x)]| \leqslant A$ when $r(y)-r(x) \neq 0$, it follows from (2.7.5), (2.7.12), and (2.7.13) that

$$\mathfrak{R}[s(y)\, e^{i\theta}] \geqslant |s(x)| \cos(\theta+\gamma) - \epsilon \geqslant |s(x)|\,(1-\mu) - \epsilon, \tag{2.7.14}$$

for $|\xi-y|<\delta$. On the other hand, if $|\pi-\gamma|<\frac{1}{2}\pi$, we can define θ so that

$$|\pi+\theta|\leqslant\tfrac{1}{2}\pi-A, \quad |\gamma+\theta|\leqslant A. \tag{2.7.15}$$

If we write

$$\delta(\epsilon, x)=\delta_2(\epsilon, x), \quad \xi(\epsilon, x)=\lambda_{n+1}-\delta_2(\epsilon, x), \quad \mu(\epsilon, x)=1-\cos A,$$

then (2.7.14) follows from (2.7.10), (2.7.12), and (2.7.15). Since $\delta=\delta(\epsilon, x)\geqslant\delta(\epsilon)>0$, and $|\xi-x|\leqslant\delta$, this completes the proof of the first part of the theorem.

(ii) Let $A=0$, $p_\nu\geqslant 0$. Let

$$\delta(\epsilon, x)=\delta_1(\epsilon, x), \quad \xi(\epsilon, x, \phi)=\lambda_n+\delta_1(\epsilon, x) \qquad (|\phi|\leqslant\tfrac{1}{2}\pi),$$

$$\delta(\epsilon, x)=\delta_2(\epsilon, x), \quad \xi(\epsilon, x, \phi)=\lambda_{n+1}-\delta_2(\epsilon, x) \quad (\tfrac{1}{2}\pi\leqslant|\phi|\leqslant\pi).$$

Plainly
$$\mathfrak{R}\left[e^{i\phi}[r(y)-r(x)]\right]\geqslant 0 \quad (|\xi-y|<\delta).$$

Hence, using (2.7.5) and (2.7.10), we have

$$\mathfrak{R}\left[e^{i\phi}[s(y)-s(x)]\right]\geqslant-\epsilon \quad (|\xi-y|<\delta),$$

and therefore $s(x)\,\epsilon\, T_\alpha$, with $\mu(\epsilon, x)=0$.

(iii) If $y\geqslant x$, then

$$\begin{aligned}|\arg[s(y)-s(x)+t(x)-t(y)]| &= |\arg[r(y)-r(x)]| \\ &\leqslant A<\tfrac{1}{2}\pi.\end{aligned}$$

Since $t(x)\,\epsilon\, S_\beta$, by Theorem 2, it is plain, on writing $\eta(x, y)=t(x)-t(y)$, that $s(x)\,\epsilon\, S$.

2.8. If $s(x)=\sum_{\lambda_n\leqslant x} a_n$, it is convenient to have conditions† under which the function $s[e^x]$ belongs to T. These can be obtained by a suitable transformation of the preceding theorems.

THEOREM 4. *Let* $0<\lambda_1<\lambda_2<\ldots$, $\lambda_\nu\to\infty$. *Let* $s(x)=\sum_{\lambda_\nu\leqslant x} a_\nu$. *Suppose that for a sequence* ν_k *we have*

$$\lambda_{\nu_k}/\lambda_{\nu_k-1}\geqslant C_0>1.$$

† See Szász (14) for further references.

Then $s[e^x] \epsilon T_\beta'$ *if one of the following conditions is satisfied.*

(*a*) $$|a_\nu| \leqslant K(\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu \quad (\nu \neq \nu_k),$$

(*b*) $$\sum_{\nu=1}^{n}{}' |a_\nu|^p \lambda_\nu{}^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} \leqslant K\lambda_n \quad (p > 1),$$

(*c*) $$\sum_{\nu=1}^{\infty}{}' |a_\nu|^p \lambda_\nu^{p-1} (\lambda_\nu - \lambda_{\nu-1})^{1-p} = K < \infty,$$

where Σ' *means that the terms in which* $\nu = \nu_k$ *are omitted. In particular,* $s[e^x] \epsilon T_\beta'$ *if*

(*d*) $$\lambda_\nu/\lambda_{\nu-1} \geqslant C_0 > 1$$

for every ν. *Moreover,* $s[e^x] \epsilon S_\beta$ *if one of the above conditions is satisfied and if also* $a_\nu \to 0$.

THEOREM 5. *Let* $0 < \lambda_1 < \lambda_2 < \ldots$, $\lambda_\nu \to \infty$. *Let*

$$s(x) = \sum_{\lambda_\nu \leqslant x} a_\nu, \quad a_\nu = c_\nu + p_\nu,$$

where

$$|\arg p_\nu| \leqslant A < \tfrac{1}{2}\pi.$$

Let

$$\lambda_{\nu_k}/\lambda_{\nu_k - 1} \geqslant C_0 > 1,$$

$$|c_{\nu-1}| \leqslant K(\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu + \eta_\nu \quad (\nu \neq \nu_k),$$

where

$$\lim_{\nu \to \infty} \eta_\nu = 0.$$

Then (i) $s[e^x] \epsilon T$ *if one of the following conditions is satisfied.*

(*a*) $$|c_\nu| \leqslant K(\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu \quad (\nu \neq \nu_k),$$

(*b*) $$\sum_{\nu=1}^{n}{}' |c_\nu|^p \lambda_\nu{}^p (\lambda_\nu - \lambda_{\nu-1})^{1-p} \leqslant K\lambda_n \quad (p > 1),$$

(*c*) $$\sum_{\nu=1}^{\infty}{}' |c_\nu|^p \lambda_\nu^{p-1} (\lambda_\nu - \lambda_{\nu-1})^{1-p} = K < \infty,$$

where Σ' *means that the terms in which* $\nu = \nu_k$ *are omitted. Moreover,* (ii) $s[e^x] \epsilon T_a$ *if* $A = 0$, (iii) $s[e^x] \epsilon S$ *if* $c_n \to 0$.

We have

$$\log \lambda_\nu - \log \lambda_{\nu-1} = \log \lambda_\nu/\lambda_{\nu-1} = -\log \lambda_{\nu-1}/\lambda_\nu$$
$$= -\log [1 - (\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu] \geqslant (\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu.$$

Since $s[e^x]$ has increments a_ν at points $\log \lambda_\nu$, we may deduce Theorems 4 and 5 by writing $(\lambda_\nu - \lambda_{\nu-1})/\lambda_\nu$ in place of $\lambda_\nu - \lambda_{\nu-1}$, and λ_ν in place of e^{λ_ν}, in the conditions of Theorems 2 and 3.

The conditions of Theorem 5 extend the "one-sided" conditions of the classical Tauberian theorems of Hardy and Littlewood. For example, if $A = 0$, condition (a) is equivalent to

$$a_n \geqslant -K(\lambda_n - \lambda_{n-1})/\lambda_n.$$

3. *General Tauberian theorems.*

3.1. *Definitions.* We define $\mathfrak{A}$ to be the class of Fourier transforms of functions of $L(-\infty, \infty)$. If $h(x) \in L$, we write†

$$H(x) = \int e^{-iyx} h(y)\, dy, \quad \|H(x)\| = \int |h(y)|\, dy,$$

and similarly for other letters. Here, and in the remainder of this paper, any integration in which the limits are not specified is to be taken over the range $(-\infty, \infty)$.

3.2. We deduce the principal results of this section from the following theorem on Fourier transforms.

THEOREM 6. *Suppose that $H(x)$ and $K(x)$ belong to $\mathfrak{A}$, that $H(x) = 0$ for $|x| > \pi/\lambda$, and that $K(x) \neq 0$ for $|x| \leqslant \pi/\lambda$. Then*

(a) $$\frac{H(x)}{K(x)} \in \mathfrak{A},$$

(b) $$\left\| \frac{H(x)}{K(x)} \right\| \leqslant \frac{\alpha(\lambda)\|H(x)\|}{|K(0)|},$$

where $\alpha(\lambda) = \alpha(\lambda, K)$ is independent of $H(x)$, and $\lim_{\lambda \to \infty} \alpha(\lambda) = 1$ for any fixed $K(x)$.

(c) If $|K(x)| \geqslant B > 0$ for $|x| \leqslant \pi/\lambda$, and if $k(x)$ is majorized by a function $k^*(x)$ of L, then

$$\frac{H(x)}{K(x)} = \int e^{-ixy} p(y)\, dy,$$

where $p(y)$ is majorized by a function $p^*(y)$ of L which depends only on λ, B, $k^*(x)$, and $H(x)$.

† It follows from the uniqueness theorem for transforms of functions of L that $h(y)$ is determined almost everywhere by $H(x)$, and that $\|H(x)\|$ is determined uniquely.

We require two elementary lemmas.

LEMMA 1. *If F_1 and F_2 belong to $\mathfrak{A}$, and c_1 and c_2 are real or complex numbers, then $F_1 . F_2$ and $c_1 F_1 + c_2 F_2$ belong to $\mathfrak{A}$, and*

(i) $$\| F_1 . F_2 \| \leqslant \| F_1 \| . \| F_2 \|,$$

(ii) $$\| c_1 F_1 + c_2 F_2 \| \leqslant |c_1| \| F_1 \| + |c_2| \| F_2 \|.$$

(i) Let

$$F_1(x) = \int e^{-ixy} f_1(y)\, dy, \qquad F_2(x) = \int e^{-iyx} f_2(y)\, dy,$$

where f_1 and f_2 belong to L. Then

$$\begin{aligned} F_1(x) . F_2(x) &= \int e^{-ixt} f_2(t)\, dt \int e^{-ixy} f_1(y)\, dy \\ &= \int f_2(t)\, dt \int e^{-ixy} f_1(y-t)\, dy \\ &= \int e^{-ixy}\, dy \int f_1(y-t) f_2(t)\, dt. \end{aligned}$$

$$\begin{aligned} \| F_1(x) . F_2(x) \| &= \int dy \left| \int f_1(y-t) f_2(t)\, dt \right| \\ &\leqslant \int |f_1(y)|\, dy \int |f_2(t)|\, dt = \| F_1 \| . \| F_2 \|. \end{aligned}$$

The second inequality is evident.

LEMMA 2. *If F and G belong to $\mathfrak{A}$, and if $\| F \| = \alpha < 1$, then*

$$G(x) \Big/ \big(1 + F(x) \big)$$

belongs to $\mathfrak{A}$, and

$$\left\| \frac{G(x)}{1+F(x)} \right\| \leqslant \frac{\| G(x) \|}{1-\alpha}.$$

We have

$$\begin{aligned} \frac{G(x)}{1+F(x)} &= G(x) \sum_{n=0}^{\infty} (-1)^n [F(x)]^n \\ &= G(x) \left[1 + \sum_{n=1}^{\infty} (-1)^n \int f_n(y)\, e^{-iyx}\, dy \right], \end{aligned}$$

where $$\int |f_n(y)|\,dy = \|F^n\| \leqslant \alpha^n,$$

by Lemma 1. Hence

$$\frac{G(x)}{1+F(x)} = G(x)\left[1+\int e^{-iyx}\left[\sum_{n=1}^{\infty}(-1)^n f_n(y)\right]dy\right],$$

$$\left\|\frac{G(x)}{1+F(x)}\right\| \leqslant \|G\|\left[1+\sum_{n=1}^{\infty}\int |f_n(y)|\,dy\right]$$

$$\leqslant \|G\| \sum_{n=0}^{\infty} \alpha^n = \frac{\|G\|}{1-\alpha}.$$

Proof of Theorem 6. (*a*) If $\epsilon > 0$, we define a function $\Omega_\epsilon(x)$ as follows:

$$\Omega_\epsilon(x) = 1 \qquad (|x| \leqslant \epsilon),$$

$$\Omega_\epsilon(x) = 2-|x|/\epsilon \quad (\epsilon < |x| \leqslant 2\epsilon),$$

$$\Omega_\epsilon(x) = 0 \qquad (|x| > 2\epsilon).$$

An easy calculation shows that

$$\Omega_\epsilon(x) = \int e^{-iyx}\,\omega_\epsilon(y)\,dy, \tag{3.2.1}$$

where

$$\omega_\epsilon(x) = \epsilon\omega(\epsilon y) = \frac{\cos y\epsilon - \cos 2y\epsilon}{\pi\epsilon y^2}. \tag{3.2.2}$$

Let N be a positive integer, and define ϵ by

$$2\pi = 3\epsilon\lambda N. \tag{3.2.3}$$

Let $x_n = 3\epsilon n - \pi/\lambda$ for $n = 0, 1, \ldots, N$. Then

$$\sum_{n=0}^{N} \Omega_\epsilon(x-x_n) = 1 \quad (|x| \leqslant \pi/\lambda),$$

and

$$\frac{H(x)}{K(x)} = \sum_{n=0}^{N} \frac{\Omega_\epsilon(x-x_n)\,H(x)}{K(x)}. \tag{3.2.4}$$

We now show that each term of this sum is a function of $\mathfrak{A}$ when N is sufficiently large. We have

$$\frac{\Omega_\epsilon(x-x_n)\,H(x)}{K(x)} = \frac{H(x)\,\Omega_\epsilon(x-x_n),}{K(x_n)+[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)}, \tag{3.2.5}$$

since $\Omega_{2\epsilon}(x) = 1$ for $|x| \leqslant 2\epsilon$ and $\Omega_{\epsilon}(x) = 0$ for $|x| > 2\epsilon$. Also

$$\begin{aligned}
&[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)\\
&\quad = \int e^{-itx}k(t)\,dt \int e^{-iyx+iyx_n}\omega_{2\epsilon}(y)\,dy - \int e^{-itx_n}k(t)\,dt \int e^{-iyx+iyx_n}\omega_{2\epsilon}(y)\,dy\\
&\quad = \int k(t)\,dt \int e^{-iyx+ix_n(y-t)}\omega_{2\epsilon}(y-t)\,dy - \int e^{-itx_n}k(t)\,dt \int e^{-iyx+iyx_n}\omega_{2\epsilon}(y)\,dy\\
&\quad = \int e^{-iyx}\,dy \int e^{ix_n(y-t)}k(t)[\omega_{2\epsilon}(y-t)-\omega_{2\epsilon}(y)]\,dt.
\end{aligned}$$

Hence

$$\begin{aligned}
(3.2.6)\quad &\|[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)\|\\
&\qquad = \int dy \left| \int e^{ix_n(y-t)}k(t)[\omega_{2\epsilon}(y-t)-\omega_{2\epsilon}(y)]\,dt \right|\\
&\qquad \leqslant \int |k(t)|\,dt \int |\omega_{2\epsilon}(y-t)-\omega_{2\epsilon}(y)|\,dy\\
&\qquad = \int |k(t)|\,dt \int |\omega(u-2\epsilon t)-\omega(u)|\,du.
\end{aligned}$$

Since $K(x)$ is continuous and not zero in the closed interval $|x| \leqslant \pi/\lambda$, $|K(x)|$ has a positive lower bound B in this interval. Therefore, since $\int |\omega(u-2\epsilon t)-\omega(u)|\,du$ is bounded and tends to 0, for any fixed t, as $\epsilon \to 0$, it follows from (3.2.3) and (3.2.6) that, if N is sufficiently large,

$$(3.2.7)\qquad \|[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)\| < B \leqslant |K(x_n)|$$

for $n = 0, 1, \ldots, N$. Hence, if N is so chosen,

$$\frac{H(x)}{K(x_n)+[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)} \in \mathfrak{A},$$

by Lemma 2, and the conclusion follows from Lemma 1.

(*b*) If $\epsilon = \pi/2\lambda$, we have

$$\frac{H(x)}{K(x)} = \frac{H(x)}{K(0)+[K(x)-K(0)]\,\Omega_{2\epsilon}(x)}.$$

The preceding argument shows that

$$\lim_{\epsilon \to 0} \|[K(x)-K(0)]\,\Omega_{2\epsilon}(x)\| = 0,$$

and the conclusion follows from Lemmas 1 and 2.

(c) We have, by (3.2.6),

$$\|[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)\| \leqslant \int k^*(t)\,dt \int |\omega(u-2\epsilon t)-\omega(u)|\,du.$$

It is plain that (3.2.7) is satisfied for a value of N which depends only on λ, B, and $k^*(t)$, and that, when N is so chosen, we can write

$$\frac{H(x)}{K(x_n)+[K(x)-K(x_n)]\,\Omega_{2\epsilon}(x-x_n)} = \int e^{-iyx} p_n(y)\,dy,$$

where $|p_n(y)| \leqslant p_n^*(y)$, and $p_n^*(y)$ belongs to L and depends only on B and $k^*(t)$. The conclusion follows now from (3.2.4), (3.2.5), and Lemma 1.

3.3. We now define a function $\Phi(x, y)$ which is used frequently in the sequel. Consider the class of functions $h(t)$ which satisfy the following conditions:

$$(3.3.1)\quad \begin{cases} \text{(i)} & h(t) \text{ is real and belongs to } L(-\infty, \infty), \\ \text{(ii)} & \int |h(t)|\,dt = 1, \\ \text{(iii)} & \int h(t)\,dt > 0, \\ \text{(iv)} & H(u) = \int e^{-itu} h(t)\,dt = 0 \quad \text{for} \quad |u| > \pi. \end{cases}$$

These conditions are satisfied, for example, if $h(y) = 2(\sin \frac{1}{2}\pi t/\pi t)^2$. Let

$$h^+(t) = h(t) \quad [h(t) \geqslant 0],$$

$$h^+(t) = 0 \quad [h(t) < 0],$$

$$h^-(t) = h(t) - h^+(t).$$

If $0 \leqslant x \leqslant 1$, $y \geqslant 0$, we define $\Phi_h(x, y)$ by

$$(3.3.2)\quad \Phi_h(x, y) = (1-x)\int_{-y}^{y} h^+(t)\,dt + \int_{-y}^{y} h^-(t)\,dt - \int_{-\infty}^{-y} + \int_{y}^{\infty} |h(t)|\,dt.$$

We define $\Phi(x, y)$ to be the upper bound of $\Phi_h(x, y)$ for all functions $h(t)$ which satisfy (3.3.1).

The functions $h(t)$ and corresponding $\Phi_h(x, y)$ play a fundamental part in what follows. Our theorems usually involve certain constants A and B, and we use different definitions of $h(t)$ for different values of A and B. Our motive for introducing the function $\Phi(x, y)$ becomes clear if we observe that, for given A and B, the best results in these theorems are obtained when we work with a function $h(t)$ for which $\Phi_h(A, B)$ is as large as possible.

THEOREM 7. (*a*) $\Phi(x, y)$ *is continuous.*

(*b*) $\Phi(x, y)$ *is an increasing function of* y *for* $0 \leqslant x < 1$ *and a decreasing function of* x *for* $y > 0$.

(*c*) $$\lim_{y\to\infty} \Phi(x, y) = 1-x.$$

(*d*) $$\Phi(0, y) > 0 \quad \text{if} \quad y > \tfrac{1}{2}.$$

(*a*) If $h(t)$ satisfies (3.3.1), we have

$$|H(u)| \leqslant \int |h(t)|\, dt = 1,$$

$$|h(t)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(u)|\, du \leqslant 1,$$

so that $\Phi_h(x, y)$ is a continuous function of (x, y) for each $h(t)$. Moreover, its modulus of continuity is independent of $h(t)$. That is, if $\epsilon > 0$, we can choose $\delta = \delta(\epsilon)$, independent of $h(t)$, so that

$$|\Phi_h(x_1, y_1)-\Phi_h(x_2, y_2)| < \epsilon \quad \text{for} \quad |x_1-x_2| < \delta, \quad |y_1-y_2| < \delta.$$

Now suppose that $|x_1-x_2| < \delta$, $|y_1-y_2| < \delta$, and choose $h_1(t)$, $h_2(t)$ so that

$$\Phi_{h_1}(x_1, y_1) \geqslant \Phi(x_1, y_1)-\epsilon, \quad \Phi_{h_2}(x_2, y_2) \geqslant \Phi(x_2, y_2)-\epsilon.$$

Then
$$\begin{aligned}\Phi(s_1, y_1) &\geqslant \Phi_{h_2}(x_1, y_1) \geqslant \Phi_{h_2}(x_2, y_2)-\epsilon\\ &\geqslant \Phi(x_2, y_2)-2\epsilon,\\ \Phi(x_2, y_2) &\geqslant \Phi_{h_1}(x_2, y_2) \geqslant \Phi_{h_1}(x_1, y_1)-\epsilon\\ &\geqslant \Phi(x_1, y_1)-2\epsilon.\end{aligned}$$

Hence
$$|\Phi(x_1, y_1)-\Phi(x_2, y_2)| \leqslant 2\epsilon$$

when $|x_1-x_2| < \delta$, $|y_1-y_2| < \delta$, and this is what we had to prove.

(*b*) is obvious.

(*c*) Let $h(t) = 2(\sin \tfrac{1}{2}\pi t/\pi t)^2$. Then

$$\Phi(x, y) \geqslant \Phi_h(x, y) = 2(1-x)\int_{-y}^{y} \left(\frac{\sin \frac{1}{2}\pi t}{\pi t}\right)^2 dt - 2\int_{-\infty}^{-y} + \int_{y}^{\infty} \left(\frac{\sin \frac{1}{2}\pi t}{\pi t}\right)^2 dt,$$

$$\lim_{y\to 0} \Phi(x, y) \geqslant 2(1-x)\int_{-\infty}^{\infty} \left(\frac{\sin \frac{1}{2}\pi t}{\pi t}\right)^2 dt = 1-x.$$

Since $\Phi(x, y) \leqslant 1-x$, on account of (3.3.1) and (3.3.2), it follows that

$$\lim_{y\to\infty} \Phi(x, y) = 1-x.$$

(*d*) Let†

$$C=\frac{2}{\pi}\int\left|\frac{\cos\pi t}{1-4t^2}\right|dt, \qquad h(t)=\frac{2\cos\pi t}{\pi C(1-4t^2)}.$$

It is easy to verify that (3.3.1) is satisfied. Also

$$\Phi_h(0,\tfrac{1}{2})=2\int_0^{\frac{1}{2}}h(t)\,dt-2\int_{\frac{1}{2}}^{\infty}|h(t)|\,dt,$$

since $\bar{h}(t)=0$ for $|t|\leqslant\frac{1}{2}$. We have

$$\begin{aligned}\int_{\frac{1}{2}}^{\infty}|h(t|\,dt&=\frac{2}{\pi C}\int_{\frac{1}{2}}^{\infty}\left|\frac{\cos\pi t}{1-4t^2}\right|dt=\frac{1}{\pi C}\int_{\frac{1}{2}}^{\infty}\frac{|\cos\tfrac{1}{2}\pi t|\,dt}{t^2-1}\\
&=\frac{1}{\pi C}\sum_{n=0}^{\infty}\left[-\int_{4n+1}^{4n+3}\frac{\cos\frac{1}{2}\pi t\,dt}{t^2-1}+\int_{4n+3}^{4n+5}\frac{\cos\frac{1}{2}\pi t\,dt}{t^2-1}\right]\\
&=\frac{1}{\pi C}\sum_{n=0}^{\infty}\left[\int_{-1}^{1}\frac{\cos\frac{1}{2}\pi t\,dt}{(t+4n+2)^2-1}+\int_{-1}^{1}\frac{\cos\frac{1}{2}\pi t\,dt}{(t+4n+4)^2-1}\right]\\
&\quad-\frac{1}{2\pi C}\int_{-1}^{1}\cos\tfrac{1}{2}\pi t\sum_{n=0}^{\infty}\left[\frac{1}{t+4n+1}-\frac{1}{t+4n+3}+\frac{1}{t+4n+3}-\frac{1}{t+4n+5}\right]dt\\
&=\frac{1}{2\pi C}\int_{-1}^{1}\frac{\cos\frac{1}{2}\pi t}{1+t}\,dt=\frac{1}{\pi C}\int_0^1\frac{\cos\frac{1}{2}\pi t}{1-t^2}\,dt=\int_0^{\frac{1}{2}}h(t)\,dt.\end{aligned}$$

Therefore $\Phi_h(0,\frac{1}{2})=0$, and since $\Phi_h(0,y)$ is a strictly increasing function of y,

$$\Phi(0,y)\geqslant\Phi_h(0,y)>0 \quad \text{for} \quad y>\tfrac{1}{2}.$$

3.4. We are now in a position to prove the following Tauberian theorem.

THEOREM 8.

HYPOTHESIS‡.

(i) $$k(x)\in L,$$

(ii) $$K(it)=\int e^{-itx}k(x)\,dx\neq 0 \quad \textit{for} \quad |t|\leqslant\pi/\lambda,$$

(iii) $$K(0)=1,$$

(iv) $$|s(x)|\leqslant C,$$

(v) $$g(x)=\int k(x-y)\,s(y)\,dy,$$

† This definition of $h(t)$ was first used by Ingham (8).

‡ Our notation here is not quite the same as in Theorem 6. The use of $K(it)$ instead of $K(t)$ is more convenient in relation to the theorems which follow.

(vi) *$s(x)$ belongs to T, and for some positive ϵ,*

$$\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda] > 0.$$

CONCLUSION. (*a*) *If we write*† $S = \overline{\lim_{x\to\infty}}\,|s(x)|$, $G = \overline{\lim_{x\to\infty}}\,|g(x)|$, *then*

$$S \leqslant \frac{\epsilon + a(\lambda)\,G}{\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda]},$$

where $a(\lambda)$ depends only on $k(x)$ and λ, and $\lim_{\lambda\to\infty} a(\lambda) = 1$.

(*b*) *Moreover, if $s(x)$ belongs also to T', with*

$$\Phi[\mu'(\epsilon),\ \delta'(\epsilon)/\lambda] > 0,$$

and we write $S' = \overline{\mathrm{bd}}\,|s(x)|$, $G' = \overline{\mathrm{bd}}\,|g(x)|$, then

$$S' \leqslant \frac{\epsilon + a(\lambda)\,G'}{\Phi[\mu'(\epsilon),\ \delta'(\epsilon)/\lambda]}.$$

(*a*) Suppose that $h(x)$ satisfies (3.3.1) and let

$$\lambda h_\lambda(x) = h(x/\lambda).$$

Then, by Theorem 6, there is a function $p_\lambda(y)$ of L such that

$$\begin{aligned}\int e^{-itx} h_\lambda(x)\,dx = H_\lambda(t) &= K(it)\int e^{-ity} p_\lambda(y)\,dy \\ &= \int e^{-ity} p_\lambda(y)\,dy \int e^{-itx} k(x)\,dx \\ &= \int e^{-itx}\,dx \int k(x-y)\,p_\lambda(y)\,dy.\end{aligned}$$

It follows from the uniqueness theorem for transforms of functions of L that

$$h_\lambda(x) = \int k(x-y)\,p_\lambda(y)\,dy$$

for almost all values of x. Therefore, since $s(x)$ is bounded,

$$\int h_\lambda(x-y)\,s(y)\,dy = \int p_\lambda(x-y)\,g(y)\,dy. \tag{3.4.1}$$

† We use S, G, S', G' in the same sense in the remainder of this paper.

If ξ, δ, and μ are the functions defined in § 2, we have

$$(3.4.2)\quad \left|\int h_\lambda(\xi-y)\,s(y)\,dy\right| \geqslant [(1-\mu)\,|s(x)|-\epsilon]\int_{-\delta}^{\delta} h_\lambda{}^{+}(y)\,dy + \overline{\mathrm{bd}}_{|\xi-y|<\delta}\,|s(y)|\int_{-\delta}^{\delta} h_\lambda{}^{-}(y)\,dy - \int_{-\infty}^{\xi-\delta} + \int_{\xi+\delta}^{\infty} |h_\lambda(\xi-y)\,s(y)|\,dy.$$

If we let $x\to\infty$ in (3.4.2), and use (3.4.1) and (3.3.1) (ii), we have

$$(3.4.3)\quad S\Phi_h[\mu(\epsilon),\,\delta(\epsilon)/\lambda] \leqslant \overline{\lim_{x\to\infty}} \int |\,p_\lambda(x-y)\,g(y)|\,dy+\epsilon \leqslant \epsilon+\alpha(\lambda)\,G,$$

since $\int|\,p_\lambda(y)|\,dy\leqslant\alpha(\lambda)$, $\lim_{\lambda\to\infty}\alpha(\lambda)=1$, by Theorem 6 (*b*) and (3.3.1) (ii). Since (3.4.3) is true for any function $h(x)$ which satisfies (3.3.1), the conclusion follows.

(*b*) If $\Phi[\mu'(\epsilon),\,\delta'(\epsilon)/\lambda]>0$, we consider the upper bounds of the two sides of (3.4.2). Using (3.4.1), we have

$$S'\Phi_h[\mu'(\epsilon),\,\delta'(\epsilon)/\lambda] \leqslant \overline{\mathrm{bd}}\int|\,p_\lambda(x-y)\,g(y)|\,dy+\epsilon \leqslant \epsilon+\alpha(\lambda)\,G',$$

and it follows that

$$S'\,\Phi[\mu'(\epsilon),\,\delta'(\epsilon)/\lambda] \leqslant \epsilon+\alpha(\lambda)\,G'.$$

3.5. The following corollary of Theorem 8 is usually sufficient for application to special Tauberian theorems.

THEOREM 9. *Suppose that conditions* (i), (iii), (iv), *and* (v) *of Theorem* 8 *are satisfied and that* $K(it)\neq 0$ *for all real* t.

(*a*) *Then if* $s(x)\,\epsilon\,T$, *and* C_1 *is a positive constant, we can define another positive constant* C_2, *depending only on* $k(x)$, $\delta(\epsilon)$, $\mu(\epsilon)$, *and* C_1, *such that*

$$S\leqslant C_1+C_2\,G.$$

(*b*) *If* $s(x)\,\epsilon\,T'$, *we can write*

$$S'\leqslant C_1+C_2\,G',$$

where C_2 *now depends only on* $k(x)$, $\delta'(\epsilon)$, $\mu'(\epsilon)$, *and* C_1.

(*a*) Since $\mu(+0)<1$, it follows from Theorem 7 (*c*) that we can choose $\epsilon>0$, $\lambda>0$, so that

$$\Phi[\mu(\epsilon),\,\delta(\epsilon)/\lambda]\geqslant\epsilon/C_1.$$

If we then write

$$C_2 = \frac{\alpha(\lambda)}{\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda]},$$

the conclusion follows from Theorem 8(a).

(b) We use Theorem 8 (b) in the same way to prove the second part.

3.6. We can now prove the two theorems which Wiener describes collectively as the "general Tauberian theorem" (**18**, Chapter 2). The first is

3.6.1. THEOREM A.

HYPOTHESIS.

(i) $k_1(x)$ *and* $k_2(x)$ *belong to* L,

(ii) $K_1(it) \neq 0$ *for real* t,

(iii) $s(x)$ *is bounded*,

(iv) $$\lim_{x\to\infty} \int k_1(x-y)\, s(y)\, dy = A \int k_1(x)\, dx.$$

CONCLUSION.

$$\lim_{x\to\infty} \int k_2(x-y)\, s(y)\, dy = A \int k_2(x)\, dx.$$

We may plainly suppose that $A = 0$, $\int k_1(x)\, dx = 1$. Let

$$S(x) = \int k_2(x-y)\, s(y)\, dy.$$

Then

$$\begin{aligned}\int k_1(x-y)\, S(y)\, dy &= \int k_1(y)\, S(x-y)\, dy \\ &= \int k_1(y)\, dy \int k_2(x-t)\, s(t-y)\, dt \\ &= \int k_2(x-t)\, dt \int k_1(y)\, s(t-y)\, dy \\ &= o(1), \quad \text{as} \quad s\to\infty,\end{aligned}$$

since $\int k_1(y)\, s(t-y)\, dy = \int k_1(t-y)\, s(y)\, dy$ is bounded and $o(1)$ as $t\to\infty$. Now

$$\overline{\lim_{0\leqslant x'-x\to 0}} |S(x')-S(x)| \leqslant \overline{\mathrm{bd}}\, |s(y)|\ \overline{\lim_{x'-x\to 0}} \int |k_2(x'-x+y)-k_2(y)|\, dy = 0,$$

by a classical theorem on Lebesgue integration. Hence $S(x)$ is slowly oscillating in Schmidt's sense, and therefore belongs to T. It follows from Theorem 9 (a) that $\lim_{t\to\infty} S(t) = 0$, which is what we have to prove.

3.6.2. Following Wiener, we define M_1 to be the class of continuous functions of L for which

$$\sum_{n=-\infty}^{\infty} \max_{n\leqslant x<n+1} |f(x)| < \infty.$$

Then Wiener's second theorem is as follows.

THEOREM B.

HYPOTHESIS.

(i) *$k_1(x)$ and $k_2(x)$ belong to M_1,*

(ii) *$K_1(it) \neq 0$ for real t,*

(iii) *$s(x)$ is of bounded variation in any finite interval, and for all values of X we have*

$$\int_X^{X+1} |ds(x)| \leqslant C.$$

(iv) $$\lim_{x\to\infty} \int k_1(x-y)\, ds(y) = A \int k_1(x)\, dx.$$

CONCLUSION. $$\lim_{x\to\infty} \int k_2(x-y)\, ds(y) = A \int k_2(x)\, dx.$$

We may suppose that $A = 0$. Let $S(x) = \int k_2(x-y)\, ds(y)$. Then

$$\begin{aligned}\int k_1(x-y)\, S(y)\, dy &= \int k_1(y)\, S(x-y)\, dy \\ &= \int k_1(y)\, dy \int k_2(x-t)\, ds(t-y) \\ &= \int k_2(x-t)\, dt \int k_1(y)\, ds(t-y) \\ &= \int k_2(x-t)\, dt \int k_1(t-y)\, ds(y) \\ &= o(1), \quad \text{as} \quad x\to\infty,\end{aligned}$$

since $\int k_1(t-y)\,ds(y)$ is bounded and $o(1)$ as $t\to\infty$.

$$|S(x')-S(x)| = \left|\int [k_2(x'-y)-k_2(x-y)]\,ds(y)\right|$$

$$\leqslant C \sum_{n=-\infty}^{\infty} \max_{n\leqslant y<n+1} |k_2(x'-y)-k_2(x-y)|$$

$$= C \sum_{n=-\infty}^{\infty} \max_{x+n-1<u\leqslant x+n} |k_2(x'-x+u)-k_2(u)|$$

$$\leqslant 2C \sum_{n=-\infty}^{\infty} \max_{n<u\leqslant n+1} |k_2(x'-x+u)-k_2(u)|.$$

If $\epsilon>0$, we can choose $N=N(\epsilon)$ so that

$$\sum_{|n|\geqslant N-1} \max_{n<u\leqslant n+1} |k_2(x'-x+u)-k_2(u)| < \epsilon \quad (|x'-x|<1).$$

Therefore

$$\overline{\lim_{0\leqslant x'-x\to 0}} |S(x')-S(x)| \leqslant \epsilon,$$

since $k_2(u)$ is continuous in $-N\leqslant u\leqslant N$. Since ϵ is arbitrary, it is plain that $S(x)$ is slowly oscillating and belongs to T. The conclusion follows, as in the preceding theorem, from Theorem 9 (a).

Approximate kernels.

3.7. The "kernel" in Wiener's theorems is a function of the form $f(x-y)$. In special Tauberian theorems we frequently have to consider transformations in which the kernel is not a function of $x-y$, but behaves effectively like such a function. We therefore introduce at this stage the notion of an *approximate kernel.*

We say that $\kappa(x, y)$ approximates to $k(x-y)$, or what is the same thing, that $\kappa(x, x-y)$ approximates to $k(y)$, if

$$\eta(x) = \int |\kappa(x, x-y)-k(y)|\,dy$$

is bounded and $\lim_{x\to\infty} \eta(x) = 0$. It is easy to see that, if $k(x)$ satisfies the conditions of Theorem 8, the conclusion (a) remains true if we replace $k(x-y)$ by any approximating kernel $\kappa(x, y)$. However, in order to consider certain less trivial extensions of Theorem 8, it is necessary to extend the definition of an approximate kernel by introducing a new parameter σ.

Definition. We say that $\kappa(x, x-y)$ approximates to $k(y)$ in the interval $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, and write

$$\kappa(x, x-y) \sim k(y) \quad (\sigma_1 \leqslant \sigma \leqslant \sigma_2),$$

if $\kappa(x, x-y)\, e^{-\sigma y}$ and $k(y)\, e^{-\sigma y}$ belong to L for $\sigma_1 \leqslant \sigma \leqslant \sigma_2$, and if

$$\eta(x) = \overline{\mathrm{bd}}_{\sigma_1 \leqslant \sigma \leqslant \sigma_2} \int |\kappa(x, x-y) - k(y)|\, e^{-\sigma y}\, dy$$

is bounded and $$\lim_{x \to \infty} \eta(x) = 0.$$

3.8. We can now prove

THEOREM 10.

HYPOTHESIS.

(i) $$\sigma_2 > \sigma_1 \geqslant 0,$$

(ii) $$\kappa(x, x-y) \sim k(y) \quad (0 \leqslant \sigma \leqslant \sigma_2)$$

(iii) $$K(\omega) = K(\sigma+it) = \int e^{-\omega y} k(y)\, dy \neq 0 \quad (0 \leqslant \sigma \leqslant \sigma_1),$$

(iv) $$K(0) = 1,$$

(v) $s(x) = O(e^{\sigma x})$ $(x > 0)$ *for each fixed* σ *in the range* $\sigma_1 < \sigma \leqslant \sigma_2$,

$$s(x) = 0 \quad (x \leqslant 0),$$

(vi) $g(x) = \int \kappa(x, y)\, s(y)\, dy$ *is bounded,*

(vii) $s(x)$ *belongs to* T.

CONCLUSION.

(*a*) $s(x)$ *is bounded.*

(*b*) *If* C_1 *is any positive constant, we can define another positive constant* C_2, *depending only on* $k(y)$, $\delta(\epsilon)$, $\mu(\epsilon)$ *and* C_1, *so that*

$$S \leqslant C_1 + C_2 G.$$

It will be observed that the boundedness of $s(x)$ is now a conclusion, being replaced in the hypothesis by the more general condition (v). This result is sufficient for most applications to special Tauberian theorems; but here it seems worth while to aim at the greatest possible generality, and we show that Theorem 10 is included in the following more general result.

THEOREM 11.

HYPOTHESIS.

(i), (ii), (iv), (v), (vi) *as in Theorem* 10,

(iii) $$K(\omega) \neq 0 \quad (0 \leqslant \sigma \leqslant \sigma_1,\ |t| \leqslant \pi/\lambda),$$

(vii) *There is a function* $\gamma = \gamma(\sigma)$, *defined for* $0 \leqslant \sigma \leqslant \sigma_1$, *such that* $\beta\sigma \leqslant \gamma \leqslant \sigma$, $\beta > 0$, *and*

$$s_\sigma(x) = e^{-\gamma x}\int_0^x e^{-(\sigma-\gamma)y}\,ds(y) = e^{-\sigma x}s(x) + (\sigma-\gamma)\,e^{-\gamma x}\int_0^x e^{-(\sigma-\gamma)y}s(y)\,dy$$

belongs to T, *with* $\delta(\epsilon)$, $\mu(\epsilon)$ *independent of* σ, *and*

$$\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda] > 0$$

for some positive ϵ.

CONCLUSION.

(*a*) $s(x)$ *is bounded.*

(*b*) $$S \leqslant \frac{\epsilon + a(\lambda)\,G}{\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda]},$$

where $a(\lambda)$ *depends only on* $k(y)$ *and* λ, *and* $\lim_{\lambda\to\infty} a(\lambda) = 1$.

Deduction of Theorem 10 *from Theorem* 11.

Let $\gamma(\sigma) = \sigma$ for $0 \leqslant \sigma \leqslant \sigma_1$, so that

$$s_\sigma(x) = e^{-\sigma x}s(x).$$

Since $s(x) \in T$, we can define ξ, δ, μ, θ to satisfy (2.2.1), (2.2.2), (2.2.3), (2.2.6). Moreover, we can always suppose that $\delta(\epsilon)$ is finite and $\delta(\epsilon, x)$ bounded. Then

$$\mathfrak{R}[s(x)\,e^{i\theta}] \geqslant (1-\mu)\,|s(x)| - \epsilon \quad (|\xi - y| < \delta).$$

Hence

$$\mathfrak{R}[s_\sigma(y)\,e^{i\theta}] = \mathfrak{R}[s(y)e^{-\sigma y}\,e^{i\theta}] \geqslant (1-\mu)\,e^{-\sigma(y-x)}\,|s(x)|\,e^{-\sigma x} - \epsilon\,e^{-\sigma y}$$
$$\geqslant (1-\mu)\,e^{-2\sigma\delta}\,|s_\sigma(x)| - \epsilon \quad (|\xi - y| < \delta).$$

This shows that $s_\sigma(x)$ belongs to T, and we can define its Tauberian characteristics δ_1 and μ_1 (independent of σ) by

$$\delta_1(\epsilon, x) = \delta(\epsilon, x), \quad \mu_1(\epsilon, x) = 1 - [1-\mu(\epsilon, x)]\,e^{-2\sigma_1 d},$$

where d is the upper bound of $\delta(\epsilon, x)$ for all x and positive ϵ. Since $\Phi[\mu_1(\epsilon), \delta_1(\epsilon)/\lambda] > 0$ for sufficiently small λ, by Theorem 7(c), the conclusion follows from Theorems 11 (a) and 9 (a).

Proof of Theorem 11.

Since $K(\omega)$ is continuous in a strip to the right of σ_1, it follows from (iii) that there is a value of σ_2 for which the hypothesis of the theorem is true and for which $K(\omega) \neq 0$ in $|t| \leqslant \pi/\lambda$, $0 \leqslant \sigma \leqslant \sigma_2$. Also, we may plainly suppose that σ_1 is the lower bound of the positive numbers σ for which $s(y)e^{-\sigma y}$ is bounded. Let

$$\sigma_1 < \sigma' < \sigma_2, \tag{3.8.1}$$

$$0 < \sigma_0 < \sigma_1 \quad (\sigma_1 > 0), \tag{3.8.2}$$

$$\sigma_0 = 0 \qquad (\sigma_1 = 0),$$

$$\sigma' - \sigma_0 < \sigma_2 - \sigma'. \tag{3.8.3}$$

We write A_r $(r = 1, 2, \ldots)$ for positive constants and $\epsilon_r(u)$ for functions which are bounded and tend steadily to zero as $u \to +0$, provided that these are independent of σ' and σ_0.

Let

$$\begin{aligned} f(x) &= s_{\sigma_0}(x) \\ &= e^{-\sigma_0 x} s(x) + (\sigma_0 - \gamma_0) \int_0^x e^{-\gamma_0(x-y)} e^{-\sigma_0 y} s(y)\, dy \quad [\gamma_0 = \gamma(\sigma_0)]. \end{aligned} \tag{3.8.4}$$

It follows from (3.8.4) (by the use of Laplace integrals) that

$$e^{-\sigma_0 x} s(x) = f(x) - (\sigma_0 - \gamma_0) \int_0^x e^{-\sigma_0(x-y)} f(y)\, dy, \tag{3.8.5}$$

so that

$$g(x)\, e^{-\sigma' x} = \int \rho(x, y) f(y)\, e^{-(\sigma' - \sigma_0) y}\, dy, \tag{3.8.6}$$

where

$$\rho(x, y) = \kappa(x, y)\, e^{-\sigma'(x-y)} - (\sigma_0 - \gamma_0)\, e^{-\sigma'(x-y)} \int_y^\infty \kappa(x, u)\, du. \tag{3.8.7}$$

It is convenient to divide the proof from this point onwards into five stages.

(1) We define a function $r(y)$ related to $\rho(x, y)$ as $k(y)$ is to $\kappa(x, y)$, and show that

$$\rho(x, x-y) \sim r(y) \quad (|\sigma| \leqslant \sigma' - \sigma_0).$$

Let
$$r(y) = k(y)\, e^{-\sigma' y} - (\sigma_0 - \gamma_0)\, e^{-\sigma' y} \int_{-\infty}^{y} k(u)\, du.$$

Then it is plain, by (3.8.2) and (3.8.3), that $\rho(x, x-y)\, e^{-\sigma y}$ and $r(y)\, e^{-\sigma y}$ belong to L for $|\sigma| \leqslant \sigma' - \sigma_0$. Moreover, if $|\sigma| \leqslant \sigma' - \sigma_0$, $\sigma_0 > 0$, we have

$$\begin{aligned}
(3.8.8) \quad & \int |\rho(x, x-y) - r(y)|\, e^{-\sigma y}\, dy \\
&= \int e^{-(\sigma'+\sigma)y} \Big| \kappa(x, x-y) - k(y) \\
&\qquad\qquad - (\sigma_0 - \gamma_0) \int_{-\infty}^{y} [\kappa(x, x-u) - k(u)]\, du \Big| dy \\
&\leqslant \int e^{-(\sigma'+\sigma)y} |\kappa(x, x-y) - k(y)|\, dy \qquad (\sigma_1 > 0) \\
&\qquad\qquad + (\sigma_0 - \gamma_0) \int |\kappa(x, x-u) - k(u)|\, du \int_{u}^{\infty} e^{-(\sigma'+\sigma)y}\, dy \\
&= \left(1 + \frac{\sigma_0 - \gamma_0}{\sigma' + \sigma}\right) \int e^{-(\sigma'+\sigma)y} |\kappa(x, x-y) - k(y)|\, dy \leqslant \epsilon_1 (1/x),
\end{aligned}$$

by (ii), since $0 < \sigma_0 \leqslant \sigma' + \sigma \leqslant \sigma_2$. The inequality is obvious when $\sigma_0 = \sigma_1 = 0$ and $\sigma = -\sigma'$.

(2) Suppose that $h(x)$ satisfies (3.3.1), and that

$$(3.8.9) \qquad \Phi_h[\mu(\epsilon), \delta(\epsilon)/\lambda] > 0.$$

Then we can show that

$$(3.8.10) \qquad h_\lambda(x) = \int r(x-y)\, p(y)\, dy, \quad |p(y)| \leqslant p^*(y),$$

where $p^*(y)$ belongs to L and is independent of σ_0 and σ'. For

$$\begin{aligned}
R(it) &= \int r(y)\, e^{-ity}\, dy = \int e^{-ity} \left[k(y)\, e^{-\sigma' y} - (\sigma_0 - \gamma_0)\, e^{-\sigma' y} \int_{-\infty}^{y} k(u)\, du \right] dy \\
&= K(\sigma' + it)\, \frac{\sigma' - \sigma_0 + \gamma_0 + it}{\sigma' + it},
\end{aligned}$$

so that, using (vii) and observing that $\sigma_0 < \sigma'$, we have

$$|R(it)| \geqslant |K(\sigma' + it)|\, [\sigma' - \sigma_0(1-\beta)]/\sigma' \geqslant \beta\, |K(\sigma' + it)|,$$

and this has a positive lower bound for $|t| \leqslant \pi/\lambda$ and the values of σ' considered. Moreover, an argument similar to that used in stage (1) shows that $r(y)$ is majorized by an integrable function which is independent of σ' and σ_0. The conclusion (3.8.10) then follows from Theorem 6 (*c*).

(3) We define $f_\nu(x)$ by

$$f_\nu(x) = f(x)\, e^{-(\sigma'-\sigma_0)x}\, x^\nu.$$

Then it follows from (3.8.4) that $S_\nu = \overline{\mathrm{bd}}\, |f_\nu(x)|$ is finite for every ν. We prove that $S_\nu(\sigma'-\sigma_0)^\nu/\nu!$ is bounded (in ν) for sufficiently small $\sigma'-\sigma_0$. This is the most troublesome part of the proof, and we divide it into two parts.

(3.*a*) We first prove an inequality [(3.8.17)] which does not depend on the Tauberian properties of $f(x)$. We have, by (3.8.6),

$$g(x)\, e^{-\sigma' x}\, x^n = \int \rho(x, y)[y+(x-y)]^n f(y)\, e^{-(\sigma'-\sigma_0)y}\, dy$$

$$= \int \sum_{\nu=0}^{n} \binom{n}{\nu} f_\nu(y)\, \rho(x, y)(x-y)^{n-\nu}\, dy,$$

so that

$$\text{(3.8.11)} \quad \left| \int \rho(x, y) f_n(y)\, dy \right| \leqslant |g(x)|\, e^{-\sigma' x} x^n + \int \sum_{\nu=0}^{n-1} \binom{n}{\nu} |f_\nu(y)\, \rho(x, y)(x-y)^{n-\nu}|\, dy.$$

Now suppose that $S_\nu(\sigma'-\sigma_0)^\nu/\nu!$ is not bounded. Then, if C is any positive constant, we can choose arbitrarily large integers n for which

$$\text{(3.8.12)} \qquad (\sigma'-\sigma_0)^n S_n \geqslant Cn!,$$

$$\text{(3.8.13)} \qquad \frac{S_\nu(\sigma'-\sigma_0)^\nu}{\nu!} \leqslant \frac{S_n(\sigma'-\sigma_0)^n}{n!} \quad (\nu = 0, 1, \ldots, n-1).$$

Then it follows that

$$\sum_{\nu=0}^{n-1} \binom{n}{\nu} S_\nu |y|^{n-\nu} \leqslant \sum_{\nu=0}^{n-1} \frac{S_n(\sigma'-\sigma_0)^{n-\nu}|y|^{n-\nu}}{(n-\nu)!}$$

$$= S_n \sum_{\nu=1}^{n} \frac{|y(\sigma'-\sigma_0)|^\nu}{\nu!} \leqslant S_n(e^{|y|(\sigma'-\sigma_0)}-1).$$

From this inequality and (3.8.11) it follows that

$$(3.8.14)\quad \Big|\int r(x-y)f_n(y)\,dy\Big| \leqslant \Big|\int \rho(x,\,y)f_n(y)\,dy\Big| + S_n\int |\rho(x,\,x-y)-r(y)|\,dy$$
$$\leqslant |g(x)|\,e^{-\sigma' x}x^n + S_n\int |\rho(x,\,x-y)|\,[e^{|y|(\sigma'-\sigma_0)}-1]\,dy + S_n\int |\rho(x,\,x-y)-r(y)|\,dy.$$

Now

$$(3.8.15)\quad \overline{\operatorname{bd}}_{x\geqslant 0}\, e^{-\sigma' x}x^n = (n/e\sigma')^n = o(n!/\sigma'^n),$$

as $n\to\infty$, by Stirling's formula. Hence, we can write

$$|g(x)|\,e^{-\sigma' x}x^n \leqslant A_1\, n!/\sigma'^n,$$

and the other terms on the right-hand side of (3.8.14) do not exceed

$$S_n\int |r(y)|\,[e^{|y|(\sigma'-\sigma_0)}-1]\,dy + S_n\int |\rho(x,\,x-y)-r(y)|\,e^{|y|(\sigma'-\sigma_0)}\,dy$$
$$\leqslant S_n\Big\{\epsilon_2(\sigma'-\sigma_0)+\int |\rho(x,\,x-y)-r(y)|\,[e^{y(\sigma'-\sigma_0)}+e^{-y(\sigma'-\sigma_0)}]\,dy\Big\}$$
$$\leqslant S_n\{\epsilon_2(\sigma'-\sigma_0)+2\epsilon_1(1/x)\},$$

by (3.8.8). Therefore

$$(3.8.16)\quad \Big|\int r(x-y)f_n(y)\,dy\Big| \leqslant A_1\, n!/\sigma'^n + S_n[\epsilon_2(\sigma'-\sigma_0)+2\epsilon_1(1/x)].$$

It follows from (3.8.10) and (3.8.16) that

$$(3.8.17)\quad \Big|\int h_\lambda(x-y)f_n(y)\,dy\Big| = \Big|\int p(x-t)\,dt\int r(t-y)f_n(y)\,dy\Big|$$
$$\leqslant [A_1\, n!/\sigma'^n + S_n\,\epsilon_2(\sigma'-\sigma_0)]\int p^*(x)\,dx + 2S_n\int p^*(x-t)\,\epsilon_1(1/t)\,dt$$
$$= A_2\, n!/\sigma'^n + S_n[\epsilon_3(\sigma'-\sigma_0)+\epsilon_4(1/x)].$$

(3.*b*) We now use the fact that $f(y)\,\epsilon\, T$. Let ξ, μ, δ, θ be defined as in §2.2. We may plainly suppose that $\lim_{x\to\infty}\delta(\epsilon,\,x)=\delta(\epsilon)$, $\lim_{x\to\infty}\mu(\epsilon,\,x)=\mu(\epsilon)$.

Then, if $\xi \geqslant \delta$, we have

$$(3.8.18)\quad \left|\int h_\lambda(\xi-y) f_n(y)\,dy\right|$$

$$\geqslant \left|\int_{\xi-\delta}^{\xi+\delta} e^{-(\sigma'-\sigma_0)y} y^n h_\lambda^+(\xi-y)\, e^{i\theta} f(y)\,dy\right|$$

$$-S_n\left\{\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}|h_\lambda(y)|\,dy-\int_{-\delta}^{\delta} h_\lambda^-(y)\,dy\right\}$$

$$\geqslant \int_{\xi-\delta}^{\xi+\delta} e^{-(\sigma'-\sigma_0)y} y^n h_\lambda^+(\xi-y)\,\Re[e^{i\theta} f(y)]\,dy$$

$$-S_n\left\{\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}|h_\lambda(y)|\,dy-\int_{-\delta}^{\delta} h_\lambda^-(y)\,dy\right\} \quad [\text{since } h^+(y)\geqslant 0]$$

$$\geqslant [|f(x)|(1-\mu)-\epsilon]\int_{\xi-\delta}^{\xi+\delta} e^{-(\sigma'-\sigma_0)y} y^n h_\lambda^+(\xi-y)\,dy$$

$$-S_n\left\{\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}|h_\lambda(y)|\,dy-\int_{-\delta}^{\delta} h_\lambda^-(y)\,dy\right\},$$

by (2.2.5). Now

$$\epsilon\int_{\xi-\delta}^{\xi+\delta} e^{-(\sigma'-\sigma_0)y} y^n h_\lambda^+(\xi-y)\,dy \leqslant A_3 n!\,(\sigma'-\sigma_0)^{-n},$$

by (3.8.15) and (3.3.1) (ii). Therefore, since

$$e^{-(\sigma'-\sigma_0)y} y^n \geqslant e^{-(\sigma'-\sigma_0)x} x^n[1-2\delta n/x)]\, e^{-2\delta(\sigma'-\sigma_0)} \quad (|y-\xi|<\delta),$$

it follows from (3.8.17) and (3.8.18) that

$$(3.8.19)\quad |f_n(x)|(1-\mu)(1-2\delta n/x)\, e^{-2\delta(\sigma'-\sigma_0)}\int_{-\delta}^{\delta} h_\lambda^+(y)\,dy$$

$$-S_n\left\{\int_{-\infty}^{-\delta}+\int_{\delta}^{\infty}|h_\lambda(y)|\,dy-\int_{-\delta}^{\delta} h_\lambda^-(y)\,dy-\epsilon_3(\sigma'-\sigma_0)-\epsilon_4(1/x)\right\}$$

$$\leqslant A_4 n!\,(\sigma'-\sigma_0)^{-n}.$$

Now it is easy to show, by (3.8.12) and (3.8.13), that $|f_n(x)|$ does not exceed the value S_n for $x < n(\sigma'-\sigma_0)^{-1}$. Hence, if we consider the upper bound of the left-hand side of (3.8.19) for $x \geqslant n(\sigma'-\sigma_0)^{-1}$, and then let $n\to\infty$, so that $x\to\infty$, $\lim\delta=\delta(\epsilon)$, $\lim\mu=\mu(\epsilon)$, we have

$$\overline{\lim_{n\to\infty}}\,(\sigma'-\sigma_0)^n (n!)^{-1} S_n[\Phi_h-\epsilon_3(\sigma'-\sigma_0)] \leqslant A_4,$$

where

$$\Phi_h = \Phi_h\Big[1-\big(1-\mu(\epsilon)\big)[1-2\delta(\epsilon)(\sigma'-\sigma_0)]e^{-2\delta(\epsilon)(\sigma'-\sigma_0)},\quad \delta(\epsilon)/\lambda\Big].$$

If we now choose σ' and σ_0 so that

$$\Phi_h - \epsilon_3(\sigma'-\sigma_0) > 0,$$

it follows that, for sufficiently large n,

$$S_n \leqslant A_5\, n!(\sigma'-\sigma_0)^{-n}$$

and this contradicts (3.8.12), since C is arbitrarily large. Therefore, $S_\nu(\sigma'-\sigma_0)^\nu/\nu!$ is bounded for sufficiently small $\sigma'-\sigma_0$.

(4) The remainder of the proof is easy. We show first that $s(y)\,e^{-\sigma y}$ is bounded for any positive σ. If $\sigma'-\sigma_0$ is sufficiently small, we can write

$$|f(x)|\,e^{-(\sigma'-\sigma_0)x}x^n(\sigma'-\sigma_0)^n \leqslant A_6\, n!.$$

Let $0<\vartheta<1$. Then

$$|f(x)|\,e^{-(\sigma'-\sigma_0)x}\sum_{n=0}^{\infty}\frac{[\vartheta x(\sigma'-\sigma_0)]^n}{n!} \leqslant A_6\sum_{n=0}^{\infty}\vartheta^n,$$

so that

$$|f(x)| \leqslant A_6(1-\vartheta)^{-1}e^{(1-\vartheta)(\sigma'-\sigma_0)x}.$$

Therefore, since $1-\vartheta$ may be arbitrarily small, $f(x)=O(e^{\sigma x})$ if $\sigma>0$, and it follows easily from this and (3.8.5) that $s(x)=O(e^{\sigma x})$ if $\sigma>\sigma_0$. But if $\sigma_1>0$, so that $\sigma_0<\sigma_1$, this contradicts our assumption that $s(x)\,e^{-\sigma x}$ is unbounded for $\sigma<\sigma_1$. Therefore $\sigma_1=0$, $\sigma_0=\gamma_0=0$, $f(y)=s(y)$, and $s(x)\,e^{-\sigma x}$ is bounded if $\sigma>0$.

(5) Let $M_\sigma = \overline{\mathrm{bd}}\,|s(x)|\,e^{-\sigma x}$, and suppose that $s(x)$ is not bounded. Then

$$\lim_{\sigma\to 0} M_\sigma = \infty, \tag{3.8.20}$$

and $|s(x)|\,e^{-\sigma x}$ does not exceed the value M_σ in a certain interval $x<X_\sigma$, where $\lim_{\sigma\to 0} X_\sigma = \infty$. If we put $n=0$, $\sigma'=\sigma$, $\sigma_0=0$ in (3.8.19), we have

$$|s(x)|\,(1-\mu)\,e^{-2\delta\sigma}\int_{-\delta}^{\delta}h_\lambda^{+}(y)\,dy$$

$$-M_\sigma\left\{\int_{-x}^{-\delta}+\int_{\delta}^{\infty}|h_\lambda(y)|\,dy-\int_{-\delta}^{\delta}h_\lambda(y)\,dy-\epsilon_3(\sigma)-\epsilon_4(1/x)\right\} \leqslant A_4.$$

On taking the upper bound of the left-hand side for $x \geqslant X_\sigma$, we have

$$M_\sigma \Phi(\sigma) \leqslant A_4,$$

where $$\lim_{\sigma \to 0} \Phi(\sigma) = \Phi_h[\mu(\epsilon), \delta(\epsilon)/\lambda] > 0,$$

and this contradicts (3.8.20). Therefore $s(x)$ is bounded, and since

$$\lim_{x \to \infty} \left\{ \int \kappa(x, y) s(y)\, dy - \int k(x-y) s(y)\, dy \right\} = 0,$$

the second part of the theorem follows from Theorem 8.

3.9. As a corollary of Theorem 11, we have the following "high indices theorem" for general kernels.

THEOREM 12.

HYPOTHESIS. (i), (ii), (iii), (iv), (v), (vi) *as in Theorem* 11,

(vii) $$s(x) = \sum_{\lambda_n \leqslant x} a_n, \qquad \varliminf_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \lambda^* \geqslant \lambda.$$

CONCLUSION. (*a*) *$s(x)$ is bounded,*

(*b*) $S \leqslant CG$,

where C is a positive constant depending only on $k(y)$ and λ^*.

Let $0 < \beta < 1$, $e^{\beta\sigma_1 \lambda^*} < 2$, $\gamma(\sigma) = \beta\sigma$. Then if

$$\lambda_n \leqslant x < \lambda_{n+1}, \quad \lambda_n \leqslant y < \lambda_{n+1},$$

we have

$$s_\sigma(y) - s_\sigma(x) = (e^{-\beta\sigma y} - e^{-\beta\sigma x}) \sum_{\nu \leqslant n} a_\nu e^{-\lambda_\nu \sigma(1-\beta)}$$

$$= (e^{\beta\sigma(x-y)} - 1)\, s_\sigma(x).$$

This shows that $s_\sigma(y)$ belongs to T_β, with $2\delta(\epsilon) = \lambda^*$ and $\mu(\epsilon) = e^{\beta\sigma_1\lambda^*} - 1$. Now since $K(\omega)$ is continuous for $0 \leqslant \sigma \leqslant \sigma_1$, we can suppose that $K(\omega) \neq 0$ for $0 \leqslant \sigma \leqslant \sigma_1$, $|t| \leqslant \pi/\lambda'$, where $\lambda' < \lambda \leqslant \lambda^*$. Then

$$\Phi[\mu(\epsilon), \delta(\epsilon)/\lambda'] = \Phi[e^{\beta\sigma_1\lambda^*} - 1, \lambda^*/2\lambda'],$$

which is positive, by Theorem 7, when β is sufficiently small. It follows from Theorem 11 that $s(x$ is bounded, and since $\mu(\epsilon)$ may be taken

arbitrarily small by suitable choice of β, we have

$$S \leqslant \overline{\lim_{\epsilon \to 0}} \frac{\epsilon + \alpha(\lambda)\, G}{\Phi[0,\ \lambda^*/2\lambda']} = CG.$$

It is interesting to observe that, if $\lim_{x \to \infty} g(x) = 0$, we obtain the conclusion $\lim_{x \to \infty} s(x) = 0$ without making any assumption about the zeros of $K(\omega)$ outside the rectangle $0 \leqslant \sigma \leqslant \sigma_1$, $|t| \leqslant \pi/\lambda$. In general, this is not the case if $s(x)$ satisfies any other Tauberian condition.

3.10. In certain cases, conditions (v) of Theorem 10 may be replaced by a "one-sided" order condition. Thus, if $s(x)$ belongs to T_α and $\kappa(x, y) \geqslant 0$, it is sufficient to suppose that, for any fixed $\sigma > \sigma_1$, the values of $s(x)\, e^{-\sigma x}$ lie in a region of the z-plane defined by

$$\arg\,[e^{i\phi}(z - z_0)] \leqslant \psi < \tfrac{1}{2}\pi,$$

ϕ, ψ being real constants. We shall not discuss these extensions in detail as they can be deduced easily from Theorem 10 by elementary arguments which show that condition (v) is, in fact, satisfied.

4. *Special Tauberian theorems.*

We illustrate the results of the last section by using them to prove some Tauberian theorems for Abel, Borel, and Hausdorff summation.

4.1. The first of these theorems includes Littlewood's theorem (**2**, **3**, **11**, **13**, **14**) and the "high-indices theorem" (**4**, **9**, **20**).

THEOREM 13.

HYPOTHESIS.

(i) $$s_\sigma(y) = \int_0^y e^{-\sigma t}\, ds(t) \quad \textit{exists for } \sigma \geqslant 0,\ y > 0.$$

(ii) $s_\sigma[e^y]$ *belongs to* T', *with* $\mu'(\epsilon)$, $\delta'(\epsilon)$ *independent of* σ, *for*

$$0 \leqslant \sigma < \sigma_0.$$

(iii) $$\int_0^\infty e^{-\sigma y}\, |ds(y)| \ \textit{converges for } \sigma > 0.$$

(iv) $$g(x) = \int_0^\infty e^{-y/x}\, ds(y) \ \textit{is bounded.}$$

CONCLUSION.

(*a*) *If C_1 is a positive constant, we can define another positive constant C_2, depending only on $\delta(\epsilon)$ and $\mu(\epsilon)$, so that*

$$S \leqslant C_1 + C_2 G.$$

(*b*) *If $\delta(\epsilon) = +\infty$ for $\epsilon > 0$, then*

$$S \leqslant \frac{G}{1-\mu(+0)}.$$

If also $\mu(+0) = 0$, then

$$S = G.$$

(*c*) *If*

$$s(x) = \sum_{\lambda_n < x} a_n, \quad \lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = l > 0,$$

then

$$S \leqslant CG,$$

where $C = C(l)$ depends only on l, and $\lim_{l\to\infty} C(l) = 1$.

Remarks. (*a*) It is plain that condition (ii) is satisfied if $s(y)$ is a step function of the type considered in Theorem 4. In particular, we obtain Littlewood's theorem on supposing that $G = 0$ and that $s(y)$ satisfies condition (*a*) of Theorem 4.

(*b*) The condition $\delta(\epsilon) = \infty$ is characteristic of Tauberian conditions of the "*o*" type. It is satisfied, for example, if $s(x) = \sum_{n \leqslant x} a_n$ and $na_n = o(1)$, if $s(x) = \sum_{\lambda_n \leqslant x} a_n$ and $\lambda_{n+1} - \lambda_n \to \infty$, or if

$$\lim_{\delta\to 0} \lim_{\substack{x \leqslant y \leqslant x(1+\delta) \\ x\to\infty}} \left(\frac{s(y)-s(x)}{\delta}\right) \geqslant 0.$$

The latter condition leads to a Tauberian theorem on oscillation due to V. Ramaswami (**12**, Theorem 2).

(*c*) The third part is the high-indices theorem. A very simple proof, which does not depend on any general theorem, has recently been given by A. E. Ingham (**9**).

These results throw some light on Ramaswami's conjecture (**12**, § 10) that $S \leqslant CG$, for some absolute constant C, if $s(x)$ belongs to S_α. He has recently shown that the conjecture is false as it stands, but Ingham's result

shows that it is true, except that C is not an *absolute* constant, if $s(y)$ satisfies a "high-indices" condition instead of the Schmidt condition.

Proof of Theorem 13.

(a) We may plainly suppose that $s(0)=0$. Let $\sigma_0>\sigma>0$. Then

$$(4.1.1)\qquad \frac{1}{x}\int_0^\infty e^{-y/x}s_\sigma(y)\,dy=\int_0^\infty e^{-y/x}\,ds_\sigma(y)$$
$$=\int_0^\infty e^{-y/x}e^{-\sigma y}\,ds(y)=g\left[\frac{x}{\sigma x+1}\right].$$

If we write

$$k(x)=e^{-x}e^{-e^{-x}},$$

it follows from (4.1.1) that

$$(4.1.2)\qquad \int k(x-y)\,s_\sigma(e^y)\,dy=g\left[\frac{1}{\sigma+e^{-x}}\right].$$

Also,
$$K(it)=\int e^{-itx}k(x)\,dx=\int_0^\infty e^{-u}u^{it}\,du=\Gamma(1+it),$$

which does not vanish for real values of t. Since $g(x)$ is bounded, it follows from Theorem 9 (b) that

$$(4.1.3)\qquad |s_\sigma(e^y)|\leqslant A,$$

where A is independent of σ. Therefore $s(y)$ is bounded, and if we put $\sigma=0$ in (4.1.2), we have

$$(4.1.4)\qquad \int k(x-y)\,s(e^y)\,dy=g(e^x).$$

The conclusion then follows from Theorem 9 (a).

(b) We have, by Theorem 8 (a),

$$S\leqslant\frac{\epsilon+\alpha(\lambda)\,G}{\Phi[\mu(\epsilon),\ \delta(\epsilon)/\lambda]}$$
$$=\frac{\epsilon+\alpha(\lambda)\,G}{1-\mu(\epsilon)}\quad[\text{Theorem 7 }(c)].$$

Therefore

$$S\leqslant\lim_{\lambda\to\infty}\lim_{\epsilon\to0}\frac{\epsilon+\alpha(\lambda)\,G}{1-\mu(\epsilon)}$$
$$=\lim_{\lambda\to\infty}\frac{\alpha(\lambda)\,G}{1-\mu(+0)}=\frac{G}{1-\mu(+0)}.$$

Since $G \leqslant S$, by an elementary argument, it follows that $S = G$ if $\mu(+0) = 0$.

(c) Here we may write $\mu(\epsilon) = 0$, $\delta(\epsilon) = \frac{1}{2}l$, so that

$$S \leqslant \frac{\epsilon + \alpha(\lambda)\, G}{\Phi[0,\ l/2\lambda]}$$

for any ϵ and $\lambda < l$, by Theorem 8 (a). On letting $\epsilon \to 0$, we obtain

$$S \leqslant C(l)\, G,$$

where $C(l)$ is the lower bound of $\alpha(\lambda)/\Phi\,[0,\ l/2\lambda]$ for positive values of λ. Then $C(l) \geqslant 1$, and since $\lim_{\lambda\to\infty} \alpha(\lambda) = 1$, it follows from Theorem 7 (c) that

$$\overline{\lim_{l\to\infty}}\ C(l) \leqslant \lim_{\lambda\to\infty}\ \overline{\lim_{l\to\infty}}\ \frac{\alpha(\lambda)}{\Phi[0,\ l/2\lambda]} = \lim_{\lambda\to\infty} \alpha(\lambda) = 1.$$

Hence $\lim_{l\to\infty} C(l) = 1$.

4.2. In Theorem 13 we were concerned with the behaviour of

$$\zeta \int_0^\infty e^{-\zeta y}\, s(y)\, dy$$

as $\zeta \to 0$ along the positive real axis. Wiener (**19**) has proved a similar theorem for the case in which $\zeta \to 0$ along the line $\zeta = re^{i\alpha}(|\alpha| < \frac{1}{2}\pi)$. We shall now extend this result to a more general class of curves. We consider only the simplest case.

A rather less general form of the same theorem was proved by Bosanquet and Miss Cartwright (**1**), who also gave an example to show that an order condition similar to (4.2.1) below is essential in a theorem of this type.

THEOREM 14. *Suppose that*

$$s(y) = O(y^{\sigma_1}) \tag{4.2.1}$$

for some positive σ_1, *that* $s(e^y) \in T$, *and that*

$$\zeta \int_0^\infty e^{-\zeta y}\, s(y)\, dy \to A \tag{4.2.2}$$

as $\zeta \to 0$ *along a curve which lies to the right of the imaginary axis and has a finite gradient at the origin. Then*

$$\lim_{y\to\infty} s(y) = A.$$

If $x > 0$, the circle $|z| = x^{-1}$ meets the curve in one or more points. We choose one such point $\zeta = \zeta(x)$ for each value of x and define $\alpha = \alpha(x)$ by

$$x\zeta(x) = e^{i\alpha(\log x)}, \quad |\alpha| < \tfrac{1}{2}\pi.$$

Then, by hypothesis,

$$\lim_{x\to\infty} \alpha(x) = \alpha_0, \quad |\alpha_0| < \tfrac{1}{2}\pi. \tag{4.2.3}$$

Let

$$g(x) = \zeta \int_0^\infty e^{-\zeta y} s(y)\, dy.$$

Then

$$g(e^x) = \int \kappa(x, y)\, s(e^y)\, dy, \tag{4.2.4}$$

where

$$\kappa(x, y) = e^{i\alpha(x)} e^{-(x-y)} e^{-e^{-(x-y)+i\alpha(x)}}. \tag{4.2.5}$$

Let

$$k(y) = e^{i\alpha_0} e^{-y} e^{-e^{-y+i\alpha_0}}. \tag{4.2.6}$$

Then

$$e^{-\sigma y}|\kappa(x, x-y)| \leqslant e^{-y(1+\sigma)} e^{-e^{-y}\cos\alpha(x)},$$

so that, if σ_2 $(> \sigma_1)$ is any fixed number, $e^{-\sigma y}\kappa(x, x-y)$ is majorized, for $0 \leqslant \sigma \leqslant \sigma_2$ and sufficiently large x, by a function of L. Since

$$\lim_{x\to\infty} \kappa(x, x-y)\, e^{-\sigma y} = k(y)\, e^{-\sigma y}$$

for every value of y, it follows that

$$\kappa(x, x-y) \sim k(y) \quad (0 \leqslant \sigma \leqslant \sigma_2).$$

Moreover,

$$K(\omega) = \int k(y)\, e^{-\omega y}\, dy = e^{i\alpha_0} \int_0^\infty e^{-ye^{i\alpha_0}} y^\omega\, dy$$

$$= e^{i\alpha_0\omega}\, \Gamma(1+\omega),$$

which does not vanish for $0 \leqslant \sigma \leqslant \sigma_1$. Since $s(e^y) = O(e^{\sigma_1 y})$, by (4.2.1), and $s(e^y) \in T$, it follows from Theorem 10 (b) that

$$\overline{\lim_{x\to\infty}}\, |s(x) - A| \leqslant C_1,$$

for any positive C_1, and therefore $s(x) \to A$,

4.3. *Hausdorff summation* (**6**).

Let μ_n be a sequence of real numbers and let

$$\mu_{\nu,\,n-\nu} = \mu_\nu - \binom{n-\nu}{1}\mu_{\nu+1} + \ldots + (-1)^{n-\nu}\binom{n-\nu}{n-\nu}\mu_n, \tag{4.3.1}$$

$$g_{n+1} = \sum_{\nu=0}^{n}\binom{n}{\nu}\mu_{\nu,\,n-\nu}s_\nu. \tag{4.3.2}$$

Hausdorff has shown that a necessary and sufficient condition for the consistency of the transformation (4.3.2) is that

$$\mu_n = \int_0^1 t^n\,d\chi(t), \tag{4.3.3}$$

where $\chi(t)$ is of bounded variation and

$$\chi(0) = \chi(+0) = 0, \quad \chi(1) = 1. \tag{4.3.4}$$

If these conditions are satisfied, we can write (4.3.1) in the form

$$\mu_{\nu,\,n-\nu} = \int_0^1 t^\nu(1-t)^{n-\nu}\,d\chi(t). \tag{4.3.5}$$

We say that s_n is summable (H, χ), to sum A, if $\lim_{n\to\infty} g_n = A$. We shall suppose in every case that $\chi(t)$ satisfies (4.3.4).

Hausdorff's method of summation includes as simple special cases the Hölder and Cesàro methods of order greater than -1.

The following Tauberian theorem for Hausdorff summation can be deduced easily from Theorem 10.

THEOREM 15.

HYPOTHESIS.

(i) $$\int_0^1 t^\omega\,d\chi(t) \neq 0 \quad (0 \leqslant \sigma \leqslant \sigma_1),$$

(ii) $$\sum_{n=0}^{\infty} a_n = C(H, \chi),$$

(iii) $$\sum_{\nu=0}^{n} a_\nu = O(n^\sigma) \quad \textit{for fixed } \sigma > \sigma_1,$$

(iv) $$s(x) = \sum_{\log n \leqslant x} a_n \quad \textit{belongs to } T,$$

CONCLUSION. $\sum_{n=0}^{\infty} a_n = C.$

We prove the theorem in the first place for the case in which $\chi(t)$ is absolutely continuous. Then we can write $\chi'(t) = \phi(t)$,

$$\mu_n = \int_0^1 t^n \phi(t)\, dt. \tag{4.3.6}$$

It is convenient to define $\phi(t)$ to be 0 for $t > 1$. Let $s_n = \sum_{\nu=0}^{n} a_\nu$, so that

$$s(x) = s_m \quad (m \leqslant e^x < m+1). \tag{4.3.7}$$

Suppose that g_n is defined by (4.3.2) and let

$$g(x) = g_m \quad (m \leqslant e^x < m+1). \tag{4.3.8}$$

We define $n = n(x)$ by

$$n+1 = [e^x]. \tag{4.3.9}$$

Let

$$\kappa(x, y) = e^y \binom{n}{\nu} \mu_{\nu, n-\nu} \quad [\log \nu \leqslant y < \log(\nu+1);\ 0 \leqslant \nu \leqslant n], \tag{4.3.10}$$

$$\kappa(x, y) = 0 \quad [y \geqslant \log(n+1)]. \tag{4.3.11}$$

Then

$$g(x) = \int \kappa(x, y)\, s(y)\, dy. \tag{4.3.12}$$

We show that, for any positive $\sigma_2\ (> \sigma_1)$,

$$\kappa(x, x-y) \sim e^{-y} \phi(e^{-y}) \quad (0 \leqslant \sigma \leqslant \sigma_2).$$

Let $\epsilon > 0$, and define a continuous function $\phi^*(t)$ so that

$$\int_0^1 |\phi(t) - \phi^*(t)|\, dt < \epsilon. \tag{4.3.13}$$

Let $\mu_n^* = \int_0^1 t^n \phi^*(t)\, dt$ and define $\kappa^*(x, y)$ by equations (4.3.10) and

(4.3.11), with μ_n replaced by $\mu_n{}^*$. Then

$$(4.3.14)\quad \int|\kappa(x,y)-\kappa^*(x,y)|\,dy=\sum_{\nu=0}^{n}\binom{n}{\nu}|\mu_{\nu,n-\nu}-\mu^*_{\nu,n-\nu}|$$

$$=\sum_{\nu=0}^{n}\binom{n}{\nu}\left|\int_0^1 t^\nu(1-t)^{n-\nu}[\phi(t)-\phi^*(t)]\,dt\right|$$

$$\leqslant\int_0^1|\phi(t)-\phi^*(t)|\sum_{\nu=0}^{n}\binom{n}{\nu}t^\nu(1-t)^{n-\nu}\,dt$$

$$=\int_0^1|\phi(t)-\phi^*(t)|\,dt\leqslant\epsilon.$$

Let $u=e^x$, $t=ue^{-y}$. Then

$$\int|\kappa^*(x,x-y)-e^{-y}\phi^*(e^{-y})|\,dy$$

$$=\int_0^u\left|\kappa^*(\log u,\log t)-\frac{t}{u}\phi^*\left(\frac{t}{u}\right)\right|\frac{dt}{t}$$

$$\leqslant\sum_{\nu=0}^{n}\int_\nu^{\nu+1}\left|\binom{n}{\nu}\mu^*_{\nu,n-\nu}-\frac{1}{u}\phi^*\left(\frac{t}{u}\right)\right|dt+\frac{1}{u}\int_{n+1}^{u}\left|\phi^*\left(\frac{t}{u}\right)\right|dt$$

$$=\sum_{\nu=0}^{n}\int_\nu^{\nu+1}\left|\binom{n}{\nu}\int_0^1\xi^\nu(1-\xi)^{n-\nu}\phi^*(\xi)\,d\xi\right.$$

$$\left.-\binom{n}{\nu}\frac{n+1}{u}\int_0^1\xi^\nu(1-\xi)^{n-\nu}\phi^*\left(\frac{t}{u}\right)d\xi\right|dt+\frac{1}{u}\int_{n+1}^{u}\left|\phi^*\left(\frac{t}{u}\right)\right|dt$$

$$\left[\text{since }(n+1)\binom{n}{\nu}\int_0^1\xi^\nu(1-\xi)^{n-\nu}\,d\xi=1\right]$$

$$\leqslant\int_0^1\left\{\sum_{\nu=0}^{n}\binom{n}{\nu}\xi^\nu(1-\xi)^{n-\nu}\int_\nu^{\nu+1}\left|\phi^*(\xi)-\frac{n+1}{u}\phi^*\left(\frac{t}{u}\right)\right|dt\right\}d\xi$$

$$+\frac{1}{u}\int_{n+1}^{u}\left|\phi^*\left(\frac{t}{u}\right)\right|dt.$$

The integrand in the first term is bounded in $0\leqslant\xi\leqslant1$, and, since $\phi^*(\xi)$ is continuous, it follows easily from Stirling's formula that it tends to 0, for any fixed ξ, as $n\to\infty$. The second term is $\int_{(n+1)/u}^{1}|\phi^*(y)|\,dy$, which tends to 0 as $n\to\infty$, since $n+1=[u]$. Hence

$$\lim_{x\to\infty}\int_0^\infty|\kappa^*(x,x-y)-e^{-y}\phi^*(e^{-y})|\,dy=0,$$

It follows from (4.3.13) and (4.3.14) that

$$\overline{\lim_{x\to\infty}} \int_0^\infty |\kappa(x, x-y) - e^{-y}\phi(e^{-y})|\,dy \leqslant 2\epsilon,$$

and since ϵ is arbitrarily small, we have

$$\lim_{x\to\infty} \int_0^\infty |\kappa(x, x-y) - e^{-y}\phi(e^{-y})|\,dy = 0.$$

Therefore, since $\kappa(x, x-y)$ and $e^{-y}\phi(e^{-y})$ vanish for $y < 0$,

$$\kappa(x, x-y) \sim e^{-y}\phi(e^{-y}) \quad (0 \leqslant y \leqslant \sigma_2).$$

If we write

$$k(y) = e^{-y}\phi(e^{-y}),$$

then

$$K(\omega) = \int_0^\infty k(y)\, e^{-\omega y}\, dy = \int_0^\infty e^{-y}\phi(e^{-y})\, e^{-\omega y}\, dy$$

$$= \int_0^1 \phi(t)\, t^\omega\, dt,$$

which does not vanish for $0 \leqslant \sigma \leqslant \sigma_1$, by hypothesis. The conclusion follows now from Theorem 10.

We can easily deduce the theorem in the general case. Using the identity

$$(n+1)\binom{n}{\nu}\int_0^1 \xi^\nu (1-\xi)^{n-\nu}\, d\xi = 1,$$

we have

$$\frac{1}{m+1}\sum_{n=0}^{m} g_{n+1} = \int_0^1 dt \sum_{n=0}^{m}\binom{m}{n} t^n (1-t)^{m-n} g_{n+1}$$

$$= \int_0^1 dt \int_0^1 d\chi(u) \sum_{n=0}^{m}\binom{m}{n} t^n (1-t)^{m-n} \sum_{\nu=0}^{n} s_\nu \binom{n}{\nu} u^\nu (1-u)^{n-\nu}$$

$$= \int_0^1 dt \int_0^1 d\chi(u) \sum_{\nu=0}^{m} s_\nu u^\nu \sum_{n=\nu}^{m}\binom{m}{n}\binom{n}{\nu} t^n (1-t)^{m-n} (1-u)^{n-\nu}$$

$$= \int_0^1 dt \int_0^1 d\chi(u) \sum_{\nu=0}^{m} s_\nu \binom{m}{\nu} u^\nu t^\nu (1-ut)^{m-\nu}$$

$$= \int_0^1 \frac{d\chi(u)}{u} \int_0^u \sum_{\nu=0}^{m} s_\nu \binom{m}{\nu} x^\nu (1-x)^{m-\nu}\, dx$$

$$= \int_0^1 dx \int_x^1 \frac{d\chi(u)}{u} \sum_{\nu=0}^{m} s_\nu \binom{m}{\nu} x^\nu (1-x)^{m-\nu},$$

Since $$\lim_{m\to\infty}\frac{1}{m+1}\sum_{n=0}^{m} g_{n+1}=\lim_{n\to\infty} g_n=C,$$

it follows that $s_n\to C(H,\chi_0)$, where

$$\chi_0(x)=\int_0^x\left(\int_\xi^1\frac{d\chi(u)}{u}\right)d\xi.$$

being a Lebesgue integral, is absolutely continuous, and

$$\chi_0'(x)=\int_x^1\frac{d\chi(u)}{u}.$$

Also,

$$\int_0^1 x^\omega\,d\chi_0(x)=\int_0^1 x^\omega\,dx\int_x^1\frac{d\chi(u)}{u}$$

$$=\int_0^1\frac{d\chi(u)}{u}\int_0^u x^\omega\,dx=\frac{1}{\omega+1}\int_0^1 u^\omega\,d\chi(u),$$

which does not vanish for $0\leqslant\sigma\leqslant\sigma_1$. The conclusion follows now from the special case.

4.4. *Borel summation.*

We say that $s(x)$ is summable (B^*), to sum A, and write

$$s(x)\to A(B^*),$$

if $$\lim_{x\to\infty}\int_0^x e^{-t}dt\int_0^\infty\frac{t^u\,ds(u)}{\Gamma(1+u)}=A,$$

the double integral being absolutely convergent for each x. If

$$s(x)=\sum_{n<x}a_n=s_{[x]},$$

we say that s_n is summable (B), to sum A, and write

$$s_n\to A(B),\qquad\sum_{n=0}^{\infty}a_n=A(B),$$

if $$\lim_{x\to\infty}e^{-x}\sum_{n=0}^{\infty}\frac{s_n x^n}{n!}=A.$$

It is easy to show that $\sum_{n=1}^{\infty}a_n=A-a_0(B^*)$ is equivalent to $\sum_{n=0}^{\infty}a_n=A(B)$,

and the Tauberian theorems for the two methods are equivalent. We shall discuss the theorem for (B^*).

THEOREM 16.

HYPOTHESIS.

(i) $s_\sigma(y) = \int_0^y e^{-\sigma t} ds(t)$ *exists for* $y > 0$, $\sigma \geqslant 0$ *and is bounded for each fixed positive* σ.

(ii) $s_\sigma[y^2]$ *belongs to* T', *with* $\mu'(\epsilon)$, $\delta'(\epsilon)$ *independent of* σ, *for* $0 \leqslant \sigma < \sigma_0$.

(iii) $$s(x) \to A (B^*).$$

CONCLUSION. $$s(x) \to A.$$

We prove the boundedness of $s(x)$ by an extension of the method used in Theorem 13. Briefly, the idea is to prove that $s_\sigma(y)$ is summable by a method which approximates to B^* as $\sigma \to 0$. We then use Theorem 9 (b) to show that the upper bound of $s_\sigma(x)$ is independent of σ when σ is sufficiently small.

[If we try to use Theorem 10, we require

$$s(y) = O(e^{\sigma_1 \sqrt{y}})$$

for some positive σ_1, instead of condition (i), which is substantially

$$s(y) = O(e^{\sigma y})$$

for every positive σ.]

Let $$g(x) = \int_0^x e^{-t} dt \int_0^\infty \frac{t^u \, ds(u)}{\Gamma(1+u)}.$$

Then $$g(x) = \frac{1}{q} \int_0^{qx} e^{-t/q} dt \int_0^\infty \frac{t^u \, ds_\sigma(u)}{\Gamma(1+u)},$$

where $q = e^\sigma$. It is not difficult to show that this can be written in the form

$$g(x^2) = \int_{-\infty}^{\infty} \kappa(x, y, \sigma) \, s_\sigma(y^2) \, dy, \tag{4.4.1}$$

where

$$\eta(x, \sigma) = \int \left| \kappa(x, x-y, \sigma) - \sqrt{\left(\frac{\pi}{2}\right)} e^{-y^2} \right| dy \tag{4.4.2}$$

is bounded and

$$\lim_{\substack{x\to\infty\\ \sigma\to\infty}} \eta(x,\sigma)=0. \tag{4.4.3}$$

Now $$\sqrt{\left(\frac{\pi}{2}\right)}\int e^{-y^2}e^{-iyt}\,dt = e^{-\frac{1}{4}t^2}\neq 0$$

for real t. We can therefore define C_1 and C_2 as in Theorem 9 (b). We define $X>0$, $\sigma_0>0$, so that

$$C_2\Big[\overline{\mathrm{bd}}_{\substack{x\geqslant X\\ 0<\sigma\leqslant\sigma_0}}\eta(x,\sigma)+\sqrt{\left(\frac{2}{\pi}\right)}\int_X^\infty e^{-y^2}\,dy\Big]\leqslant\tfrac{1}{2}. \tag{4.4.4}$$

Let $$C_3=\int_0^{2X}|ds(t)|,$$

so that

$$|s_\sigma(x)|\leqslant C_3 \quad (x\leqslant 2X,\ \sigma\geqslant 0). \tag{4.4.5}$$

If $x\geqslant X$, it follows from (4.4.1), (4.4.2), and (4.4.4) that

$$\left|\sqrt{\left(\frac{\pi}{2}\right)}\int e^{-(x-y)^2}s_\sigma(y)\,dy\right|\leqslant G+S_\sigma\eta(x,\sigma),$$

where $$G=\overline{\mathrm{bd}}\,|g(x)|,\quad S_\sigma=\overline{\mathrm{bd}}\,|s_\sigma(x)|.$$

On the other hand, if $x\leqslant X$,

$$\left|\sqrt{\left(\frac{\pi}{2}\right)}\int e^{-(x-y)^2}s_\sigma(y)\,dy\right|\leqslant C_3+S_\sigma\sqrt{\left(\frac{\pi}{2}\right)}\int_{2X}^\infty e^{-(x-y)^2}dy$$

$$\leqslant C_3+S_\sigma\sqrt{\left(\frac{\pi}{2}\right)}\int_{-\infty}^{-X}e^{-y^2}dy.$$

Therefore, using (4.4.4),

$$\left|\sqrt{\left(\frac{\pi}{2}\right)}\int e^{-(x-y)^2}s_\sigma(y)\,dy\right|\leqslant C_3+G+S_\sigma/2C_2 \tag{4.4.5}$$

for all real values of x.

If we now use Theorem 9 (b), we have

$$S_\sigma\leqslant C_1+C_2[C_3+G+S_\sigma/2C_2],$$

$$S_\sigma\leqslant 2C_1+2C_2[C_3+G],$$

and since the right-hand side is independent of σ and $\lim_{\sigma\to 0} s_\sigma(x) = s(x)$, it is plain that $s(x)$ is bounded.

If we let $\sigma \to 0$ and then $x \to \infty$ in (4.4.1), we have

$$\lim_{x\to\infty} \sqrt{\left(\frac{\pi}{2}\right)} \int e^{-(x-y)^2} s(y^2)\, dy = \lim_{x\to\infty} g(x^2) = A,$$

and the conclusion follows from Theorem 9 (a).

As an immediate corollary, we have the following "high indices theorem" for Borel summation.

THEOREM 17.

HYPOTHESIS.

(i) $$0 < \lambda_1 < \lambda_2 < \ldots, \quad (\lambda_{n+1} - \lambda_n)/\sqrt{\lambda_n} \geqslant \lambda > 0,$$

(ii) $$s(y) = \sum_{\lambda_n \leqslant y} a_n,$$

(iii) $$s(y) = O(e^{\sigma y}) \text{ for each fixed positive } \sigma,$$

(iv) $$s(y) \to A \, (B^*).$$

[*Or* (B), *if the* λ_n*'s are integers.*]

CONCLUSION. $$s(y) \to A.$$

References.

1. L. S. Bosanquet and M. L. Cartwright, "Some Tauberian theorems", *Math. Zeitschrift*, 37 (1933), 416–423.
2. G. H. Hardy and J. E. Littlewood, "Abel's theorem and its converse", *Proc. London Math. Soc.* (2), 18 (1920), 205–235.
3. ————, "Abel's theorem and its converse (II)", *Proc. London Math. Soc.* (2), 22 (1924), 254–269.
4. ————, "A further note on the converse of Abel's theorem", *Proc. London Math. Soc.* (2), 25 (1926), 219–236.
5. ————, "Theorems concerning the summability of series by Borel's exponential method", *Rend. di Palermo*, 41 (1916), 36–53.
6. F. Hausdorff, "Summationsmethoden und Momentfolgen", *Math. Zeitschrift*, 9 (1921), 74–109.
7. J. M. Hyslop, "On the summability of series by a method of Valiron", *Proc. Edinburgh Math. Soc.*, 4 (1935), 218–223.
8. A. E. Ingham, "Some trigonometrical inequalities with applications to the theory of series", *Math. Zeitschrift*, 41 (1936), 367–379.
9. ————, "On the 'high-indices theorem' of Hardy and Littlewood", *Quart. J. of Math.*, 8 (1937), 1–7.

10. J. Karamata, "Über die Hardy-Littlewoodschen Umkehrungen des Abelschen Stetigkeitsatzes", *Math. Zeitschrift*, 32 (1930), 319–320.
11. J. E. Littlewood, "On the converse of Abel's theorem on power series", *Proc. London Math. Soc.* (2), 9 (1910), 434–444.
12. V. Ramaswami, "Some Tauberian theorems on oscillation", *Journal London Math. Soc.*, 10 (1935), 294–308.
13. R. Schmidt, "Über divergente Folgen und lineare Mittelbildungen", *Math. Zeitschrift*, 22 (1925), 89–152.
14. O. Szasz, "Converse theorems of summability for Dirichlet series", *Trans. American Math. Soc.*, 39 (1936), 117–130.
15. G. Valiron, "Remarques sur la sommation des séries divergentes par les méthodes de M. Borel", *Rend. di Palermo*, 42 (1917), 267–284.
16. N. Wiener, "A new method in Tauberian theorems", *Journal Math. Phys., M.I.T.*, 7 (1928), 161–184.
17. ———, "Tauberian theorems", *Annals of Mathematics*, 33 (1932), 1–100.
18. ———, *The Fourier integral and certain of its applications* (Cambridge, 1933).
19. ———, "A one-sided Tauberian theorem", *Math. Zeitschrift*, 36 (1933), 787–798.
20. ———, "A Tauberian gap-theorem of Hardy and Littlewood", *Chinese Journal of Math.*, 1 (1936), 15–22.

Peterhouse,
Cambridge.

GENERAL GAP TAUBERIAN THEOREMS: I

By NORMAN LEVINSON.

[Received 16 August, 1937.—Read 18 November, 1937.]

1. In 1924 Hardy and Littlewood* proved the following remarkable Tauberian theorem:

Let

$$0<\mu_1<\mu_2<\ldots<\mu_k<\ldots; \quad \mu_{k+1}/\mu_k\geqslant\theta>1 \quad (k=1,\ 2,\ \ldots),$$

and let

$$\sum_1^\infty a_k e^{-\mu_k x}=f(x) \tag{1.0}$$

converge for $x>0$ *and*

$$\lim_{x\to+0} f(x)=s.$$

Then

$$\sum_1^\infty a_k=s.$$

This theorem differs from the usual Tauberian theorem in that there is no order condition on the coefficients a_k (or no condition that can trivially be shown to imply such an order restriction on the a_k's).

This theorem can be stated in somewhat different form. If we put $\log\mu_n=\lambda_n$ and assume that $\mu_1>1$, which is no real restriction, the theorem can be stated as follows:

Let $\{\lambda_k\}$ *be an increasing sequence of positive numbers such that*

$$\lambda_{k+1}-\lambda_k\geqslant D>0.$$

Let

$$f(x)=\sum_1^\infty a_k\int_{\lambda_k-x}^\infty e^{-e^y}e^y\,dy,$$

* Hardy and Littlewood, *Proc. London Math. Soc.* (2), 22 (1924), 254–269.

where the series converges absolutely for any $x < \infty$. *If*

$$\lim_{x \to \infty} f(x) = s,$$

then

$$\sum_1^\infty a_k = s.$$

We concern ourselves here with this theorem when

$$f(x) = \sum_1^\infty a_k \int_{\lambda_k - x}^\infty K(y)\, dy$$

for some $K(y)$ belonging to a general class of functions. The existence of such a general theorem was suspected by Wiener and myself on the basis of a proof given by Wiener* of the Hardy-Littlewood theorem stated above. There is a section in Wiener's proof that appears to have a wide range of applicability in theorems of this type. This is the section that makes use of an interpolation function whose Fourier transform vanishes outside a finite interval. That such an interpolation function is of basic importance in a general theory will soon be evident. But the rest of the analysis that must be developed in the general theorem is either completely missing or else so trivially simplified in the case $K(y) = e^y e^{-e^y}$ that everything else in the Wiener proof is quite misleading as regards generalizations. Basically, the difficulty turns out to be that the convergence of the Dirichlet series (1.0) for any $x > 0$ implies that

$$\sum_1^\infty a_k e^{-\epsilon \mu_k} e^{-\mu_k x}$$

is absolutely convergent for any $x > 0$. This allows the fundamental inversion of order of integration and summation in Wiener's proof. However, in the general theory no such simple device is used. Instead, certain complex variable properties of the Fourier transform of $K(y)$ play the fundamental rôle.

It is convenient to assume that we are dealing with a normal kernel, that is one for which

$$\int_{-\infty}^\infty K(y)\, dy = 1.$$

The general gap Tauberian theorem proved here is

THEOREM I. *Let*

$$\sum_1^\infty a_k \int_{\lambda_k - x}^\infty K(y)\, dy = f(x) \tag{1.1}$$

* Wiener, *Journal Chinese Math. Soc.*, 1 (1936), 15–22.

and let

$$\max_{x<X} \sum_{1}^{\infty} \left| a_k \int_{\lambda_k}^{\infty}{}_{x} K(y)\,dy \right| \leqslant M(X), \tag{1.2}$$

where $M(X) < \infty$ *for any* $X < \infty$.

Let $\lambda_0 = 0$ *and*

$$\lambda_{k+1} - \lambda_k \geqslant D > 0 \quad (k \geqslant 0). \tag{1.3}$$

Also let

$$K(x) \in L(-\infty, \infty), \tag{1.4}$$

$$xK(x) \in L(-\infty, \infty). \tag{1.5}$$

The Fourier transform of $K(x)$ *is*

$$k(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x)\, e^{-iux}\,dx. \tag{1.6}$$

Suppose that it is possible to find an analytic function of $w = u+iv$ *in the half-plane* $v > 0$, *continuous for* $v \geqslant 0$ *and equal to* $k(u)$ *on the real axis* $v = 0$. *This function we call* $k(u+iv)$.

Let $c = 2\pi/D$ *and let* A *be a positive constant. Suppose that* ξ *is a real variable. Let*

$$k(u+iv) \neq 0 \quad (v \geqslant 0), \tag{1.7}$$

$$\max_{|\xi|<c} \left| \frac{k(u+\xi)}{k(u)} \right| \leqslant e^{\theta(u)}, \tag{1.8}*$$

$$\max_{|\xi|<c} \left| \frac{k(w+\xi)}{k(w)} \right| \leqslant A e^{A|w|} \quad (v \geqslant 0), \tag{1.9}*$$

$$\left| \frac{k'(u)}{k(u)} \right| \leqslant e^{\theta(u)}, \tag{1.10}$$

where $\theta(u)$ *is a positive even function of* u, *monotone non-decreasing for* $u > 0$, *and*

$$\int_1^{\infty} \frac{\theta(u)}{u^2}\,du < \infty. \tag{1.11}$$

Then

$$\lim_{x \to \infty} f(x) = s \tag{1.12}$$

* Obviously, if (1.8) and (1.9) hold for one value of c, similar relations hold for any other value of c.

implies

$$\sum_1^\infty a_k = s. \tag{1.13}$$

The statement of this theorem is by no means simple. *Condition* (1.5) *is not necessary*, but it allows the proof to be considerably simplified. Condition (1.10) probably is not essential. Although it does not exclude any of the usual kernels from the scope of the theorem, the aesthetic qualities of the result would be considerably improved if it were removed.

Given (1.7) and (1.8), condition (1.9) is implied by what are apparently very much weaker requirements. However, since the theorem is no more general with this apparent weakening of (1.9) we shall not consider it here.

Condition (1.7) is basic. Let us suppose that it does not hold. Then, proceeding heuristically, we have

$$\int_{-\infty}^{\infty} K(t)\, e^{-iat+bt}\, dt = 0$$

for some $a+ib$, $b \geqq 0$. By an integration by parts (heuristically at least)

$$\int_{-\infty}^{\infty} e^{-iat+bt}\, dt \int_t^{\infty} K(y)\, dy = 0,$$

which can also be written as

$$\int_{-\infty}^{\infty} e^{-iat+bt}\, dt \int_{t-x}^{\infty} K(y)\, dt = 0.$$

But (1.1) can be written as

$$\int_{-\infty}^{\infty} ds(t) \int_{t-x}^{\infty} K(y)\, dy = f(x),$$

where $s(t)$ is defined in an obvious manner. A consideration of these last two formulae makes (1.7) seem reasonable.

Condition (1.8) excludes certain of the usual kernels from the scope of this theorem. For example, e^{-y^2} is excluded. The gap theorem for kernels of this type and other types excluded here will be discussed in future papers. It appears that the gap condition (1.3) is too lax for this type of kernel, but with somewhat larger gaps a gap Tauberian result does exist*.

---

* *Added April*, 1938. It can be shown by a simple example that with gap condition (1.3) and no restriction on $\{a_n\}$, a Tauberian result for $K(x) = e^{-x^2}$ does not hold. However, if (1.3) is replaced by $\sum_1^\infty \frac{1}{\lambda_n} < \infty$, such a result does hold.

Incidentally, in proving Theorem I, we shall also prove*

THEOREM II. *If* (1.12) *in Theorem I is replaced by*

$$\overline{\lim_{x\to\infty}}\,\{f(x)-s\} = \pm\Omega, \tag{1.14}$$

then

$$\overline{\lim_{n\to\infty}}\left|\sum_1^n a_k - s\right| \leqslant C_0\,\Omega, \tag{1.15}$$

where C_0 is a constant depending only on D and $K(x)$. In particular, if (1.3) *is replaced by*

$$\lim_{k\to\infty}(\lambda_{k+1}-\lambda_k) = \infty, \tag{1.16}$$

then $C_0 = 1$ in (1.15).

Alternative statement of Theorem I:

Let

$$0 < \mu_1 < \mu_2 < \ldots < \mu_k < \ldots; \quad \mu_{k+1}/\mu_k \geqslant e^D \quad (D > 0).$$

Let

$$\sum_1^\infty a_k N(x\mu_k) = f(x),$$

$$\max_{x>\delta}\sum_1^\infty |a_k N(x\mu_k)| \leqslant M(\delta),$$

where $M(\delta) < \infty$ for any $\delta > 0$. Let

$$N'(x) \in L(0,\infty),$$

$$N'(x)\log x \in L(0,\infty).$$

Let

$$k(u) = \frac{1}{\sqrt{2\pi}}\int_0^\infty N'(x)\,x^{-iu}\,dx.$$

If $k(u+iv)$ satisfies the requirements of Theorem I, then

$$\lim_{x\to+0} f(x) = s$$

implies

$$\sum_1^\infty a_k = \frac{s}{N(0)}.$$

* Ingham, *Quart. J. of Math.*, 8 (1937), 1. Ingham obtains the results of this theorem for the series (1.0).

In particular then, the gap Tauberian theorem holds for Lambert series, for here

$$N(x) = \frac{xe^{-x}}{1-e^{-x}}$$

and

$$k(u) = \frac{iu}{\sqrt{2\pi}}\,\Gamma(1-iu)\,\zeta(1-iu).$$

That $k(u)$ satisfies the requirements of Theorem I follows from well-known results on the gamma and zeta functions.

2. The proof of Theorems I and II depend on the following lemma. C_1, C_2, ... denote positive numbers which are constant for any particular D and $K(x)$.

LEMMA 1. *There exists an $R_n(x, B)$, $B > 2$, for every $n \geqslant 1$, such that*

$$\int_{-\infty}^{\infty} |R_n(x, B)|\,dx < C_1, \tag{2.1}$$

$$R_n(x, B) = 0 \quad (x > C_2 B), \tag{2.2}$$

$$\int_{-\infty}^{\frac{1}{2}\lambda_n} |R_n(x, B)|\,dx \leqslant \frac{C_3}{\lambda_n^{\frac{1}{2}}}, \tag{2.3}$$

$$\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy = 0 \quad (k \neq n,\ k \geqslant 0), \tag{2.4}$$

$$\lim_{B\to\infty} \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = 1. \tag{2.5}$$

Before proving Lemma 1 we shall use it to prove Theorem I. This proof we shall divide into two parts, (*a*) and (*b*).

Proof of Theorem I. Part (a). Here we prove that

$$a_n = o(1) \quad (n \to \infty).$$

By (1.2) and (2.2) the order of summation and integration can be inverted in what follows to give

$$\int_{-\infty}^{\infty} R_n(x, B)\{f(x) - s\}\,dx$$

$$= \sum_1^{\infty} a_k \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_k - x}^{\infty} K(y)\,dy - s\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_0 - x}^{\infty} K(y)\,dy.$$

By (2.4) this gives

$$\int_{-\infty}^{\infty} R_n(x, B)\{f(x)-s\}\,dx = a_n \int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy.$$

But by (2.1) and (2.3)

$$\left|\int_{-\infty}^{\infty} R_n(x, B)\{f(x)-s\}\,dx\right|$$

$$\leqslant \max|f(x)-s| \int_{-\infty}^{\frac{1}{2}\lambda_n} |R_n(x, B)|\,dx + \max_{x>\frac{1}{2}\lambda_n}|f(x)-s| \int_{\frac{1}{2}\lambda_n}^{\infty} |R_n(x, B)|\,dx$$

$$\leqslant C_3\lambda_n^{-\frac{1}{2}} \max|f(x)-s| + C_1 \max_{x>\frac{1}{2}\lambda_n}|f(x)-s|.$$

Thus

$$|a_n|\left|\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy\right|$$

$$\leqslant C_3\lambda_n^{-\frac{1}{2}} \max|f(x)-s| + C_1 \max_{x>\frac{1}{2}\lambda_n}|f(x)-s|.$$

By (2.5), letting $B\to\infty$, this becomes

$$|a_n| \leqslant C_3\lambda_n^{-\frac{1}{2}} \max|f(x)-s| + C_1 \max_{x>\frac{1}{2}\lambda_n}|f(x)-s|. \tag{2.6}$$

It is clear from (1.2) and (1.12) that $f(x)-s$ is uniformly bounded for all x, and that it tends to zero with increasing x. Thus, from (2.6), $a_n = o(1)$ as $n\to\infty$.

Part (b). Here we prove that

$$\sum_1^{\infty} a_k = s.$$

This part is quite elementary and really "Tauberian" in the sense of being of the same depth as the result originally proved by Tauber.

Let us take $x = \nu_n = \frac{1}{2}(\lambda_n+\lambda_{n+1})$ in (1.1). Then

$$\sum_1^{n} a_k \int_{\lambda_k - \nu_n}^{\infty} K(y)\,dy + \sum_{n+1}^{\infty} a_k \int_{\lambda_k - \nu_n}^{\infty} K(y)\,dy = f(\nu_n).$$

This can be written as

$$\sum_1^{n} a_k - \sum_1^{n} a_k \int_{-\infty}^{\lambda_k - \nu_n} K(y)\,dy + \sum_{n+1}^{\infty} a_k \int_{\lambda_k - \nu_n}^{\infty} K(y)\,dy = f(\nu_n). \tag{2.7}$$

Clearly

$$\left|\sum_1^n a_k \int_{-\infty}^{\lambda_k-\nu_n} K(y)\,dy\right|$$

$$\leqslant \sum_1^{[\frac{1}{2}n]} |a_k| \int_{-\infty}^{\lambda_k-\nu_n} |K(y)|\,dy + \sum_{[\frac{1}{2}n]+1}^{n} |a_k| \int_{-\infty}^{\lambda_k-\nu_n} |K(y)|\,dy$$

$$\leqslant \max|a_k| \sum_{k=1}^{[\frac{1}{2}n]} \int_{-\infty}^{-(n-k)D} |K(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \sum_{[\frac{1}{2}n]}^{n} \int_{-\infty}^{-(n+\frac{1}{2}-k)D} |K(y)|\,dy$$

$$\leqslant \max|a_k|\, n \int_{-\infty}^{-\frac{1}{2}nD} |K(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \sum_{l=0}^{n} \int_{-\infty}^{-(\frac{1}{2}+l)D} |K(y)|\,dy$$

$$\leqslant 3\max|a_k|\, n \int_{-\infty}^{-\frac{1}{2}nD} |K(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \sum_{l=0}^{n} (l+1) \int_{-(\frac{3}{2}+l)D}^{-(\frac{1}{2}+l)D} |K(y)|\,dy$$

$$\leqslant \max|a_k| \frac{6}{D} \int_{-\infty}^{-\frac{1}{2}nD} |yK(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \frac{2}{D} \sum_{l=0}^{n} \int_{-(\frac{3}{2}+l)D}^{-(\frac{1}{2}+l)D} |yK(y)|\,dy.$$

From this it follows at once that

$$(2.8)\quad \left|\sum_1^n a_k \int_{-\infty}^{\lambda_k-\nu_n} K(y)\,dy\right|$$

$$\leqslant \max|a_k| \frac{6}{D} \int_{-\infty}^{-\frac{1}{2}nD} |yK(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \frac{2}{D} \int_{-\infty}^{-\frac{1}{2}D} |yK(y)|\,dy.$$

Also

$$\left|\sum_{n+1}^{\infty} a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy\right| \leqslant \max_{k>n}|a_k| \sum_{l=0}^{\infty} \int_{(\frac{1}{2}+l)D}^{\infty} |K(y)|\,dy$$

$$\leqslant \max_{k>n}|a_k| \sum_{l=0}^{\infty} (l+1) \int_{(\frac{1}{2}+l)D}^{(\frac{3}{2}+l)D} |K(y)|\,dy.$$

And this gives

$$(2.9)\qquad \left|\sum_{n+1}^{\infty} a_k \int_{\lambda_k-\nu_n}^{\infty} K(y)\,dy\right| \leqslant \max_{k>n}|a_k| \frac{2}{D} \int_{\frac{1}{2}D}^{\infty} |yK(y)|\,dy.$$

(2.7), (2.8), and (2.9) give

$$(2.10)\quad \left|\sum_1^n a_k - f(\nu_n)\right|$$

$$\leqslant \max|a_k| \frac{6}{D} \int_{-\infty}^{-\frac{1}{2}nD} |yK(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \frac{2}{D} \int_{-\infty}^{\infty} |yK(y)|\,dy.$$

By Part (a) and (1.5) it follows that

$$\lim_{n\to\infty} \sum_1^n a_k = \lim_{n\to\infty} f(\nu_n),$$

and this completes Part (b).

We have used nothing of the strong hypothesis of Theorem I in this proof. It is in the proof of Lemma 1 that this hypothesis will be needed.

Proof of Theorem II. It follows from(1.14) and (2.6) that

$$\overline{\lim_{n\to\infty}} |a_n| \leqslant C_1 \Omega. \tag{2.11}$$

Clearly $$\left|\sum_1^n a_k - s\right| \leqslant \left|\sum_1^n a_k - f(\nu_n)\right| + |f(\nu_n) - s|.$$

With (2.10) this gives

$$\left|\sum_1^n a_k - s\right| \leqslant |f(\nu_n)-s| + \max|a_k| \frac{6}{D}\int_{-\infty}^{-\frac{1}{2}nD} |yK(y)|\,dy + \max_{k\geqslant \frac{1}{2}n} |a_k| \frac{2}{D}\int_{-\infty}^{\infty} |yK(y)|\,dy. \tag{2.12}$$

By (1.14) and (2.11) it follows that

$$\lim_{n\to\infty} \left|\sum_1^n a_k - s\right| \leqslant \Omega + \frac{2C_1\Omega}{D}\int_{-\infty}^{\infty} |yK(y)|\,dy$$

from which (1.15) follows at once.

We now consider the case in which (1.16) holds. From (2.12)

$$\left|\sum_1^n a_k - s\right| \leqslant |f(\nu_n)-s| + \max|a_k| \frac{8}{D}\int_{-\infty}^{\infty} |yK(y)|\,dy. \tag{2.13}$$

If, instead of (1.1), we consider, for any $N > 0$,

$$\sum_{N+1}^{\infty} a_k \int_{\lambda_k - x}^{\infty} K(y)\,dy = f_N(x), \tag{2.14}$$

$$s_N = s - \sum_1^N a_k, \quad D_N = \min(\lambda_{k+1} - \lambda_k) \quad (k > N),$$

then, applying (2.13) to (2.14), we have, for $n > N$,

$$\left|\sum_{N+1}^n a_k - s_N\right| \leqslant |f_N(\nu_n) - s_N| + \frac{8}{D_N}\max|a_k| \int_{-\infty}^{\infty} |yK(y)|\,dy. \tag{2.15}$$

But
$$\sum_{N+1}^{n} a_k - s_N = \sum_{1}^{n} a_k - s,$$

$$\lim_{n\to\infty} \left(|f_N(\nu_n) - s_N| - |f(\nu_n) - s| \right) = 0.$$

Thus (2.15) gives

$$\overline{\lim_{n\to\infty}} \left| \sum_{1}^{n} a_k - s \right| \leqslant \overline{\lim_{n\to\infty}} |f(\nu_n) - s| + \frac{8}{D_N} \max |a_k| \int_{-\infty} |yK(y)|\, dy.$$

But by (2.11) a_k is bounded. By taking N sufficiently large, we can make D_N arbitrarily large by (1.16). Thus

$$\overline{\lim_{n\to\infty}} \left| \sum_{1}^{n} a_k - s \right| \leqslant \overline{\lim_{n\to\infty}} |f(\nu_n) - s| = \Omega,$$

which proves that $C_0 = 1$ in this case.

3. In proving Lemma 1 we require several auxiliary results. The first of these is an interpolation result of the type already mentioned. It will be clear that the result itself is quite crude.

LEMMA 2. *For every $n > 0$ there exists an $H_n(s) \in L(-\infty, \infty)$, such that*

$$H_n(\lambda_n) = 1; \quad H_n(\lambda_k) = 0 \quad (k \neq n,\ k \geqslant 0), \tag{3.1}$$

where $\{\lambda_n\}$ are as in Theorem I. If

$$h_n(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H_n(s)\, e^{-isu}\, ds \tag{3.2}$$

then $h_n(u) = 0$, $|u| > c$, and

$$h_n(u) = g_n(u)\, e^{\,iu\lambda_n}, \tag{3.3}$$

where

$$|g_n(u)| < C_4, \quad |g_n'(u)| < C_5. \tag{3.4}$$

Proof of Lemma 2. Throughout this proof n is some fixed positive integer. For $-\infty < k < \infty$ let

$$\sigma_k = \tfrac{1}{2}kD \quad \text{if} \quad |\tfrac{1}{2}kD - \lambda_m + \lambda_n| > \tfrac{1}{4}D \quad \text{for all } m \geqslant 0,$$

$$\sigma_k = \tfrac{1}{2}kD \quad \text{if} \quad \tfrac{1}{2}kD - \lambda_m + \lambda_n = \tfrac{1}{4}D \quad \text{for some } m,$$

$$\sigma_k = \lambda_m - \lambda_n \quad \text{if} \quad |\tfrac{1}{2}kD - \lambda_m + \lambda_n| < \tfrac{1}{4}D,$$

$$\sigma_k = \lambda_m - \lambda_n \quad \text{if} \quad \tfrac{1}{2}kD - \lambda_m + \lambda_n = -\tfrac{1}{4}D.$$

By (1.3) this defines σ_k uniquely and

$$\sigma_{k+1}-\sigma_k \geqslant \tfrac{1}{4}D.$$

Let s be a complex variable and let

$$G(s) = \prod_1^\infty (1-s/\sigma_k)(1-s/\sigma_{-k}).$$

Then
$$\left|\frac{G(s)}{\sin 2\pi s/D}\right| = \frac{D}{2\pi s}\left|\prod_1^\infty \frac{1-s/\sigma_k}{1-2s/kD}\right|\left|\frac{1-s/\sigma_{-k}}{1+2s/kD}\right|.$$

Clearly

$$\left|\frac{1-s/\sigma_k}{1-2s/kD}\right| \leqslant 1+\left|\frac{s(\sigma_k-\frac{1}{2}kD)}{\sigma_k(s-\frac{1}{2}kD)}\right| \leqslant 1+\left|\frac{\frac{1}{4}Ds}{(\frac{1}{2}kD-\frac{1}{4}D)(s-\frac{1}{2}kD)}\right|$$

$$\leqslant 1+\left|\frac{s}{(2k-1)(s-\tfrac{1}{2}kD)}\right| \leqslant \exp\left(\left|\frac{s}{k(s-\frac{1}{2}kD)}\right|\right).$$

Thus
$$\left|\frac{G(s)}{\sin 2\pi s/D}\right| \leqslant \left|\frac{D}{2\pi s}\right| \exp\left(\Sigma'\left|\frac{s}{k(s-\frac{1}{2}kD)}\right|\right),$$

where Σ' denotes the sum from $-\infty$ to ∞ with $k=0$ omitted.

If $|s-\frac{1}{2}kD| \geqslant \frac{1}{8}D$ for every k, then

$$\Sigma'\left|\frac{s}{k(s-\frac{1}{2}kD)}\right| \leqslant 20\sum_{-\infty}^{\infty}\frac{|s|}{(1+|k|)(|s-\frac{1}{2}kD|+D)}$$

$$\leqslant 40\sum_{0\leqslant k\leqslant 2|s|/D}\frac{|s|}{D+|s|-\frac{1}{2}kD}\frac{1}{1+k} + 40\sum_{2|s|/D<k}\frac{|s|}{D+\frac{1}{2}kD-|s|}\frac{1}{1+k}.$$

From this it follows readily that

$$\Sigma'\left|\frac{s}{k(s-\frac{1}{2}kD)}\right| \leqslant C_6\log(2+|s|).$$

Thus
$$|G(s)| \leqslant (2+|s|)^{C_6}\left|\frac{D}{2\pi s}\sin\frac{2\pi s}{D}\right|,$$

and from this, if $\Im s = t$,

$$|G(s)| \leqslant C_7(2+|s|)^{C_6}e^{c|t|},$$

if $|s-\frac{1}{2}kD| \geqslant \frac{1}{8}D$ for all k. But, since $G(s)$ is analytic and therefore takes its maximum value in any circle $|s-\frac{1}{2}kD| \leqslant \frac{1}{8}D$ on the boundary of that circle, it follows that for all s

$$|G(s)| \leqslant C_8(2+|s|)^{C_6}e^{c|t|}.$$

Obviously we may assume that $C_6 > 4$ in this last inequality.

From the definition of σ_k it follows that in a large interval more than one-third of the zeros of $G(s)$ are not of the form $\lambda_m-\lambda_n$, $0\leqslant m<\infty$. Let the zeros not of this form be denoted by $\{\tau_k\}$. Since the number of τ_k such that $|\tau_k|\leqslant 2C_6D$ is at least $\frac{1}{3}(8C_6-2)$, it follows for $|s|>4C_6D$ that

$$\left|G(s)\Big/\prod_{|\tau_k|\leqslant 2C_6D}\left(1-\frac{s}{\tau_k}\right)\right|\leqslant C_8(2+|s|)^{C_6}e^{c|t|}\Big/\prod_{|\tau_k|\leqslant 2C_6D}\left|1-\frac{s}{\tau_k}\right|$$

$$\leqslant\frac{C_9e^{c|t|}}{(1+|s|)^{C_6}}.$$

But, by the maximum modulus theorem, this can be extended to all values of s, whence we have

$$\left|G(s)\Big/\prod_{|\tau_k|\leqslant 2C_6D}\left(1-\frac{s}{\tau_k}\right)\right|\leqslant\frac{C_{10}e^{c|t|}}{(1+|s|)^{C_6}}.$$

Let $$H_n(s)=G(s-\lambda_n)\Big/\prod_{|\tau_k|\leqslant 2C_6D}\left(1-\frac{s-\lambda_n}{\tau_k}\right).$$

Then, since $C_6>4$,

$$|H_n(s)|\leqslant\frac{C_{10}e^{c|t|}}{(1+|s-\lambda_n|)^4}. \tag{3.5}$$

Obviously $H_n(s)$ satisfies (3.1). If

$$g_n(u)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}H_n(s+\lambda_n)\,e^{-isu}\,ds,$$

then, by (3.5),

$$|g_n(u)|<C_4.$$

Also $$g_n'(u)=-\frac{i}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}H_n(s+\lambda_n)\,se^{-us}\,ds;$$

and again, by (3.5),

$$|g_n'(u)|\leqslant C_5.$$

By the Cauchy integral theorem and (3.5) it follows easily that

$$g_n(u)=0,\quad |u|>c.$$

If $h_n(u)$ is defined by (3.2) then

$$h_n(u)=\frac{1}{\sqrt{(2\pi)}}\int_{-\infty}^{\infty}H_n(s+\lambda_n)\,e^{-iu(s+\lambda_n)}\,ds=e^{-iu\lambda_n}g_n(u).$$

This completes the proof of Lemma 2.

LEMMA 3. *There exists a function $\phi(u+iv)$ analytic in the upper half-plane $v \geqslant 0$ and such that*

$$|\phi(u+iv)| \leqslant C_{12} e^{c_{11} v} \quad (v \geqslant 0), \tag{3.6}$$

$|\phi(u)|$ *is even, and*

$$|\phi(u)| \leqslant C_{13} \frac{e^{-2\theta(u)}}{1+u^4}, \tag{3.7}$$

where $\theta(u)$ satisfies the conditions given in Theorem I. Moreover, if

$$\Phi(s) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \phi(u) e^{isu} du, \tag{3.8}$$

then

$$|\Phi(s)| \leqslant C_{14}, \quad |\Phi'(s)| \leqslant C_{15} \tag{3.9}$$

and

$$\Phi(0) = 1. \tag{3.10}$$

Proof of Lemma 3. Let $\theta_1(u)$ be an even function of u defined for $u > 0$ by

$$\theta_1(u) = 4 \int_u^{u+1} \theta(u)\, du + 2u^{\frac{1}{2}} \geqslant 4\theta(u) + 2u^{\frac{1}{2}}.$$

Clearly $\theta_1(u)$ is continuous and is monotone increasing for $u > 0$. Let

$$\lambda(u, v) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(v+1)\,\theta_1(\xi)}{(u-\xi)^2 + (v+1)^2}\, d\xi.$$

By (1.11) the above integral defines $\lambda(u, v)$ as an harmonic function in the half-plane $v > -1$. For $u > 0$

$$\lambda(u, 0) \geqslant \frac{1}{\pi} \int_u^{\infty} \frac{\theta_1(\xi)}{(u-\xi)^2+1}\, d\xi \geqslant \frac{\theta_1(u)}{\pi} \int_0^{\infty} \frac{d\xi}{1+\xi^2}$$

$$\geqslant \tfrac{1}{2}\theta_1(u) \geqslant 2\theta(u) + u^{\frac{1}{2}}.$$

Since $\lambda(u, 0)$ is even, it follows that, for all u,

$$\lambda(u, 0) \geqslant 2\theta(u) + u^{\frac{1}{2}}. \tag{3.11}$$

It is well known that there exists a function $\mu(u, v)$ conjugate to $\lambda(u, v)$. Let $w = u+iv$. Then

$$g(w) = \lambda(u+iv) + i\mu(u, v)$$

is analytic in the half-plane $v > -1$. Since $\lambda(u, v) \geqq 0$,

$$|e^{-g(w)}| \leqq 1. \tag{3.12}$$

Along the real axis, by (3.11),

$$|e^{-g(u)}| \leqq e^{-2\theta(u)-|u|}. \tag{3.13}$$

Let
$$\Phi_1(s) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-g(u)} e^{isu}\, du.$$

$\Phi_1(s)$ is not identically zero since it is the Fourier transform of $e^{-g(u)}$. Thus for some value of s, $\Phi_1(s) \neq 0$. Let such a value of s be a. Then

$$\frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-g(u)} e^{-iau}\, du = b \neq 0. \tag{3.14}$$

Let
$$\phi(w) = \frac{e^{-g(w)+iaw}}{b}.$$

Then by (3.12), (3.6) follows and by (3.13), (3.7) follows. Let $\Phi(s)$ be defined as in (3.8). Then by (3.14), (3.10) is satisfied. (3.9) follows easily from (3.7). This completes the proof of Lemma 3.

4. Lemma 1 actually involves solving a certain integral equation for $R_n(x, B)$. The five conditions which $R_n(x, B)$ must fulfil make this a much harder problem to handle than is the case with integral equations usually associated with Tauberian theorems. In fact, the major difficulty here is in finding just what this integral equation should be.

Proof of Lemma 1. Using the terminology in the statements of Theorem I and Lemmas 2 and 3, let us define

$$r_n(u, B) = -\frac{iuBe^{-iu\lambda_n}}{\sqrt{(2\pi)}\, k(u)} \int_{u-c}^{u+c} k(y)\, \phi(By)\, g_n(u-y)\, e^{i\lambda_n y}\, dy. \tag{4.1}$$

By (1.8) and (3.4)

$$|r_n(u, B)| \leqq C_4 B|u| e^{\theta(u)} \int_{u-c}^{u+c} |\phi(By)|\, dy \leqq C_4 |u| e^{\theta(u)} \int_{B(u-c)}^{B(u+c)} |\phi(y)|\, dy.$$

Thus for $|u| < 2c$ it follows from (3.7) that

$$|r_n(u, B)| \leqq C_{16}|u|. \tag{4.2}$$

For $|u| > 2c$ and $B > 2$ it follows from (3.7) that

$$|r_n(u, B)| \leqq C_4 |u| e^{\theta(u)} \int_{|u|}^{\infty} |\phi(y)|\, dy \leqq \frac{C_{17}}{1+u^2}. \tag{4.3}$$

Thus

$$\int_{-\infty}^{x} |r_n(u, B)|\,du < C_{18}. \tag{4.4}$$

From (4.1) we also have

$$\frac{d}{du}\{e^{iu\lambda_n} r_n(u, B)\} = -\frac{iuB}{\sqrt{(2\pi)}\,k(u)} \int_{u-c}^{u+c} k(y)\,\phi(By)\,g_n'(u-y)\,e^{i\lambda_n y}\,dy$$

$$-\frac{iB}{\sqrt{(2\pi)}\,k(u)}\left(1-\frac{uk'(u)}{k(u)}\right)\int_{u-c}^{u+c} k(y)\,\phi(By)\,g_n(u-y)\,e^{i\lambda_n y}\,dy.$$

Treating each of the terms on the right in a manner similar to that used in (4.1) it follows easily that

$$\left|\frac{d}{du}\{e^{iu\lambda_n} r_n(u, B)\}\right| \leq \frac{C_{19}}{1+u^2}. \tag{4.5}$$

If we define

$$R_n(x, B) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} r_n(u, B)\,e^{iux}\,du,$$

it follows from (4.4) that

$$|R_n(X, B)| < C_{18}. \tag{4.6}$$

Also $$R_n(x+\lambda_n, B) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} r_n(u, B)\,e^{iu\lambda_n} e^{iux}\,du,$$

and, integrating by parts,

$$-ix\,R_n(x+\lambda_n, B) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} \frac{d}{du}\{r_n(u, B)\,e^{iu\lambda_n}\}\,e^{iux}\,du.$$

By the Parseval theorem for Fourier transforms and (4.5),

$$\int_{-\infty}^{\infty} x^2 |R_n(x+\lambda_n, B)|^2\,dx \leq C_{20}. \tag{4.7}$$

From this and (4.6) it follows by the Schwartz inequality that

$$\int_{-\infty}^{\infty} |R_n(x, B)|\,dx < C_1.$$

This is (2.1). Again by the Schwartz inequality

$$\int_{-\infty}^{\frac{1}{2}\lambda_n} |R_n(x, B)|\,dx \leq \left[\int_{-\infty}^{\infty} (x-\lambda_n)^2 |R_n(x, B)|^2\,dx\right]^{\frac{1}{2}} \left[\int_{-\infty}^{\frac{1}{2}\lambda_n} \frac{dx}{(x-\lambda_n)^2}\right]^{\frac{1}{2}}.$$

Using (4.7), we have (2.3) at once,

Clearly, by (4.1), for real w

$$r_n(w, B) = -\frac{iBw}{\sqrt{(2\pi)}\,k(w)} \int_{-c}^{c} k(w-y)\,\phi\{B(w-y)\}\,h_n(y)\,dy.$$

But this definition shows that $r_n(w, B)$ is analytic in the half-plane $v > 0$ and continuous in $v \geqslant 0$. For $v \geqslant 0$, by (1.9), (3.6), and (3.4)

$$|r_n(w, B)| \leqslant B|w|\,A e^{A|w|}\,C_{12}\,e^{C_{11}Bv}\,2c\,C_4,$$

i.e, for $B > 2$,

$$|r_n(B, w)| \leqslant B\,C_{21}\,e^{BC_{22}|w|}. \tag{4.8}$$

By (4.2), (4.3), and (4.8)

$$r_n(w, B)\,e^{2BC_{22}iw}(w+i)^2$$

is bounded along the entire real axis and the positive imaginary axis. From this and (4.8), by the well-known theorem of Phragmén-Lindelöf*, we have at once

$$|r_n(w, B)\,e^{2BC_{22}iw}(w+i)^2| \leqslant C_{23}$$

in the half-plane $v \geqslant 0$. Thus

$$|r_n(w, B)\,e^{2BC_{22}iw}| \leqslant \frac{C_{23}}{1+|w|^2}.$$

From the definition of $R_n(x, B)$ and the Cauchy integral theorem it now follows at once that

$$R_n(x, B) = 0, \quad x > 2BC_{22},$$

and this proves (2.2).

(4.1) can be written as

$$-\frac{r_n(u, B)\,k(u)}{iu} = \frac{B}{\sqrt{(2\pi)}} \int_{-c}^{c} k(u-y)\,\phi\{B(u-y)\}\,h_n(y)\,dy. \tag{4.9}$$

By the definition of $k(u)$ we have

$$\begin{aligned}\int_{-\infty}^{\infty} r_n(u, B)\,k(u)\,e^{iuy}\,du &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} r_n(u, B)\,e^{iuy}\,du \int_{-\infty}^{\infty} K(x)\,e^{-iux}\,dx \\ &= \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} K(x)\,dx \int_{-\infty}^{\infty} r_n(u, B)\,e^{iu(y-x)}\,du \\ &= \int_{-\infty}^{\infty} K(x)\,R_n(y-x, B)\,dx.\end{aligned}$$

---

* Titchmarsh, *Theory of functions* (1932), 177.

The repeated integral here is absolutely integrable, so that inversion of the order of integration is permissible. The result can also be written as

$$\int_{-\infty}^{\infty} r_n(u, B)\, k(u)\, e^{iuy}\, du = \int_{-\infty}^{\infty} R_n(x, B)\, K(y-x)\, dx.$$

Integrating with respect to y, we have

$$(4.10)\quad \int_{-\infty}^{\infty} r_n(u, B)\, k(u)\, \frac{e^{iua}-e^{ius}}{iu}\, du = \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{s}^{a} K(y-x)\, dy.$$

Since $r_n(u, B)/u \in L(-\infty, \infty)$ by (4.2) and (4.3) and since $k(u)$ is bounded, it follows by the theorem of Riemann and Lebesgue that

$$\lim_{a\to\infty} \int_{-\infty}^{\infty} \frac{r_n(u, B)\, k(u)}{iu}\, e^{iua}\, du = 0.$$

Letting $a\to\infty$ in (4.10), we have

$$(4.11)\quad -\int_{-\infty}^{\infty} \frac{r_n(u, B)\, k(u)}{iu}\, e^{ius}\, du = \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{s-x}^{\infty} K(y)\, dy,$$

since the repeated integral on the right is absolutely integrable.

Let

$$(4.12)\quad J_n(s) = \int_{-\infty}^{\infty} e^{ius}\, du\, \frac{B}{\sqrt{(2\pi)}} \int_{-c}^{c} k(u-y)\, \phi\{B(u-y)\}\, h_n(y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ius}\, du \int_{-c}^{c} B\phi\{B(u-y)\}\, h_n(y)\, dy \int_{-\infty}^{\infty} K(x)\, e^{-ix(u-y)}\, dx.$$

That this repeated integral exists and is absolutely integrable is clear if it is written as

$$J_n(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(x)\, dx \int_{-c}^{c} h_n(y)\, e^{iyx}\, dy \int_{-\infty}^{\infty} B\phi\{B(u-y)\}\, e^{iu(s-x)}\, du.$$

It follows at once that

$$J_n(s) = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\infty} K(x)\, dx \int_{-c}^{c} h_n(y)\, e^{iyx}\, \Phi\left(\frac{s-x}{B}\right) e^{iy(s-x)}\, dy$$

$$= H_n(s) \int_{-\infty}^{\infty} K(x)\, \Phi\left(\frac{s-x}{B}\right) dx.$$

But with (4.9), (4.11), and (4.12) this gives

$$(4.13)\quad \int_{-\infty}^{\infty} R_n(x, B)\, dx \int_{s-x}^{\infty} K(y)\, dy = H_n(s) \int_{-\infty}^{\infty} K(x)\, \Phi\left(\frac{s-x}{B}\right) dx.$$

(2.4) now follows at once from (3.1). (3.1) also gives

$$\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = \int_{-\infty}^{\infty} K(x)\,\Phi\left(\frac{\lambda_n - x}{B}\right) dx$$

$$= \int_{-\infty}^{\infty} K(x) \left\{\Phi(0) + \int_0^{(\lambda_n - x)/B} \Phi'(y)\,dy\right\} dx.$$

By (3.10) this gives

$$\int_{-\infty}^{\infty} R_n(x, B)\,dx \int_{\lambda_n - x}^{\infty} K(y)\,dy = 1 + \int_{-\infty}^{\infty} K(x)\,dx \int_0^{(\lambda_n - x)/B} \Phi'(y)\,dy.$$

By (3.9) and (1.4), (2.5) follows at once. This completes the proof of Lemma 1. The integral equation handled for this Tauberian gap theorem, is (4.13).

Massachusetts Institute of Technology,
Cambridge, Massachusetts, U.S.A.

ON GOLDBACH'S PROBLEM : PROOF THAT ALMOST ALL EVEN POSITIVE INTEGERS ARE SUMS OF TWO PRIMES

By T. Estermann.

[Received 10 July, 1937.—Read 18 November, 1937.]

Introduction.

By combining ideas of Hardy and Littlewood (**1**) with a recent discovery of Vinogradow (**2**, **3**, Theorem 1), I obtain the following result:

Let $Q(n)$ denote the number of those even positive integers less than n which are not representable as sums of two primes. Then, as $n \to \infty$,

$$Q(n) = O(n \log^{-a} n)$$

for any positive number a.

1. *Notation.*

η may be any number greater than 4;

$A_1, A_2, \ldots$ are suitable (sufficiently large) positive numbers depending at most on η;

h and l are integers;

k, m, n, q, r, s, t, and v are positive integers, and

$$n > A_1; \tag{1}$$

p is a prime;

x and y are real numbers.

I put

$$\xi_q = e^{2\pi i/q}, \tag{2}$$

$$c_q(l) = \sum_{\substack{r \leqslant q \\ (r,q)=1}} \xi_q^{rl}, \tag{3}$$

$$f(x) = \sum_{p \leqslant n} e^{2\pi i x p}, \tag{4}$$

$$g_1(x) = \sum_{2 \leqslant s \leqslant n} \frac{e^{2\pi i x s}}{\log s}, \tag{5}$$

$$g_2(x, q) = \sum_{\substack{r \leqslant q \\ (r,q)=1}} g_1{}^2(x - r/q), \tag{6}$$

and

$$g_3(x) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} g_2(x, q), \tag{7}$$

where μ and ϕ denote Möbius's and Euler's functions respectively.

2. The aim of §§ 2–3 is the inequality

$$|f^2(x) - g_3(x)| \leqslant A_2 n^2 \log^{-\eta} n. \tag{8}$$

Putting for brevity

$$G = \sum_{q \leqslant \log^{3\eta} n} \frac{\mu^2(q)}{\phi^2(q)} g_2(x, q), \tag{9}$$

and noting that the function g_1 has period 1, we have, by (6),

$$G = \sum_{\substack{q, l \\ (11)(12)(13)}} \frac{\mu^2(q)}{\phi^2(q)} g_1{}^2(x - l/q), \tag{10}$$

with the conditions of summation

$$q \leqslant \log^{3\eta} n, \tag{11}$$

$$(l, q) = 1, \tag{12}$$

and

$$-\tfrac{1}{2} < x - l/q \leqslant \tfrac{1}{2}. \tag{13}$$

Choose h and k so that

$$(h, k) = 1, \tag{14}$$

$$k \leqslant n \log^{-3\eta} n, \tag{15}$$

and

$$|kx-h| \leqslant n^{-1}\log^{3\eta} n. \tag{16}$$

This is possible by a well-known elementary theorem, since, by (1), $n\log^{-3\eta} n > 1$. There are now two cases to be distinguished.

First case. Suppose that

$$k \leqslant \log^{3\eta} n. \tag{17}$$

Then

$$\left| f^2(x) - \frac{\mu^2(k)}{\phi^2(k)}\, g_1^2(x-h/k)\right| \leqslant A_3 n^2 \log^{-\eta} n. \tag{18}$$

This can be proved in essentially the same way as Estermann **4**, (34).

Now it follows from (14), (16), (1), and (17) that h and k are values of l and q respectively which satisfy (11), (12), and (13). Hence, by (10),

$$G - \frac{\mu^2(k)}{\phi^2(k)}\, g_1^2(x-h/k) = \sum_{\substack{q,\,l \\ (11)(12)(13)(20)}} \frac{\mu^2(q)}{\phi^2(q)}\, g_1^2(x-l/q), \tag{19}$$

with

$$l/q \neq h/k. \tag{20}$$

It is easily seen [*cf.* Estermann **4**, (37)] that

$$|g_1(y)| < |y|^{-1} \quad (0 < |y| \leqslant \tfrac{1}{2}), \tag{21}$$

and it follows from (20), (17), and (11) that $|h/k - l/q| \geqslant 1/(kq) \geqslant \log^{-6\eta} n$. This, together with (16) and (1), implies that

$$|x-l/q| \geqslant |h/k-l/q| - |x-h/k| \geqslant \log^{-6\eta} n - n^{-1}\log^{3\eta} n \geqslant \tfrac{1}{2}\log^{-6\eta} n.$$

From this and (13) and (21) it follows that $|g_1^2(x-l/q)| < 4\log^{12\eta} n$. Hence, by (19) and (1),

$$\begin{aligned} \left| G - \frac{\mu^2(k)}{\phi^2(k)}\, g_1^2(x-h/k)\right| &\leqslant 4\log^{12\eta} n \sum_{\substack{q,\,l \\ (11)(13)}} 1 \\ &= 4\log^{12\eta} n \sum_{\substack{q \\ (11)}} q < 4\log^{18\eta} n < n^2 \log^{-\eta} n. \end{aligned} \tag{22}$$

From this and (18) we obtain

$$|f^2(x) - G| \leqslant A_4 n^2 \log^{-\eta} n. \tag{23}$$

Second case. Suppose that

$$k > \log^{3\eta} n. \tag{24}$$

Then (16), (14), (24), and (15) imply

$$|f(x)| \leqslant A_5 n \log^{2-\eta} n. \tag{25}$$

This (apart from the notation) is Theorem 1 of Vinogradow **2**, **3**.

In order to obtain an inequality for G, return to (10). From the pairs of numbers q, l which satisfy (11), (12), and (13), choose that (or one of those, if there are two) for which $|x-l/q|$ is least. Let the chosen pair be q', l'. Then

$$q' \leqslant \log^{3\eta} n, \tag{26}$$

$$|x-l'/q'| \leqslant \tfrac{1}{2}, \tag{27}$$

and, by (10),

$$G-\frac{\mu^2(q')}{\phi^2(q')}\, g_1{}^2(x-l'/q') = \sum_{\substack{q,l \\ (11)(12)(13)(29)}} \frac{\mu^2(q)}{\phi^2(q)}\, g_1{}^2(x-l/q), \tag{28}$$

with

$$l/q \neq l'/q', \tag{29}$$

and q' and l' have been so chosen that (11), (12), and (13) imply

$$|x-l/q| \geqslant |x-l'/q'|.$$

From this and (29), (26), and (11) it follows that

$$|x-l/q| \geqslant \tfrac{1}{2}(|x-l/q|+|x-l'/q'|) \geqslant \tfrac{1}{2}|l/q-l'/q'| \geqslant 1/(2qq') \geqslant \tfrac{1}{2}\log^{-6\eta} n.$$

Hence, applying to (28) an argument similar to that which led from (19) to (22), we obtain

$$\left| G-\frac{\mu^2(q')}{\phi^2(q')}\, g_1{}^2(x-l'/q') \right| < n^2 \log^{-\eta} n. \tag{30}$$

Now, by (26), (24), and (14), $h/k \neq l'/q'$, so that

$$|h/k-l'/q'| \geqslant 1/(kq'). \tag{31}$$

Also, by (16), (26), and (1),

$$|x-h/k| \leqslant (nk)^{-1} \log^{3\eta} n \leqslant (nkq')^{-1} \log^{6\eta} n < 1/(2kq').$$

Hence, by (31), $|x-l'/q'| > 1/(2kq')$, and hence, by (21) and (27),

$$|g_1{}^2(x-l'/q')| < (2kq')^2.$$

From this and (15) and (26) it follows that

$$|\phi^{-2}(q')g_1^2(x-l'/q')| \leqslant 4n^2 \log^{-6\eta} n\{q'/\phi(q')\}^2 < A_6 n^2 \log^{-\eta} n,$$

which, together with (30) and (25), again establishes (23).

3. It remains to estimate $|g_3(x)-G|$.

Now, by (5),

$$g_1^2(y) = \sum_{m \leqslant 2n} b(m) e^{2\pi i y m}, \tag{32}$$

where

$$b(m) = \sum_{\substack{s,t \\ (34)(35)(36)}} \frac{1}{\log s \log t}, \tag{33}$$

with

$$2 \leqslant s \leqslant n, \tag{34}$$

$$2 \leqslant t \leqslant n, \tag{35}$$

and

$$s+t = m. \tag{36}$$

By (32) and (2),

$$g_1^2(x-r/q) = \sum_{m \leqslant 2n} b(m) \xi_q^{-rm} e^{2\pi i x m},$$

and hence, by (6) and (3) [since $c_q(m) = c_q(-m)$],

$$g_2(x, q) = \sum_{m \leqslant 2n} b(m) c_q(m) e^{2\pi i x m}. \tag{37}$$

Now it follows from (33) that $0 \leqslant b(m) \leqslant 2n$, and it is known (Landau **5**, Theorem 220) that

$$c_q(m) = \sum_{r \mid (q, m)} r\mu(q/r),$$

which implies

$$|c_q(m)| \leqslant \sum_{r \mid (q, m)} r = (q, m) \sum_{s \mid (q, m)} s^{-1} \leqslant (q, m) \sum_{s \leqslant q} s^{-1} \leqslant (q, m)(1+\log q).$$

Hence, by (37),

$$|g_2(x, q)| \leqslant 2n(1+\log q) \sum_{m \leqslant 2n} (q, m). \tag{38}$$

Also $$\sum_{m\leqslant 2n}(q,m)=\sum_{v\mid q}\sum_{\substack{m\leqslant 2n\\(q,m)=v}}v\leqslant\sum_{v\mid q}\sum_{\substack{m\leqslant 2n\\v\mid m}}v$$

$$=\sum_{v\mid q}v[2n/v]\leqslant\sum_{v\mid q}2n=2nd(q).$$

Hence, by (38),

(39) $$|\phi^{-2}(q)g_2(x,q)|\leqslant 4n^2(1+\log q)d(q)\phi^{-2}(q)\leqslant A_7 n^2 q^{-\frac{5}{3}},$$

and hence, by (7) and (9),

$$|g_3(x)-G|\leqslant A_7 n^2\sum_{q>\log^{3\eta}n}q^{-\frac{5}{3}}<A_8 n^2\log^{-2\eta}n.$$

From this and (23) we obtain (8).

4. We now estimate $\int_0^1|f^2(x)-g_3(x)|^2dx$.

By (4),

(40) $$\int_0^1|f^2(x)|dx=\sum_{p\leqslant n}1<n.$$

By (5),

$$\int_0^1|g_1{}^2(x-r/q)|dx=\int_{-r/q}^{1-r/q}|g_1{}^2(y)|dy=\sum_{2\leqslant s\leqslant n}\log^{-2}s<A_9 n\log^{-2}n.$$

Hence, by (6),

$$\int_0^1|g_2(x,q)|dx<A_9 n\log^{-2}n\phi(q),$$

so that $$\phi^{-2}(q)\int_0^1|g_2(x,q)|dx<A_{10}nq^{-1}\log^{-1}n\quad(q\leqslant n^2),$$

and hence

(41) $$\int_0^1\Big|\sum_{q\leqslant n^2}\frac{\mu^2(q)}{\phi^2(q)}g_2(x,q)\Big|dx<A_{11}n.$$

Also, by (39),

$$\int_0^1\Big|\sum_{q>n^2}\frac{\mu^2(q)}{\phi^2(q)}g_2(x,q)\Big|dx\leqslant A_7 n^2\sum_{q>n^2}q^{-\frac{5}{3}}<2A_7 n^{\frac{2}{3}}.$$

From this and (7) and (41) it follows that

(42) $$\int_0^1|g_3(x)|dx<A_{12}n,$$

so that, by (40),

$$\int_0^1 |f^2(x)-g_3(x)|\,dx < (1+A_{12})\,n,$$

and hence, by (8),

$$(43)\quad \int_0^1 |f^2(x)-g_3(x)|^2\,dx \leqslant A_2 n^2 \log^{-\eta} n \int_0^1 |f^2(x)-g_3(x)|\,dx \leqslant A_{13} n^3 \log^{-\eta} n.$$

5. Put

$$(44)\quad S(m) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)}\, c_q(m).$$

Then, by (7) and (37),

$$(45)\quad g_3(x) = \sum_{m\leqslant 2n} b(m)\, S(m)\, e^{2\pi i x m}.$$

Also, by (4),

$$(46)\quad f^2(x) = \sum_{m\leqslant 2n} a(m)\, e^{2\pi i x m},$$

where $a(m)$ is the number of representations of m as a sum of two primes which do not exceed n, so that $a(m) = 0$ if m is not representable as a sum of two primes.

6. By (46) and (45),

$$(47)\quad \int_0^1 |f^2(x)-g_3(x)|^2\,dx = \sum_{m\leqslant 2n} \{a(m)-b(m)\,S(m)\}^2.$$

Now it is obvious from (33) that $b(m) \geqslant (m-3)\log^{-2} n$ if $m \leqslant n$, so that, by (1),

$$(48)\quad b(m) \geqslant n \log^{-2-\frac{1}{3}\eta} n \quad (2n \log^{-\frac{1}{3}\eta} n \leqslant m \leqslant n),$$

and it is known [Landau **5**, 226–227, formula (239)] that

$$(49)\quad S(m) \geqslant A_{14}^{-1} \quad (2\,|\,m).$$

Now let Q_1 denote the number of those even numbers m between $2n \log^{-\frac{1}{3}\eta} n$ and n which are not representable as sums of two primes. For any such number m we have, by the remark at the end of § 5, $a(m) = 0$, and hence, by (48) and (49),

$$\{a(m)-b(m)\,S(m)\}^2 \geqslant A_{14}^{-2} n^2 \log^{-4-\frac{2}{3}\eta} n.$$

It follows that

$$\sum_{m \leqslant 2n} \{a(m) - b(m)\, S(m)\}^2 \geqslant Q_1 A_{14}^{-2} n^2 \log^{-4-\frac{8}{3}\eta} n,$$

so that, by (47) and (43),

$$Q_1 \leqslant A_{13} A_{14}^2 n \log^{4-\frac{1}{3}\eta} n. \tag{50}$$

Now it is obvious that $Q(n) \leqslant Q_1 + 2n \log^{-\frac{1}{3}\eta} n$. Hence, by (50),

$$Q(n) \leqslant A_{15} n \log^{4-\frac{1}{3}\eta} n,$$

which proves the result stated in the introduction.

References.

1. G. H. Hardy and J. E. Littlewood, "Some problems of partitio numerorum (V): a further contribution to the study of Goldbach's problem", *Proc. London Math. Soc.* (2), 22 (1924), 46–56.
2. I. M. Vinogradow, "Representation of an odd number as a sum of three primes", *Comptes rendus (Doklady) de l'Académie des Sciences de l'U.R.S.S.*, 15 (1937), 169–172.
3. I. Vinogradow, "Some theorems concerning the theory of primes", *Recueil Mathématique* T. 2 (44), 2 (1937), 179–195.
4. T. Estermann, "A new result in the additive prime-number theory", *Quart. J. of Math.*, 8 (1937), 32–38.
5. E. Landau, *Vorlesungen über Zahlentheorie*, I (Leipzig, 1927).

University College,
London.

A LOCAL PROPERTY OF THE ALLIED SERIES OF A FOURIER SERIES

By S. P. BHATNAGAR.

[Received 6 September, 1937.—Read 18 November, 1937.]

Let a periodic function $f(t)$ of period 2π be integrable L in the interval $(-\pi, \pi)$ and let its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)=\tfrac{1}{2}A_0+\sum_{n=1}^{\infty}A_n(t).$$

Then the allied series is

$$\sum_{n=1}^{\infty}(b_n\cos nt-a_n\sin nt)=\sum_{n=1}^{\infty}B_n(t).$$

We know that whether the series $\Sigma B_n=\Sigma B_n(x)$ is summable (C, α) or not, where $\alpha\geqslant 0$, depends only upon the properties of $f(t)$ near the point $t=x$. But when $\alpha<0$ this is not true. The only reason for the failure of the property in the latter case is that the Lebesgue integrability of the function does not imply anything more than $B_n=o(1)$. In fact, in this paper we show that, for the class of allied series for which $B_n=o(n^\gamma)$*, $-1<\gamma<0$, as $n\to\infty$, summability (C, γ) does depend only upon the local properties of the function. The corresponding problem for Fourier series has already been considered by Bosanquet and Offord†, and where the analysis in the two problems is similar details may accordingly be omitted.

We first give some results in the form of lemmas. In Lemma 4 we prove that a necessary condition for the summability of the allied series to the sum S by a Cesàro mean of negative order is that

$$\chi(t)=\frac{1}{\pi}\int_t^{\pi}\psi(u)\cot\tfrac{1}{2}u\,du\to S \qquad (1)$$

as $t\to+0$, where $\psi(t)=\frac{1}{2}\{f(x+t)-f(x-t)\}$.

* This is a necessary condition for summability (C, γ).

† Bosanquet and Offord, 2. This paper will be referred to as B.O.

There will thus be no loss of generality if we assume that condition (1) is satisfied when we obtain conditions for such summability. We also remark in Lemma 6 that whenever condition (1) is satisfied we have at the same time

$$\psi_1(t) = \frac{1}{t}\int_0^t \psi(u)\,du = o(1) \tag{2}$$

as $t \to +0$.

LEMMA 1. *If ΣB_n is summable $(C, -a)$ to the sum S, where $0 < a < 1$, then $\psi_1(t) = o(1)$ as $t \to +0$.*

We have to show that

$$\psi_1(t) = \sum_{n=1}^{\infty} B_n \frac{1-\cos nt}{nt} \to 0 \quad \text{as} \quad t \to +0.$$

The proof is similar to one given by Hardy and Littlewood, and the reader may be referred to their paper*.

We now write†, for $t > 0$,

$$\chi_1(t) = \frac{1}{t}\int_0^t \chi(u)\,du.$$

LEMMA 2‡. *If ΣB_n converges to the sum S, then $\chi_1(t) \to S$ as $t \to +0$.*

LEMMA 3.

$$\chi_1(t) - \chi(t) - \frac{2}{\pi}\psi_1(t) = o(1) \quad \text{as} \quad t \to +0.$$

This follows from the identity§

$$\chi_1(t) - \chi(t) - \frac{2}{\pi}\psi_1(t) = \frac{2}{\pi}\,\frac{1}{t}\int_0^t (\tfrac{1}{2}u\cot\tfrac{1}{2}u - 1)\psi(u)\,du.$$

LEMMA 4‖. *If ΣB_n is summable $(C, -a)$ to the sum S, where $0 < a < 1$, then $\chi(t) \to S$ as $t \to +0$.*

---

* Hardy and Littlewood, **5**, Theorem 2. Here the function $(1-\cos nt)/nt$ plays the rôle of $\sin nt/nt$ in their theorem.

† $\chi(t)$ is integrable L. Hardy, **4**.

‡ Hardy and Littlewood, **7**. Paley, **8**.

§ Since $t\cot t - 1 = O(t^2)$ as $t \to +0$ and $|\psi(t)|$ is integrable L. See Bosanquet, **1**, 21. Paley, **8**.

‖ $\chi(t) \to S$ as $t \to +0$ is not a consequence of mere convergence. Paley, **8**.

This follows from Lemmas 1, 2, and 3.

LEMMA 5. *If* $\chi(t) \to S$ *as* $t \to +0$, *then* $\chi_1(t) \to S$ *as* $t \to +0$.

This is a particular case of the consistency theorem for Cesàro limits.

LEMMA 6*. *If* $\chi(t) \to S$ *as* $t \to +0$, *then* $\psi_1(t) = o(1)$ *as* $t \to +0$.

This follows from Lemmas 5 and 3.

LEMMA 7†. *Suppose that* $0 < \alpha < 1$, *that*

$$\kappa^{-\alpha}(n, t) = \frac{2}{\pi} \frac{1}{A_n^{-\alpha}} \sum_{\nu=1}^{n} A_{n-\nu}^{-\alpha} \sin \nu t$$

and that

$$K^{-\alpha}(n, t) = -\frac{2}{\pi} \frac{1}{A_n^{-\alpha}} \frac{\cos(n\,;\, t)}{(2 \sin \frac{1}{2}t)^{1-\alpha}}, \tag{3}$$

where

$$A_n^{\sigma} = \frac{\Gamma(n+1+\sigma)}{\Gamma(n+1)\,\Gamma(1+\sigma)},$$

and $(n\,;\, t)$ *is written for*

$$(n + \tfrac{1}{2} - \tfrac{1}{2}\alpha)t + \tfrac{1}{2}\alpha\pi.$$

Then, for $0 < t < \pi$, *we have*

$$|\kappa^{-\alpha}(n, t)| < An, \tag{4}$$

$$\left| \frac{d}{dt} \kappa^{-\alpha}(n, t) \right| < An^2, \tag{5}$$

$$\left| \frac{d^2}{dt^2} \kappa^{-\alpha}(n, t) \right| < An^3, \tag{6}$$

$$\left| \kappa^{-\alpha}(n, t) - K^{-\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t \right| < An^{-1}t^{-2}, \tag{7}$$

$$\left| \frac{d}{dt} \left\{ \kappa^{-\alpha}(n, t) - K^{-\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t \right\} \right| < An^{-1}t^{-3}, \tag{8}$$

where the A's are independent of n and t.

---

* See also Paley, 8.

† Compare Gergen, 3. Zygmund, 10, 48.

The first three inequalities are trivial and the proof is immediate. For the proof of (7) and (8) reference may be made to Gergen's paper*.

THEOREM 1. *Necessary and sufficient conditions for the allied series of the Fourier series of $f(t)$ to be summable $(C, -\alpha)$, $0<\alpha<1$, to sum S for $t=x$ are that* (i) $B_n = o(n^{-\alpha})$ *as* $n\to\infty$, (ii) $\chi(t)\to S$ *as* $t\to+0$, *and* (iii)

$$n^{\alpha}\int_0^{\delta}\psi(t)\left\{1-\left(\frac{t}{\delta}\right)^{\beta}\right\}\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt = o(1) \tag{9}$$

as $n\to\infty$, *where* $0<\delta<\pi$ *but* δ *is otherwise arbitrary and* $\beta\geqslant 1$.

Alternatively, (iii) *can be replaced by* (iii)′

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{1/n}^{\delta}\psi(t)\left\{1-\left(\frac{t}{\delta}\right)^{\beta}\right\}\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt\right| = 0, \tag{10}$$

and (iii)′ *can again be replaced by* (iii)″

$$\lim_{\delta\to 0}\overline{\lim_{\tau\to\infty}}\,\overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{\tau/n}^{\delta}\psi(t)\left\{1-\left(\frac{t}{\delta}\right)^{\beta}\right\}\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt\right| = 0. \tag{11}$$

Proof. Since conditions (i) and (ii) are necessary for summability $(C, -\alpha)$ to S, we shall assume them to be satisfied.

We know that

$$\sigma_n^{-\alpha}(x) = \int_0^{\pi}\psi(t)\kappa^{-\alpha}(n, t)\,dt,$$

where $\sigma_n^{-\alpha}(x)$ is the n-th Cesàro mean of order $-\alpha$ of the allied series at the point $t=x$.

We write

$$\sigma_n^{-\alpha}(x) = \int_0^{1/n}\psi(t)\kappa^{-\alpha}(n, t)\,dt+\int_{1/n}^{\pi}\psi(t)\kappa^{-\alpha}(n, t)\,dt.$$

By (2), (4) and (5) the first integral, on integration by parts, is seen to be $o(1)$, and so

$$\sigma_n^{-\alpha}(x)-\chi\left(\frac{1}{n}\right) = \int_{1/n}^{\pi}\psi(t)\left\{\kappa^{-\alpha}(n, t)-\frac{1}{\pi}\cot\tfrac{1}{2}t\right\}dt+o(1).$$

* When we take the imaginary part of $\sum_{\nu=0}^{n} A_{\nu}^{(\rho)}e^{i\nu t}$ instead of the real part, we get $-\cos(n; t)$ instead of Gergen's $\sin(n; t)$, and, corresponding to Gergen's $V_n^{(\rho)}(t)$, we have

$$A_n^{(\rho)}\sin t-\sum_{\nu=1}^{\infty}A_{n+\nu+1}^{(\rho-2)}\sin\nu t.$$

The integral on the right-hand side can be replaced by

$$\int_{1/n}^{\pi} \psi(t)\, K^{-\alpha}(n, t)\, dt.$$

To prove this, writing $\Psi(t) = \int_0^t \psi(u)\, du$, we have

$$\int_{1/n}^{\pi} \psi(t) \left\{\kappa^{-\alpha}(n, t) - K^{-\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t\right\} dt$$

$$= \left[\Psi(t) \left\{\kappa^{-\alpha}(n, t) - K^{-\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t\right\}\right]_{1/n}^{\pi}$$

$$- \int_{1/n}^{\pi} \Psi(t)\, \frac{d}{dt} \left\{\kappa^{-\alpha}(n, t) - K^{-\alpha}(n, t) - \frac{1}{\pi} \cot \tfrac{1}{2}t\right\} dt.$$

By (2) and (7), the integrated term is $o(1)$, and by (2) and (8), the integral on the right-hand side is

$$\int_{1/n}^{\pi} o(t)\, O(n^{-1}t^{-3})\, dt = o(1).$$

We thus find that necessary and sufficient conditions for summability $(C, -\alpha)$ to S are that (i) $B_n = o(n^{-\alpha})$, (ii) $\chi(t) \to S$ as $t \to +0$ and (iii)

$$\int_{1/n}^{\pi} \psi(t)\, K^{-\alpha}(n, t)\, dt = o(1). \tag{12}$$

To show now that (12) can be replaced by

$$\int_0^{\pi} \psi(t)\, K^{-\alpha}(n, t)\, dt = o(1) \tag{13}$$

we write

$$\int_0^{1/n} \psi(t)\, K^{-\alpha}(n, t)\, dt = \left[\Psi(t)\, K^{-\alpha}(n, t)\right]_0^{1/n} - \int_0^{1/n} \Psi(t)\, \frac{d}{dt}\, K^{-\alpha}(n, t)\, dt.$$

By (2) and (3), the integrated term is $o(1)$, and

$$\int_0^{1/n} \Psi(t)\, \frac{d}{dt}\, K^{-\alpha}(n, t)\, dt = \int_0^{1/n} o(t) \{O(n^{\alpha}t^{\alpha-2}) + O(n^{\alpha+1}t^{\alpha-1})\}\, dt$$

$$= o(1).$$

We then show that (13) may be replaced by

$$n^{\alpha}\int_0^{\pi}\psi(t)\,\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt=o(1), \tag{14}$$

and that, if $0<\delta<\pi$, $\beta\geqslant 1$

$$n^{\alpha}\int_0^{\delta}\psi(t)\left(\frac{t}{\delta}\right)^{\beta}\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt+n^{\alpha}\int_{\delta}^{\pi}\psi(t)\,\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt=o(1) \tag{15}$$

as $n\to\infty$.

The proofs of (14) and (15) are similar to the proofs of (12) and (15) in B.O. On combining (14) and (15) we obtain (9).

Finally to obtain (10) and (11) we use (2) and proceed as in B.O.

THEOREM 2. (i) *If*

$$\lim_{\delta\to 0}\overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{1/n}^{\delta}\psi(t)\,\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt\right|=0, \tag{16}$$

then (10) *is true in the case* $\beta=1$.

(ii) *If*

$$\lim_{\delta\to 0}\overline{\lim_{\tau\to\infty}}\,\overline{\lim_{n\to\infty}}\left|n^{\alpha}\int_{\tau/n}^{\delta}\psi(t)\,\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt\right|=0 \tag{17}$$

then (11) *is true in the case* $\beta=1$.

To prove this theorem we proceed as in B.O.

THEOREM 3*. *If* $0<\alpha<1$, $\eta>0$, *and*

(i) $B_n=o(n^{-\alpha})$ *and* (ii)

$$\int_0^{t}|d\{u\psi(u)\}|=O(t) \tag{18}$$

in the interval $(0,\eta)$, *then* ΣB_n *is either summable* $(C,-\alpha)$ *or not summable* (A). *A necessary and sufficient condition for summability to* S *is that* $\chi(t)\to S$ *as* $t\to+0$.

Proof. First suppose that $\chi(t)\to S$ as $t\to+0$. Then by Theorems 1 and 2, summability $(C,-\alpha)$ will be established if we show that (17) is a consequence of (18).

---

* Compare Hardy and Littlewood, 5, Theorem 1. This is, in fact, an extension of the analogue for the allied series of Hardy and Littlewood's result.

We suppose that $0<\delta<\eta$, and write

$$J=\int_{\tau/n}^{\delta}\psi(t)\,\frac{\cos(n\,;\,t)}{t^{1-\alpha}}\,dt$$

$$=\left[\frac{\psi(t)}{t^{1-\alpha}}\,\frac{\sin(n\,;\,t)}{n+\frac{1}{2}-\frac{1}{2}\alpha}\right]_{\tau/n}^{\delta}-\frac{1}{n+\frac{1}{2}-\frac{1}{2}\alpha}\int_{\tau/n}^{\delta}\sin(n\,;\,t)\,d\left\{\frac{\psi(t)}{t^{1-\alpha}}\right\}.$$

It follows from (18) that $\psi(t)=O(1)$ in $(0,\,\eta)$, and so the integrated term is

$$O(\delta^{\alpha-1}n^{-1})+O[(\tau/n)^{\alpha-1}n^{-1}].$$

Again

$$\frac{1}{n}\int_{\tau/n}^{\delta}\sin(n\,;\,t)\,d\left\{\frac{t\psi(t)}{t^{2-\alpha}}\right\}$$

$$=\frac{1}{n}\int_{\tau/n}^{\delta}\sin(n\,;\,t)\,t^{\alpha-2}\,d\{t\psi(t)\}+\frac{\alpha-2}{n}\int_{\tau/n}^{\delta}\sin(n\,;\,t)\,t^{\alpha-2}\psi(t)\,dt$$

$$=J_1+J_2,$$

where

$$|J_1|\leqslant\frac{1}{n}\int_{\tau/n}^{\delta}t^{\alpha-2}|\,d\{t\psi(t)\}|$$

$$=\frac{1}{n}\left[t^{\alpha-2}O(t)\right]_{\tau/n}^{\delta}+\frac{2-\alpha}{n}\int_{\tau/n}^{\delta}t^{\alpha-3}O(t)\,dt$$

$$=O(\delta^{\alpha-1}n^{-1})+O\{(\tau/n)^{\alpha-1}n^{-1}\},$$

by (18), and

$$|J_2|\leqslant\frac{2-\alpha}{n}\int_{\tau/n}^{\delta}t^{\alpha-2}|\psi(t)|\,dt=O(\delta^{\alpha-1}n^{-1})+O\{(\tau/n)^{\alpha-1}n^{-1}\},$$

since $\psi(t)=O(1)$.

It follows that $n^{\alpha}|J|$ is annihilated by the operation $\lim_{\tau\to\infty}\overline{\lim_{n\to\infty}}$, and therefore (17) is satisfied. This proves the sufficiency part of the theorem.

Finally it is known* that if $\psi(t)=O(1)$ in $(0,\,\eta)$, and if ΣB_n is summable (A) to S then $\chi(t)\to S$ as $t\to+0$.

This completes the proof of the theorem.

I wish to express my best thanks to Dr. L. S. Bosanquet for his constant help and advice in my work.

---

* Hardy and Littlewood, **6**. Prasad, **9**. The fact that Prasad takes $\eta=\pi$ is immaterial.

References.

1. L. S. Bosanquet, " On the Cesàro summation of Fourier series and allied series ", *Proc. London Math. Soc.* (2), 37 (1934), 17–32.
2. L. S. Bosanquet and A. C. Offord, " A local property of Fourier series ", *Proc. London Math. Soc.* (2), 40 (1936), 273–280.
3. J. J. Gergen, " Convergence and summability criteria for Fourier series ", *Quart. J. of Math.* (*Oxford series*), 1 (1930), 252–275.
4. G. H. Hardy, " Notes on some points in the integral calculus (LXVI); The arithmetic mean of a Fourier constant ", *Messenger of Math.*, 58 (1928), 50–52.
5. G. H. Hardy and J. E. Littlewood, " Notes on the theory of series (VII); On Young's convergence criteria for Fourier series ", *Proc. London Math. Soc.* (2), 28 (1928), 301–311.
6. ———, " Notes on the theory of series (XV); On the series conjugate to the Fourier series of a bounded function ", *Journal London Math. Soc.*, 6 (1931), 278–281.
7. ———, " The allied series of a Fourier series ", *Proc. London Math. Soc.* (2), 24 (1925), 211–246.
8. R. E. A. C. Paley, " On the Cesàro summability of Fourier series and allied series ". *Proc. Cambridge Phil. Soc.*, 26 (1930), 173–203.
9. B. N. Prasad, " A theorem on the Cesàro summability of the allied series of a Fourier series ", *Journal London Math. Soc.*, 6 (1931), 274–278.
10. A Zygmund, *Trigonometrical series* (1935).

University College,
London.

THE CONTINUED FRACTIONS ASSOCIATED WITH CHAINS OF QUADRATIC FORMS

By A. OPPENHEIM.

[Received 2 November, 1937.—Read 18 November, 1937.]

1. Let (g_i) be a set of positive integers $(i = \ldots, -1, 0, 1, \ldots)$, and let F_i, H_i denote the simple continued fractions

$$F_i = [g_i, g_{i+1}, \ldots], \quad H_i = [g_{i-1}, g_{i-2}, \ldots]. \tag{1}$$

For any sequence f_i associated with F_i, H_i let f^* denote the upper bound. We may ask to determine the lower bounds of f^* for all sets (g_i). Naturally some restriction on f_i is essential.

A set (g_i) gives rise to a chain of equivalent reduced indefinite quadratic forms

$$\Phi_i(x, y) = (-1)^i A_i x^2 + B_i xy + (-1)^{i+1} A_{i+1} y^2 \tag{2}$$

of discriminant

$$D^2 = B_i^2 + 4A_i A_{i+1} > 0 \quad (D > 0), \tag{3}$$

or to its opposite chain. Any indefinite form

$$\phi(x, y) = ax^2 + bxy + cy^2 \quad (b^2 - 4ac = D^2) \tag{4}$$

is equivalent to any member of a chain of such reduced forms. A reduced form equivalent to ϕ belongs to this chain. (For the properties of reduced forms reference may be made to **1**.)

The roots of $\phi_i(1, t) = 0$ are $(-1)^i F_i$, $(-1)^{i+1}/H_i$ and we have

$$\frac{F_i}{A_i} = \frac{F_i H_i - 1}{B_i} = \frac{H_i}{A_{i+1}} = \frac{F_i H_i + 1}{D} = \frac{2}{D - B_i}, \tag{5}$$

so that any property of $f_i = f(F_i, H_i)$ will give rise to a property of the coefficients A_i, B_i, A_{i+1}.

The classical example is Markoff's

$$K_i = F_i + 1/H_i = D/A_{i+1}. \tag{6}$$

Markoff (see *e.g.* **2**) determined all sets (g_i) for which $K_i \leqslant 3$ and hence all forms ϕ of discriminant D^2 for which the lower bound of $|\phi|$ is at least $\frac{1}{3}D$, since the lower bound of $|\phi|$ is the lower bound of the numbers A_i.

In this paper I consider $f_i = f(F_i, H_i)$, where f is symmetric in F and H, and I obtain results applicable to many but not to all symmetric functions. The results are obtained by studying the special sequences

$$L_i = F_i + H_i, \quad M_i = F_i H_i, \quad N_i = L_i(1+1/M_i), \tag{7}$$

for which, if λ_i denotes max (L_i, L_{i+1}, L_{i+2}), and similarly for μ_i, ν_i, etc., I prove the following two theorems.

THEOREM 1. *For any* i,

$$\lambda_i \geqslant 2\alpha, \quad \mu_i \geqslant \alpha^2, \quad \nu_i \geqslant 4\alpha - 2, \tag{8}$$

where

$$\alpha^2 = \alpha + 1, \quad \alpha = \tfrac{1}{2}(1+\sqrt{5}), \quad \alpha = \overset{\times}{[1]} = [1_\infty]. \tag{9}$$

If equality holds in any case for some i, *then equality holds in each case for every* i *and the set* (g_i) *is the ubiquitous set* $\overset{\times}{(1)}$ *leading to the form* x^2+xy-y^2.

THEOREM 2. *Except for the sets below there is always at least one* i *such that simultaneously*

$$L_i > 2\alpha + 1, \quad M_i > 2\alpha + 1, \quad N_i > 2\alpha + 2. \tag{10}$$

The exceptional sets are

$$(1_\infty), \quad (\overset{\times}{2}, \overset{\times}{1}_{2n-1}) \ (n = 1, 2, \ldots), \quad (1_\infty, 2, 1_\infty). \tag{11}$$

For the set (1_∞),

$$L_i = 2\alpha, \quad M_i = \alpha^2, \quad N_i = 4\alpha - 2 \quad (\textit{for all } i). \tag{12}$$

For the set $(\overset{\times}{2}, \overset{\times}{1}_{2n-1})$,

$$L^* = C_{2n}, \quad M^* = D_{2n}, \quad N^* = C_{2n}(1+1/D_{2n}), \tag{13}$$

where

$$C_n = \chi(p_n/q_n), \quad \chi(x) = \{1+(1+x)^{\frac{1}{2}}\}(1+1/x), \tag{14}$$

$$D_n = (p_{n-1}^{\frac{1}{2}} + p_{n+1}^{\frac{1}{2}})^2/p_n, \tag{15}$$

and p_i/q_i is the i-th convergent to α. It is to be noted that

$$C_{2n} < 2\alpha+1, \quad D_{2n} < 2\alpha+1, \quad C_{2n}(1+1/D_{2n}) < 2\alpha+2. \tag{16}$$

For the set $(1_\infty, 2, 1_\infty)$,

$$L^* = 2\alpha+1, \quad M^* = 2\alpha+1, \quad N^* = 2\alpha+2, \tag{17}$$

equality occurring for just two suffixes.

The forms corresponding to these sets are multiples of

$$\phi_0 = x^2+xy-y^2, \quad \phi_\infty = x^2+2xy-\alpha y^2, \tag{18}$$

$$\phi_{2n} = x^2 p_{2n-1}+2xyp_{2n-1}-y^2 p_{2n} \quad (n = 1, 2, \ldots).$$

Except for ϕ_0 none of these forms is equivalent to its negative.

The precise lower bounds of L^*, M^*, N^* are $2\alpha+1$, $2\alpha+1$, $2\alpha+2$.

Suppose now that $f(F, H)$ is a symmetric function of F and H which can be written in the form $u(L, M, N)$, where $u(x, y, z)$ is defined in the domain

$$2\sqrt{y} \leqslant x < y+1, \quad y \geqslant 1, \quad x < z < 2x, \tag{19}$$

and is such that

$$u(x', y', z') > u(x, y, z) \tag{20}$$

whenever

$$x' > x, \quad y' > y, \quad z' > z. \tag{21}$$

We then have from Theorem 1

THEOREM 3. $\max(f_i, f_{i+1}, f_{i+2}) \geqslant u(2\alpha, \alpha^2, 4\alpha-2)$

for every i; and equality for any i implies $(g_i) = (1_\infty)$.

If in addition $u(x, y, z)$ is continuous at $x = 2\alpha+1$, $y = 2\alpha+1$, $z = 2\alpha+2$, then a simple argument leads from Theorem 2 to

THEOREM 4. *Except for the sets in Theorem 2 there is always a suffix i for which*

$$f_i > u(2\alpha+1, 2\alpha+1, 2\alpha+2) = f(\alpha+1, \alpha). \tag{22}$$

For the excepted sets, f^* has the respective values $u(L^*, M^*, N^*)$. The precise lower bound of f^* for all sets (g_i) is $f(\alpha+1, \alpha)$.

The conditions on $f(F, H) = u(L, M, N)$ given above are sufficient but not necessary for the truth of Theorem 4. Thus Theorem 4 holds also

for the sequence $L_i(1+M_i^{-\frac{1}{2}})$: its truth can be established by Theorem 2 but a more elaborate argument is needed.

There exist also symmetric increasing functions for which Theorem 4 is false. An example which I hope to discuss in another paper is given by

$$f_i = (F_i-1)(H_i-1). \tag{23}$$

Here it is still true that $\underline{\lim} f^* = f(\alpha+1, \alpha) = 1$, but we now get additional sets which are to be excepted, *e.g.*

$$(\overset{\times}{1}, 2, 1, 3, 1, 2, 1, 3, 1, \overset{\times}{3}), \quad (\overset{\times}{2}, 1_7, 2, 1_7, 2, \overset{\times}{1_5}), \quad (\ldots, 1, 4, 1, 4, 1, 5, 1, 4, 1, 4, \ldots).$$

I have determined all the exceptional sets, the discussion required being very similar to Markoff's for $K_i = F_i + 1/H_i$.

A particularly interesting case of Theorems 3 and 4 is afforded by

$$B_i/D = (F_iH_i-1)/(F_iH_i+1) = (M_i-1)/(M_i+1), \tag{24}$$

so that, *for any* i,

$$\max(B_i, B_{i+1}, B_{i+2}) \geqslant D/\sqrt{5}, \tag{25}$$

and equality for any i *implies* $\phi(x, y) \sim x^2+xy-y^2$. *Except for the forms or their negatives in Theorem* 2, *there is always a reduced form in a chain with the property*

$$B_i > \tfrac{1}{2}(\sqrt{5}-1)\,D. \tag{26}$$

These inequalities are quite useful when computing the integral reduced forms for a given discriminant or the chains of reduced ideals in a real quadratic field.

An equally interesting case is afforded by $M_i/(M_i+1)$ which is intimately connected with a geometrical problem in a lattice plane considered by Szekeres (**3**) and arising out of a more general problem due to Mordell (**4**).

Given a pair of lines $\phi(x, y) = 0$ we can construct parallelograms with centre O and sides parallel to these lines. Let P denote any such parallelogram with the further property: P has no interior lattice points except O; each pair of its sides contains at least one inner pair of lattice points.

Let $E(\phi)$ denote the set of such parallelograms P, Δ the area of P, Δ^* the upper bound of Δ. Mordell showed that $\Delta^* \geqslant 2$, Szekeres improved this to $\Delta^* \geqslant 2(1+1/\sqrt{5})$, and added a proof by Chao Ko that this is the best possible result.

Now Szekeres shows that by a transformation T of determinant unity,

$$x' = rx+sy, \quad y' = tx+uy \quad (ru-st = 1), \tag{27}$$

any P in $E(\phi)$ can be transformed into a parallelogram P'

$$|x'+ay'| \leqslant 1, \quad |bx'+y'| \leqslant 1, \tag{28}$$

with

$$|a| < 1, \quad |b| < 1, \quad ab \leqslant 0. \tag{29}$$

Naturally

$$4\Delta^{-1} = 1-ab. \tag{30}$$

Clearly T carries $\phi(x, y)$ into $C(x'+ay')(bx'+y') = \psi$ say, where C is a constant. But the conditions (29) show that ψ is reduced and hence $\psi(x, y) = \Phi_i(x, y)$ for some i, where (Φ_i) is the chain of equivalent reduced forms determined by $\phi(x, y)$. It follows that a, b are $\pm 1/F_i$, $\mp 1/H_i$, so that (30) gives

$$\Delta = 4M_i/(M_i+1). \tag{31}$$

Conversely, to each Φ_i of the chain corresponds a parallelogram P in $E(\phi)$ with area given by (31).

If we suppose that the roots of $\phi(t, 1) = 0$ are irrational (the case when either root is rational is trivial here) then the set (g_i) extends to infinity in both directions. Numbering the Δ's by (31) we deduce from Theorems 3 and 4

THEOREM 5. *Except for the quadratic forms $\phi(x, y)$ given below, there is always a P in $E(\phi)$ such that*

$$\Delta > 1+\sqrt{5}.$$

If $\phi \sim \phi_0$, then every $\Delta = \Delta^* = 2(1+1/\sqrt{5})$;

if $\phi \sim \pm\phi_{2n}$, then $\Delta^* = 4D_{2n}/(D_{2n}+1)$;

if $\phi \sim \pm\phi_\infty$, then $\Delta^* = 1+\sqrt{5}$.

For every i and any form $\phi(x, y)$

$$\max(\Delta_i, \Delta_{i+1}, \Delta_{i+2}) \geqslant 2(1+1/\sqrt{5}),$$

and equality for some i implies that $\phi \sim \phi_0$, $\Delta_i = 2(1+1/\sqrt{5})$ always.

The precise lower bound of Δ^* is $1+\sqrt{5}$.

It may be noted that $N_i = D(1/A_i+1/A_{i+1})$, so that we can write down a theorem for $1/A_i+1/A_{i+1}$.

2. The proof of Theorems 1 and 2 is quite elementary. The essential point is this: in the continued fraction

$$x = [a_1, a_2, \ldots, a_n] \quad (a_i > 0),$$

we increase x by decreasing a_{2i} or by increasing a_{2i+1}. The odd convergents increase towards x; the even convergents decrease towards x.

3. *Proof of Theorem* 1. It is sufficient to consider μ_i only, since

$$L_i \geqslant 2M_i^{\frac{1}{2}}, \quad N_i \geqslant 2M_i^{\frac{1}{2}} + 2M_i^{-\frac{1}{2}}, \tag{1}$$

equality occurring only for $F_i = H_i$, and $M_i^{\frac{1}{2}} + M_i^{-\frac{1}{2}}$ is an increasing function of M_i.

LEMMA. *If* $\mu = \mu_i = \max(M_i, M_{i+1}, M_{i+2})$, *then*

$$(\mu-1)^2 \geqslant \mu g_i g_{i+1}, \quad (\mu-1)^2 \geqslant \mu g_{i+1} g_{i+2}. \tag{2}$$

It is enough to prove this when $i = 0$. Drop the suffix zero. Then

$$F = [g, F_1], \quad H_1 = [g, H], \quad FH \leqslant \mu, \quad F_1 H_1 \leqslant \mu, \tag{3}$$

so that

$$g + 1/H \leqslant \mu(F-g), \tag{4}$$

since $F > g$. From (4) and $FH \leqslant \mu$, elimination of F results in

$$Hg \leqslant \mu - 1, \tag{5}$$

$$\mu g \leqslant (\mu-1) F, \quad [(4), (5)]. \tag{6}$$

$$\mu g \leqslant (\mu-1) H_1, \quad [(3), (5)]. \tag{7}$$

From (6), applied to $F_1 H_1$, $F_2 H_2$,

$$\mu g_1 \leqslant (\mu-1) F_1, \tag{8}$$

and so, since

$$F_1 H_1 \leqslant \mu,$$

$$(\mu-1)^2 \geqslant \mu g_0 g_1, \tag{9}$$

which proves the first part of the Lemma for $i = 0$. Also *equality in* (9) *implies equality in* (3)–(8). The second part of the lemma follows likewise.

A fortiori the Lemma shows that

$$(\mu-1)^2 \geqslant \mu, \quad \mu \geqslant \tfrac{1}{2}(3+\sqrt{5}) = \alpha^2 = \alpha+1, \tag{10}$$

which proves Theorem 1 for μ_i and so by (1) for λ_i, ν_i. Also equality entails $g_i = g_{i+1} = 1$, $H_i = \mu - 1 = \alpha = [\overset{\times}{1}]$, $F_i = \mu/(\mu-1) = \alpha = [\overset{\times}{1}]$, $F_j H_j = \alpha^2$ (for all j).

Theorem 3 is, of course, an obvious deduction from Theorem 1. But it should be noted that Theorem 3 also holds whenever

$$f(F_i, H_i) \geqslant v(M_i), \tag{11}$$

where $v(M)$ is a steadily increasing function of M, and equality occurs when and only when $F_i = H_i$. Thus Theorem 3 holds for F^2+H^2, F^3+H^3, etc.

4. *Proof of Theorem 2 for the excepted sets*

$$(1_\infty), \quad (\overset{\times}{2}, \overset{\times}{1}_{2n-1}), \quad (1_\infty, 2, 1_\infty). \tag{1}$$

For the set (1_∞) the results are obvious. For the set $(1_\infty, 2, 1_\infty)$, if the 2 is called g_0, then

$$F_0 = 1+\alpha = H_1, \; H_0 = \alpha = F_1, \tag{2}$$

$$L_0 = L_1 = 2\alpha+1, \; M_0 = M_1 = 2\alpha+1, \; N_0 = N_1 = 2\alpha+2.$$

For $i \geqslant 2$ we have also

$$F_i = [1_\infty] = \alpha, \quad H_i = [1_{i-1}, 2, 1_\infty] < 2, \tag{3}$$

$$L_i < \alpha+2 < 2\alpha+1, \quad M_i < 2\alpha < 2\alpha+1,$$

$$N_i < \alpha+2+1/\alpha+\tfrac{1}{2} = 2\alpha+\tfrac{3}{2} < 2\alpha+2,$$

since N_i is an increasing function of F_i and of H_i.

Similarly if $i \leqslant -1$.

Consider now the set $(\overset{\times}{2}, \overset{\times}{1}_{2n-1})$, 2 being g_0. Here calculation gives

$$L_0 = L_1 = \chi(p_{2n}/q_{2n}) = C_{2n}, \tag{4}$$

where

$$\chi(x) = \{1+(1+x)^{\frac{1}{2}}\}(1+1/x), \tag{5}$$

$$M_0 = M_1 = \{(q_{2n}/p_{2n})^{\frac{1}{2}} + (p_{2n+1}/q_{2n+1})^{\frac{1}{2}}\}^2 = D_{2n}, \tag{6}$$

and

$$N_0 = N_1 = 2(1+x)^{\frac{1}{2}}(1+1/x), \quad x = p_{2n}/q_{2n}, \tag{7}$$

where p_i/q_i is the i-th convergent to $\alpha = [\overset{\times}{1}]$.

Since $q_{2n}/p_{2n} < 1/\alpha$, $p_{2n+1}/q_{2n+1} < \alpha$, we see at once that

$$D_{2n} < \alpha^{-1}+2+\alpha = 2\alpha+1. \tag{8}$$

Next $2(1+x)^{\frac{1}{2}}(1+1/x)$ is a decreasing function of x if $x \leqslant 2$, and so, since $2 \geqslant x = p_{2n}/q_{2n} > \alpha$,

$$N_0 = N_1 < 2(1+\alpha)^{\frac{1}{2}}(1+1/\alpha) = 2\alpha+2.$$

Again $\chi(x)$ is a decreasing function of x if $x \leqslant 2$, and so

$$C_{2n} < \chi(\alpha) = \left(1+(1+\alpha)^{\frac{1}{2}}\right)(1+1/\alpha) = 2\alpha+1. \tag{9}$$

It remains to show, that for $g_i = 1$ preceded by $g_{i-1} = 1$,

$$L_i < C_{2n}, \quad M_i < D_{2n}, \quad N_i < C_{2n}(1+1/D_{2n}). \tag{10}$$

Here we may take

$$F_i = [1_r, F_0], \quad H_i = [1_s, F_0), \tag{11}$$

where

$$r \geqslant 1, \quad s \geqslant 1, \quad r+s = 2n-1 \quad \text{(so that } n \geqslant 2). \tag{12}$$

Clearly $F_i < F_0$, $H_i < F_0$. Next, if s is odd so that r is even,

$$H_0 = [1_{2n-1}, F_0] = [1_s, F_i] > [1_s, F_0] = H_i; \tag{13}$$

while if s is even so that r is odd

$$H_0 = [1_r, H_i] > [1_r, F_0] = F_i. \tag{14}$$

In any case, therefore,

$$L_i < L_0, \quad M_i < M_0, \quad N_i < N_0. \tag{15}$$

It should be noted that $C_{2n} > D_{2n}$ and that C_{2n}, D_{2n} tend to $2\alpha+1$ when n tends to infinity.

That part of Theorem 2 which relates to the exceptional sets is therefore proved.

5. *Proof that, the sets* (1_∞), $(\overset{\times}{2}, \overset{\times}{1}_{2n-1})$, $(1_\infty, 2, 1_\infty)$ *being excepted, there exists a suffix* i *such that*

$$L_i > 2\alpha+1, \quad M_i > 2\alpha+1, \quad N_i > 2\alpha+2.$$

Suppose the contrary, *i.e. that*

$$\min(L_i, M_i, N_i-1) \leqslant 2\alpha+1 \tag{R}$$

for every i.

A. *If* $g_i \geqslant 3$, *then both* $g_{i-1} = 1$, $g_{i+1} = 1$.

For, if $g_{i-1} \geqslant 2$, then

$$L_i = F_i + H_i > 3 + 2 > 2\alpha + 1, \quad M_i = F_i H_i > 2\alpha + 1,$$

$$N_i > 3 + 2 + \tfrac{1}{3} + \tfrac{1}{2} > 2\alpha + 2,$$

which contradicts (**R**). Similarly if $g_{i+1} \geqslant 2$.

B. *Every* $g_i \leqslant 3$. For, if $g_i \geqslant 4$ occurs, then by **A**

$$F_i = [g_i, 1, \ldots] > [4, 2] = 9/2, \quad H_i > 1,$$

$$L_i > 2\alpha + 1, \quad M_i > 2\alpha + 1, \quad N_i > 2\alpha + 2,$$

again contradicting (**R**).

C. *Every* $g_i \leqslant 2$. For, if $g_i > 2$ occurs, then $g_i = 3$ by **B** and so (**A**, **B**)

$$F_i = [3, 1, \ldots] > [3, 2] = 7/2,$$

$$H_i = [1, g_{i-2}, \ldots] > [1, 4] = 5/4,$$

$$L_i > 2\alpha + 1, \quad M_i > 2\alpha + 1, \quad N_i > 2\alpha + 2,$$

which contradicts (**R**).

D. *If* $g_i = 2$, *then* $g_{i-1} = 1$, $g_{i+1} = 1$. For suppose that $g_{i-1} = 2$. Then

$$F_i = [2, g_{i+1}, \ldots] > [2, 3] = 7/3; \quad H_i = [2, g_{i-2}, \ldots] > 7/3;$$

$$L_i > 2\alpha + 1, \quad M_i > 2\alpha + 1, \quad N_i > 2\alpha + 2,$$

contradicting (**R**). Similarly F_{i+1} and H_{i+1} dispose of $g_{i+1} = 2$. Thus **D** is proved.

By **A**–**D**, the sets (1_∞), $(1_\infty, 2, 1_\infty)$ being excluded, the set (g_i) must be of one of the types,

(1) $$\ldots, 2, 1_n, 2, 1_\infty; \quad 1_\infty, 2, 1_n, 2, \ldots;$$

or

(2) $$\ldots, 2, 1_{n_i}, 2, 1_{n_{i+1}}, 2, \ldots,$$

the n_i being integers greater than or equal to 1 and 2's occurring indefinitely both to right and to left.

E. *The situation* $\ldots, 2, 1_n, 2, 1_\infty$ *contradicts* (**R**). Call the last 2 g_0. If n is even, then

$$F_0 = [2, 1_\infty] = 1 + \alpha; \quad H_0 = [1_n, 2, \ldots] > [1_n, 1_\infty] = \alpha.$$

If n is odd, then

$$F_1 = [1_\infty] = a; \quad H_1 = [2, 1_n, 2, \ldots] > [2, 1_n, 1_\infty] = 1+a.$$

In either case (**R**) is contradicted. Thus **E** is proved. Similarly $1_\infty, 2, 1_n, 2, \ldots$ contradicts (**R**).

F. *The situation* $\ldots 2, 1_{2n-1}, 2, 1_{2m}, 2, \ldots$ *contradicts* (**R**). Let g_0 be the central 2. Then

$$F_1 = [1_{2m}, 2, \ldots] > [1_{2m}, 2] = [1_{2m+2}] > a,$$

$$H_1 = [2, 1_{2n-1}, 2, \ldots] > [2, 1_{2n-1}, 1_\infty] = 1+a,$$

contradicting (**R**).

Similarly $\ldots 2, 1_{2n}, 2, 1_{2m-1}, 2, \ldots$ contradicts (**R**).

From **F** it follows that *in the set* (2) *the* n_i *are all even or all odd.*

G. *The situation* $\ldots, 2, 1_{2n}, 2, 1_{2m}, 2, \ldots$ *implies* $m = n$ *if* (**R**) *is satisfied.*

Suppose that $n \leqslant m-1$. Then

$$F = [2, 1_{2m}, 2, \ldots] > [2, 1_{2m}] = 1+[1_{2m+1}] = 1+u, \tag{3}$$

$$H = [1_{2n}, 2, 1, \ldots] > [1_{2n}, 2] = [1_{2n+2}] \geqslant [1_{2m}] = u', \tag{4}$$

and so

$$L > 1+u+u' = 2+u'+1/u' > 2+a+1/a = 2a+1; \tag{5}$$

$$M > (1+u)u' = 2u'+1 > 2a+1; \tag{6}$$

$$N > 1+u+u'+1/(1+u)+1/u' = 2u^3/(u^2-1) > 2a+2, \tag{7}$$

since $u' > a > u > 1$ (and $a^2 < 3$). Thus (**R**) is contradicted. Thus we have $n \geqslant m$ and likewise $m \geqslant n$, so that $m = n$.

H. *The situation* $\overset{\times\times}{2}\,1_{2n}$ *contradicts* (**R**).

If $F = [\overset{\times}{2}, \overset{\times}{1}_{2n}]$, $H = [\overset{\times}{1}_{2n}, \overset{\times}{2}]$, straightforward calculation gives

$$L = \chi(p_{2n+1}/q_{2n+1}), \quad \chi(x) = \{1+(1+x)^{\frac{1}{2}}\}(1+1/x), \tag{8}$$

$$M = \{(q_{2n+1}/p_{2n+1})^{\frac{1}{2}} + (p_{2n+2}/q_{2n+2})^{\frac{1}{2}}\}^2, \tag{9}$$

$$N = 2(1+y)^{\frac{1}{2}}(1+1/y), \quad y = p_{2n+1}/q_{2n+1}, \tag{10}$$

whence, since $p_{2n+1}/q_{2n+1} < a < p_{2n}/q_{2n}$, it follows easily that

$$L > 2a+1, \quad M > 2a+1, \quad N > 2a+2, \tag{11}$$

which contradicts (**R**). Thus **H** is proved.

By **F**, **G**, **H** the only sets in (2) left for consideration are those in which the n_i are all *odd*. Since the sets $(\overset{\times}{2}, \overset{\times}{1}_{2n-1})$ have been excluded from this discussion it remains merely to settle the situation

(12) $$\ldots, 2, 1_{2n-1}, 2, 1_{2m-1}, 2, \ldots$$

with m and n *unequal*.

I. *In the situation* (12), $n \leqslant m-2$ *contradicts* (**R**); $m \leqslant n-2$ *contradicts* (**R**).

Consider $n \leqslant m-2$. Take here

(13) $$F = [1_{2m-1}, 2, \ldots] > [1_{2n+1}, 2],$$

(14) $$H = [2, 1_{2n-1}, 2, \ldots] > [2, 1_{2n-1}, 2] = 1+[1_{2n+2}] = 1+u,$$

so that

(15) $$L > [1, u]+1+u = 2+u+1/u > 2a+1,$$

(16) $$M > (1+1/u)(1+u) = 2+u+1/u > 2a+1,$$

(17) $$N > 1+1/u+1/(1+1/u)+1+u+1/(1+u) \geqslant 3+u+1/u > 2a+2,$$

since $u = [1_{2n+2}] > a$.

Thus $n \leqslant m-2$ contradicts (**R**). So likewise (or by reversing the set) does $m \leqslant n-2$ contradict (**R**).

J. *In the situation* (12), $n = m-1$ *contradicts* (**R**); *also* $m = n-1$ *contradicts* (**R**).

Consider $n = m-1$ and take

(18) $$F = [1_{2n+1}, 2, 1, \ldots] > [1_{2n+1}, 2, 1] = [1_{2n+1}, 3] = v,$$

(19) $$H = [2, 1_{2n-1}, 2, 1, H'] > [2, 1_{2n-1}, 2, 1, 1] = 1+[1_{2n+1}, \tfrac{2}{3}] = 1+v',$$

so that

(20) $$L > 1+v+v' = X, \quad M > v(1+v'),$$

(21) $$N > X+Y, \quad Y = 1/v+1/(1+v').$$

It is sufficient to show that

(22) $$X > 2a+1, \quad Y > 1, \quad v(1+v') > 2a+1.$$

But $Y>1$ if

$$1+v+v'>v(1+v'),$$

i.e. if $$v'(v-1)<1, \quad v'<1/(v-1),$$

i.e. if $$[1_{2n+1}, \tfrac{2}{3}]<[1_{2n}, 3], \ [1, \tfrac{2}{3}]<3,$$

which is true.

Next, let p, q, p', q' denote p_{2n+1}, ..., q_{2n}, so that $p/q<\alpha<p'/q'$, $p'=q'$. Then, for any t,

$$[1_{2n+1}, t]-\alpha=[1_{2n+1}, t]-[1_{2n+1}, \alpha]=\frac{\alpha-t}{(tq+q')(\alpha q+q')}, \tag{23}$$

and therefore, since $v=[1_{2n+1}, 3]$, $v'=[1_{2n+1}, \tfrac{2}{3}]$,

$$\begin{aligned} X-2\alpha-1&=(v'-\alpha)-(\alpha-v)\\ &=\frac{\alpha-\frac{2}{3}}{(\frac{2}{3}q+q')(\alpha q+q')}-\frac{3-\alpha}{(3q+q')(\alpha q+q')}, \end{aligned}$$

which is positive if

$$\frac{3q+q'}{2q+3q'}(3\alpha-2)-(3-\alpha)>0. \tag{24}$$

But $q/q'=p'/q'>\alpha$, so that (24) is true if

$$x=(3\alpha+1)(3\alpha-2)-(2\alpha+3)(3-\alpha)\geqslant 0, \tag{25}$$

which is obvious since $x=5\alpha$. Thus $X>2\alpha+1$ and so, since $Y>1$, $L>2\alpha+1$, $N>2\alpha+2$.

Finally

$$v(1+v')-2\alpha-1=v(v'-\alpha)-(1+\alpha)(\alpha-v), \tag{26}$$

which is positive if

$$\frac{3p+p'}{2q+3q'}(3\alpha-2)-(1+\alpha)(3-\alpha)>0. \tag{27}$$

But $2q+3q'=3p-p'$ and $p/p'=p/q<\alpha$, so that

$$\frac{3p+p'}{2q+3q'}=\frac{3p+p'}{3p-p'}>\frac{3\alpha+1}{3\alpha-1}. \tag{28}$$

Hence (27) holds if

$$y=(3\alpha-2)(3\alpha+1)-(1+\alpha)(3-\alpha)(3\alpha-1)\geqslant 0, \tag{29}$$

which is the case since $y = 6-2a > 0$. Hence $M > 2a+1$.

Thus $n = m-1$ contradicts (**R**). Likewise $m = n-1$ contradicts (**R**). Thus **J** is proved.

I and **J** together contradict the condition $m \neq n$ attached to the sets (12).

The proof of the statement at the beginning of this section is therefore completed. This ends the proof of Theorem 2.

Theorem 4, of course, is an immediate deduction from Theorem 2.

References.

1. L. E. Dickson, *Introduction to the theory of numbers* (Chicago, 1929).
2. L. E. Dickson, *Studies in the theory of numbers* (Chicago, 1930).
3. G. Szekeres, "On a problem of the lattice-plane", *Journal London Math. Soc.*, 12 (1937), 88–93.
4. L. J. Mordell, "Note on an arithmetical problem on linear forms", *ibid.*, 34–36.

Raffles College,
Singapore, S.S.

ON INFINITE SOLUBLE GROUPS (II)

By K. A. HIRSCH.

[Received 16 July, 1937.—Read 18 November, 1937.]

Introduction.

In a previous paper* we introduced a class of infinite soluble groups, which we called "S-groups", with the idea of extending some of the main elementary results of the theory of finite soluble groups to the infinite case. (The defining condition was simply that an infinite soluble group is an S-group, if and only if the Abelian factor-groups of the derived series all have finite numbers of generators.) In particular we were able to show that the "natural" extensions of the Jordan-Hölder and related uniqueness theorems hold for the infinite "constituents" of S-groups†.

To carry the analogy between finite and infinite soluble groups a step further we now turn to the most important subclass of finite soluble groups, viz. the p-groups (*i.e.* those whose order is a power of a prime p) and their direct products. These groups have, of course, many "arithmetical" properties, typically expressed in the "Anzahl" theorems (giving congruences for the number of subgroups of a specified kind). But apart from that they can be characterized, as Burnside was the first to observe‡, by purely group-theoretical means, *e.g.* by the fact that they, and among finite groups they alone, coincide with their hypercentral or possess a "central series". This conception of a central series can readily be adapted to the case of the S-groups and leads to an interesting subclass of S-groups which forms the subject of this paper. As is to be expected

---

* "On infinite soluble groups (I)", *Proc. London Math. Soc.* (2), 44 (1938), 53–60. This paper will be quoted in the following as I.S.G. (I). We assume the reader to be familiar with it.

† *Cf.* I.S.G. (I), Theorem 1.42.

‡ *Cf.* Burnside, *Theory of groups* (2nd ed., 1911), 166.

the bulk of the classical results in p-group theory can be extended to S-groups with a central series.

For commutator subgroups which will frequently occur in the following we use the notation introduced by Hall*. We write $(H, K) = H^{-1}K^{-1}HK$ for the commutator of H and K, and $(\mathfrak{H}, \mathfrak{K})$ for the group generated by all elements (H, K), where $H \in \mathfrak{H}$ and $K \in \mathfrak{K}$, $\mathfrak{H}$ and $\mathfrak{K}$ being subgroups of a group $\mathfrak{G}$. $(\mathfrak{H}, \mathfrak{K}) = 1$ means that every element of $\mathfrak{H}$ is permutable with every element of $\mathfrak{K}$.

2.1. In the definitions and theorems of this section the group $\mathfrak{G}$ may be any group whatsoever, not necessarily an S-group. The proofs are analogous to those in the case of finite groups†.

Definition. A group $\mathfrak{G}$ is said to possess a "central series" of length m, if there exists in $\mathfrak{G}$ a monotonic sequence of self-conjugate subgroups

$$1 = \mathfrak{K}_0 \subseteqq \mathfrak{K}_1 \subseteqq \ldots \subseteqq \mathfrak{K}_m = \mathfrak{G} \tag{2.11}$$

leading from the 1-element to $\mathfrak{G}$ in a finite number of steps‡, with the property that

$$(\mathfrak{K}_i, \mathfrak{G}) \subseteqq \mathfrak{K}_{i-1}.$$

In words, the commutator of any element of $\mathfrak{K}_i$ with any element of $\mathfrak{G}$ belongs to $\mathfrak{K}_{i-1}$, or again $\mathfrak{K}_i/\mathfrak{K}_{i-1}$ belongs to the central of $\mathfrak{G}/\mathfrak{K}_{i-1}$.

If $\mathfrak{G}$ has a central series and $\mathfrak{H}$ is any subgroup of $\mathfrak{G}$, then the meets of $\mathfrak{H}$ with the terms of (2.11) give a central series for $\mathfrak{H}$, and, if $\mathfrak{H}$ is self-conjugate in $\mathfrak{G}$, then the joins of $\mathfrak{H}$ with the terms of (2.11) give rise to a central series for $\mathfrak{G}/\mathfrak{H}$. Thus the system of groups with a central series is closed with respect to subgroups and factor-groups.

The following two central series are of the greatest importance:

2.12. (i) Let $\mathfrak{Z}_0 = 1$, $\mathfrak{Z}_1$ be the central of $\mathfrak{G}$, and let $\mathfrak{Z}_2$, $\mathfrak{Z}_3$, ... be defined by recurrence so that $\mathfrak{Z}_i/\mathfrak{Z}_{i-1} = \mathfrak{Z}_1(\mathfrak{G}/\mathfrak{Z}_{i-1})$, the central of $\mathfrak{G}/\mathfrak{Z}_{i-1}$. The series

$$1 = \mathfrak{Z}_0 \subset \mathfrak{Z}_1 \subset \ldots \subset \mathfrak{Z}_c = \mathfrak{G}$$

is called the "upper central series" of $\mathfrak{G}$. It can be shown that, for any central series (2.11) of $\mathfrak{G}$, $\mathfrak{K}_i \subseteqq \mathfrak{Z}_i$.

* P. Hall, *Proc. London Math. Soc.* (2), 36 (1932), 42–45.

† *Cf.* Hall, *loc. cit.*, 49–50.

‡ For S-groups the maximal condition [*cf.* I.S.G. (I), Theorem 1.3] ensures the finiteness of any ascending series of subgroups.

2.13. (ii) Let $\mathfrak{H}_1 = \mathfrak{G}$, $\mathfrak{H}_2 = (\mathfrak{G}, \mathfrak{G})$, the derived group of $\mathfrak{G}$, and by recurrence $\mathfrak{H}_{i+1} = (\mathfrak{H}_i, \mathfrak{G})$. It can be shown that $\mathfrak{H}_i \supset \mathfrak{H}_{i+1}$, that the series $\mathfrak{G} = \mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \ldots$ terminates in the 1-element, and that it is a central series, the "lower central series" of $\mathfrak{G}$. For any central series (2.11) of $\mathfrak{G}$ we have $\mathfrak{H}_i \subseteqq \mathfrak{K}_{m+1-i}$. In particular $\mathfrak{H}_i$ is contained in $\mathfrak{Z}_{c-i+1}$, but not in $\mathfrak{Z}_{c-i}$.

The two central series have, therefore, the same length c which is called the "class" of $\mathfrak{G}$.

2.2. In this section we are concerned with the elements of finite order. In an S-group with a central series these are, as will be seen, much simpler to deal with than in the general case. We shall be able, *e.g.*, to prove for minimal A- and B-series in such groups the complete Jordan-Hölder and chief-series theorems, whereas the uniqueness of the finite factor-groups of an arbitrary S-group remains an open problem*.

Let

$$\mathfrak{G} = \mathfrak{K}_m \supseteqq \mathfrak{K}_{m-1} \supseteqq \ldots \supseteqq \mathfrak{K}_0 = 1 \tag{2.211}$$

be any central series of $\mathfrak{G}$ subject only to the condition that none of the Abelian factor-groups in it is a mixed group, *i.e.* that they are either finite or free Abelian groups. Such a series can always be obtained from an arbitrary central series by certain refinements.

THEOREM 2.21. *If in* (2.211) $\mathfrak{K}_{i+1}/\mathfrak{K}_i$ *is a finite, and* $\mathfrak{K}_i/\mathfrak{K}_{i-1}$ *a free Abelian group, then* $\mathfrak{K}_{i+1}/\mathfrak{K}_{i-1}$ *is a central factor of* $\mathfrak{G}$†, *i.e.* $(\mathfrak{G}, \mathfrak{K}_{i+1}) \subseteqq \mathfrak{K}_{i-1}$.

In particular $\mathfrak{K}_{i+1}/\mathfrak{K}_{i-1}$ is Abelian.

Proof. Let P and Q be arbitrary elements of $\mathfrak{K}_{i+1}$ and $\mathfrak{G}$, respectively. The commutator $R = (P, Q)$ is certainly contained in $\mathfrak{K}_i$, by definition of a central series. We have to show that it is also contained in $\mathfrak{K}_{i-1}$. Now, for any integer k, we have $(P^k, Q) = R^k . S_k$, where S_k belongs to $\mathfrak{K}_{i-1}$. But $\mathfrak{K}_{i+1}/\mathfrak{K}_i$ is of finite order; therefore, for a certain exponent n, P^n belongs to $\mathfrak{K}_i$. For this n, the commutator $(P^n, Q) = R^n . S_n$ must belong to $\mathfrak{K}_{i-1}$. This implies that R^n must lie in $\mathfrak{K}_{i-1}$; but $\mathfrak{K}_i/\mathfrak{K}_{i-1}$ is a free Abelian group and has no elements of finite order, except the 1-element. We thus see that R itself is contained in $\mathfrak{K}_{i-1}$. This concludes the proof.

* For the definitions of the terms "A-series", "B-series" and "minimal", *cf.* I.S.G. (I), 57, 58, 60.

† $\mathfrak{K}/\mathfrak{L}$ is called a central factor of $\mathfrak{G}$, if $(\mathfrak{K}, \mathfrak{G}) \subseteqq \mathfrak{L}$, *i.e.* if $\mathfrak{K}/\mathfrak{L}$ belongs to the central of $\mathfrak{G}/\mathfrak{L}$.

THEOREM 2.22. *If $\mathfrak{G}$ is an S-group with a central series, then its elements of finite order form a finite characteristic subgroup $\mathfrak{F}$.*

Proof. This is a corollary of Theorem 2.21. If in a central series for $\mathfrak{G}$ of type 2.211 $\mathfrak{R}_{i-1}/\mathfrak{R}_i$ is a finite and $\mathfrak{R}_i/\mathfrak{R}_{i-1}$ a free Abelian group, we may replace $\mathfrak{R}_i$ by a term $\mathfrak{R}_i'$ such that $\mathfrak{R}_{i+1}/\mathfrak{R}_i'$ is infinite and $\mathfrak{R}_i'/\mathfrak{R}_{i-1}$ is the finite factor of the Abelian group $\mathfrak{R}_{i+1}/\mathfrak{R}_{i-1}$ (or else is the 1-element). Thus we have shifted one finite factor-group of 2.211 towards the 1-element. This process may be repeated as often as necessary, until there is no finite factor-group left which precedes an infinite one immediately. At this stage all infinite factor-groups occur at the beginning all finite ones at the end of the central series. If this is

$$\mathfrak{G} = \mathfrak{R}_{m'}' \supseteq \mathfrak{R}_{m'-1}' \supseteq \ldots \supseteq \mathfrak{R}_0' = 1 \quad (m' \leqslant m),$$

and if $\mathfrak{R}_i'/\mathfrak{R}_{i-1}'$ is the first finite factor-group in it, then $\mathfrak{R}_i' = \mathfrak{F}$ is the required finite subgroup of $\mathfrak{G}$. Since $\mathfrak{G}/\mathfrak{R}_i'$ has no elements of finite order other than the 1-element, $\mathfrak{F}$ consists of all elements of finite order in $\mathfrak{G}$. Now $\mathfrak{F}$ must be transformed into itself by any automorphism of $\mathfrak{G}$ and is, therefore, a characteristic subgroup of $\mathfrak{G}$. Since it has a central series*, it is a direct product of p-groups.

If the finite part of an S-group $\mathfrak{G}$ is $\mathfrak{F}$, then the finite part of any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is precisely the meet $\mathfrak{H} \cap \mathfrak{F}$.

The S-group $\mathfrak{G} = \{P, Q\}$ defined by the relations

$$(P, Q) = R, \quad (R, P) = (R, Q) = R^p = 1$$

shows† that $\mathfrak{F}$ need not be a direct factor, nor even a "general factor" or "complemented subgroup" of $\mathfrak{G}$.

We are now in a position to prove for S-groups with a central series the strict analogue to the Jordan-Hölder and the chief-series theorems of finite group theory.

THEOREM 2.23. *In any minimal A- or B-series of an S-group with a central series the set of the finite and infinite factor-groups is, apart from isomorphisms, uniquely determined.*

Proof. Let

$$\mathfrak{G} = \mathfrak{G}_0 \supset \mathfrak{G}_1 \supset \ldots \supset \mathfrak{G}_m = 1 \tag{2.231}$$

* *Cf.* p. 337.

† *Cf.* B. H. Neumann, *Journal London Math. Soc.*, 10 (1935), 3–6; P. Hall, *ibid.*, 12 (1937), 200–204. $\mathfrak{G}$ has no subgroup $\mathfrak{H}$ such that $\{\mathfrak{F}, \mathfrak{H}\} = \mathfrak{G}$, $\mathfrak{F} \cap \mathfrak{H} = 1$.

be any minimal A-series of $\mathfrak{G}$. The number of infinite factor-groups in (2.231) is an invariant of $\mathfrak{G}$ by I.S.G. (I), Theorem 1.42. Let $\mathfrak{F}$ be the finite part of $\mathfrak{G}$ as defined in (2.22). The composition factors of $\mathfrak{F}$ must certainly occur among the finite factor-groups in (2.231). For, if F is any element of $\mathfrak{F}$ and F belongs to $\mathfrak{G}_i$, but not to $\mathfrak{G}_{i+1}$, then $\mathfrak{G}_i = \{F, \mathfrak{G}_{i+1}\}$, and $\mathfrak{G}_i/\mathfrak{G}_{i+1} \simeq \{F\}$. Further finite factor groups cannot occur in (2.231); for then it could not be a minimal A-series, because a suitably chosen A-series through $\mathfrak{F}$ would be shorter. So we see that the only finite factor-groups of any minimal A-series are just the composition factors of $\mathfrak{F}$, and they are uniquely determined.

The corresponding proof for minimal B-series can be omitted.

2.3. In this section we proceed to establish for S-groups with a central series* some of the more important elementary properties. These are at once recognised as corresponding exactly to well-known results in the theory of finite p-groups. Most of the proofs have been omitted, since they do not differ essentially from those in the finite case.

2.31. *The structure of the B-series of* $\mathfrak{G}$.

THEOREM 2.311. *A minimal B-series for* $\mathfrak{G}$ *can be chosen such that its factor-groups are of A-length* 1 (*or that the B-series is also an A-series*).

THEOREM 2.312. *If* $\mathfrak{K}$ *is any self-conjugate subgroup of* $\mathfrak{G}$, *then there is such a minimal B-series for* $\mathfrak{G}$ *through* $\mathfrak{K}$.

COROLLARY 2.313. *Every self-conjugate subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$ *of A-length* $\alpha \geqslant 1$ *contains some self-conjugate subgroup of A-length* 1 (*or of any A-length less than or equal to* α).

Proof. I. If

$$1 = \mathfrak{K}_0 \subseteq \mathfrak{K}_1 \subseteq \ldots \subseteq \mathfrak{K}_m = \mathfrak{G}$$

is a central series for $\mathfrak{G}$, one term of which is the finite part $\mathfrak{F}$ of $\mathfrak{G}$, it can be refined by inserting further terms between each $\mathfrak{K}_i$ and the next, until an A-series for $\mathfrak{G}$ is arrived at. Since $\mathfrak{K}_{i+1}/\mathfrak{K}_i$ belongs to the central of $\mathfrak{G}/\mathfrak{K}_i$, all groups between $\mathfrak{K}_{i+1}$ and $\mathfrak{K}_i$ are self-conjugate in $\mathfrak{G}$, and the A-series obtained in this way is thus a B-series, and, since it passes through $\mathfrak{F}$, it can be shortened, if necessary, into a minimal B-series.

* Throughout this section $\mathfrak{G}$ denotes an S-group with a central series.

II. Let $\mathfrak{N}$ be any self-conjugate subgroup of $\mathfrak{G}$. First, we choose a minimal B-series of type 2.311 for $\mathfrak{G}/\mathfrak{N}$. Secondly, we choose any minimal B-series for $\mathfrak{G}$ of type 2.311 and form the meets of its terms with $\mathfrak{N}$. Of these meets the different ones lead from $\mathfrak{N}$ to the 1-element. Moreover, they give a minimal B-series for $\mathfrak{N}$. For the factor-groups are infinite until the finite subgroup $\mathfrak{F}$ of $\mathfrak{G}$ is arrived at. And the meet of $\mathfrak{F}$ and $\mathfrak{N}$ gives just the finite part of $\mathfrak{N}$. The result of combining the two series is, then, a minimal B-series of type 2.311 through $\mathfrak{N}$.

2.32. *The upper central series of* $\mathfrak{G}$.

THEOREM 2.321. *The central* $\mathfrak{Z}_1$ *of* $\mathfrak{G}$ *contains all self-conjugate subgroups of* $\mathfrak{G}$ *of A-length* 1.

THEOREM 2.322. *The* α*-th term* $\mathfrak{Z}_\alpha$ *of the upper central series contains all self-conjugate subgroups of* $\mathfrak{G}$ *of A-length less than or equal to* α.

THEOREM 2.323. *A self-conjugate subgroup of* $\mathfrak{G}$, *if not contained in* $\mathfrak{Z}_\alpha$, *contains elements of* $\mathfrak{Z}_{\alpha+1}$ *which do not lie in* $\mathfrak{Z}_\alpha$.

Proof. (1). Let $\{P\}$ be any self-conjugate subgroup of A-length 1, and Q any element of $\mathfrak{G}$. Then $Q^{-1}PQ = P^x$, and we have to show that $x = 1$ for every Q. If P is of infinite order, the only alternative is $x = -1$. But then the subgroup $\{P, Q\}$ of $\mathfrak{G}$ would contain $\{Q^2\}$ self-conjugately, and the factor-group $\{P, Q\}/\{Q^2\}$ would be the infinite dihedral group which has no central. This is impossible*.

If P is of finite order p (p a prime), then $1 \leqslant x \leqslant p-1$. Suppose that $x \neq 1$, and let the k-th be the smallest power of Q which is permutable with P. Then the subgroup $\{P, Q\}$ of $\mathfrak{G}$ contains $\{Q^k\}$ self-conjugately, and the factor-group $\{P, Q\}/\{Q^k\}$, (the metacyclical group of order $k.p$) has no central. This is a contradiction*.

(2). The smaller that we choose α for a given self-conjugate subgroup of $\mathfrak{G}$, the more is asserted by Theorem 2.322. We therefore prove it for minimal A-lengths. We have just proved it for $\alpha = 1$.

By $\mathfrak{N}^{(\alpha)}$ we denote a typical self-conjugate subgroup of $\mathfrak{G}$ of minimal A-length α. Let us now proceed by induction and assume that we have proved Theorem 2.322 for a certain index α, *i.e.* $\mathfrak{N}^{(\alpha)} \subseteq \mathfrak{Z}_\alpha$. Then, in $\mathfrak{N}^{(\alpha+1)}$, we choose a maximal self-conjugate subgroup of $\mathfrak{G}$ of minimal A-length α (*i.e.* a $\mathfrak{N}^{(\alpha)}$), such that the factor-group $\mathfrak{N}^{(\alpha+1)}/\mathfrak{N}^{(\alpha)}$ is of A-length 1.

* *Cf.* p. 337.

That this is always possible follows from the proof of Theorem 2.312. Now we have by Theorem 2.321

$$\mathfrak{K}^{(\alpha+1)}/\mathfrak{K}^{(\alpha)} \subseteq \mathfrak{Z}_1(\mathfrak{G}/\mathfrak{K}^{(\alpha)}),$$

i.e.

$$(\mathfrak{G}, \mathfrak{K}^{(\alpha+1)}) \subseteq \mathfrak{K}^{(\alpha)} \subseteq \mathfrak{Z}_\alpha.$$

Therefore $\mathfrak{K}^{(\alpha+1)} \subseteq \mathfrak{Z}_{\alpha+1}$, and Theorem 2.322 is proved.

(3) Theorem 2.323 follows now with the help of Theorems 2.313 and 2.321 exactly as in the finite case.

2.33. *The lower central series of* $\mathfrak{G}$.

THEOREM 2.331. *The first factor-group* $\mathfrak{G}/\mathfrak{H}_2$ *of the lower central series cannot be cyclical. A group of A-length less than or equal to* 2 *is Abelian.*

THEOREM 2.332. *The* α*-th term* $\mathfrak{H}_\alpha$ *of the lower central series of* $\mathfrak{G}$ *is contained in every self-conjugate subgroup of* $\mathfrak{G}$ *the factor-group of which is of A-length less than or equal to* α *(for* $\alpha \geqslant 2$*).*

THEOREM 2.333. *If a self-conjugate subgroup* $\mathfrak{F}$ *of* $\mathfrak{G}$ *does not contain* $\mathfrak{H}_\alpha$*, then neither does the join* $\{\mathfrak{F}, \mathfrak{H}_{\alpha+1}\}$ *contain* $\mathfrak{H}_\alpha$.

Proof. (2) The smaller that we choose α for a given self-conjugate subgroup of $\mathfrak{G}$, the more is asserted by Theorem 2.332. We therefore prove it for minimal A-lengths. The case $\alpha = 2$ is contained in Theorem 2.331.

By $\mathfrak{F}^{(\alpha)}$ we denote a typical self-conjugate subgroup of $\mathfrak{G}$ the factor-group of which is of minimal A-length α. Let us proceed now by induction and assume that we have already proved Theorem 2.332 for a certain index α, *i.e.* $\mathfrak{H}_\alpha \subseteq \mathfrak{F}^{(\alpha)}$. Then, in $\mathfrak{F}^{(\alpha+1)}$, we choose a self-conjugate subgroup of A-length 1, say $\mathfrak{F}^{(\alpha)}/\mathfrak{F}^{(\alpha+1)}$, such that the factor-group $\mathfrak{G}/\mathfrak{F}^{(\alpha)}$ is of minimal A-length α. That this is always possible follows from the proof of Theorem 2.312.

Now $\mathfrak{F}^{(\alpha)}/\mathfrak{F}^{(\alpha+1)}$ belongs to the central of $\mathfrak{G}/\mathfrak{F}^{(\alpha+1)}$ by Theorem 2.321. That is

$$\mathfrak{H}_{\alpha+1} = (\mathfrak{H}_\alpha, \mathfrak{G}) \subseteq (\mathfrak{F}^{(\alpha)}, \mathfrak{G}) \subseteq \mathfrak{F}^{(\alpha+1)}. \qquad \text{Q.E.D.}$$

(3) Again Theorem 2.323 follows with the help of Theorem 2.313 and Theorem 2.321 exactly as in the finite case.

2.34. *The normalizers of subgroups* $\mathfrak{G}$.

THEOREM 2.341. *If* $\mathfrak{K}$ *is any proper subgroup of* $\mathfrak{G}$ *and* $\mathfrak{N}$ *is its normalizer, then* $\mathfrak{N}$ *contains* $\mathfrak{K}$ *as a proper subgroup.*

COROLLARY 2.342. *Every subgroup* $\mathfrak{K}$ *of* $\mathfrak{G}$ *occurs as a term in a suitably chosen A-series for* $\mathfrak{G}$. *Every subgroup of A-length* α *is self-conjugate in some subgroup of A-length* $\alpha+1$.

COROLLARY 2.343. *All maximal subgroups of* $\mathfrak{G}$ *are self-conjugate in* $\mathfrak{G}$ *and are of index* p (p *a prime*).

The proofs can be omitted.

2.35. *The derived series of* $\mathfrak{G}$*.

THEOREM 2.351. *If* $\mathfrak{R}$ *is any subgroup of* $\mathfrak{G}$ *which does not contain* θ_α, *then neither does* $\{\mathfrak{R}, \theta_{\alpha+1}\}$ *contain* θ_α.

This follows with the help of Theorem 2.342 as in the finite case.

2.352. Theorem 2.351 can be expressed in a different, but equivalent, form which is known, for finite p-groups, as "Burnside's basis theorem".

Definition. By the "principal subgroup" $\mathfrak{D}$ of a group $\mathfrak{G}$ we mean the meet of all maximal subgroups of $\mathfrak{G}$†. Since any automorphism of $\mathfrak{G}$ must permute among themselves the maximal subgroups, their meet is left unchanged by it, and $\mathfrak{D}$ is a characteristic subgroup of $\mathfrak{G}$.

Definition. An element G of a group $\mathfrak{G}$ will be called a "non-generator" if any set of elements of $\mathfrak{G}$ which, together with G, generates $\mathfrak{G}$, does so already without G; *i.e.* if G can be omitted from any basis of $\mathfrak{G}$ in which it may occur. The join of all non-generators is a subgroup $\mathfrak{F}$ of $\mathfrak{G}$.

Frattini has proved for finite groups that $\mathfrak{D}$ coincides with $\mathfrak{F}$‡. Using Zermelo's axiom of choice Neumann recently extended this result to any group whatsoever§.

Theorem 2.351 now states, for $\alpha = 0$, that the derived group θ_1 of $\mathfrak{G}$ consists only of "non-generators". It is, therefore, contained in the principal subgroup $\mathfrak{D}$ of $\mathfrak{G}$. This follows also from the fact that all maximal subgroups of $\mathfrak{G}$ are self-conjugate of prime index (2.343) and contain, therefore, the derived group θ_1 of $\mathfrak{G}$. Thus $\mathfrak{G}/\mathfrak{D}$ is an Abelian group. This Abelian group, moreover, is the direct product of a free Abelian group with elementary Abelian groups. Otherwise there would be an element in $\mathfrak{G}/\mathfrak{D}$ of order p^2, say (p a prime). Let P be a representative in $\mathfrak{G}$ of such an element. Then P^p is certainly contained in all maximal subgroups of

---

* This is defined as follows: $\theta_0 = \mathfrak{G}$, $\theta_1 = (\mathfrak{G}, \mathfrak{G})$, $\theta_{i+1} = (\theta_i, \theta_i)$.

† If G has no maximal proper subgroups, we put $\mathfrak{D} = \mathfrak{G}$.

‡ G. Frattini, *Atti Acad. Lincei*, (4), *Rendiconti*, 1 (1885), 281–285.

§ B. H. Neumann, *Journal London Math. Soc.*, 12 (1937), 120–127. Since S-groups satisfy the maximal condition for subgroups, the result can, in this case, be proved without reference to the axiom of choice.

index p; but since it is, by assumption, not contained in $\mathfrak{D}$, there is at least one maximal subgroup $\mathfrak{M}$ of index q ($q \neq p$) in which P^p is not contained. Now $\mathfrak{G}/\mathfrak{M}$ has, apart from the 1-element, only elements of order q; therefore, if P^p is not contained in $\mathfrak{M}$, then neither is $P^{p^2} = (P^p)^p$ contained in $\mathfrak{M}$. This is a contradiction. So we see that the orders of the elements of $\mathfrak{G}/\mathfrak{D}$ are, if finite, *quadratfrei*.

If the dimension of the free Abelian factor of $\mathfrak{G}/\mathfrak{D}$ is $\mathfrak{F}$, and the order of the finite factor is $\prod_\nu p_\nu^{e_\nu}$, and $e = \max_\nu (e_\nu)$, and $d = e+f$, then $\mathfrak{G}/\mathfrak{D}$ (and consequently $\mathfrak{G}$) can be generated by d, but not by fewer than d, elements. If, conversely, from any set of generators of $\mathfrak{G}$ we omit those contained in $\mathfrak{D}$, the remaining ones must represent generators of $\mathfrak{G}/\mathfrak{D}$. The maximal number of independent generators of $\mathfrak{G}$ or $\mathfrak{G}/\mathfrak{D}$ is, of course, $d' = e'+f$, where $e' = \sum_\nu e_\nu$.

University College,
Leicester.

A TRIAD OF QUADRINODAL CUBIC SURFACES CONTAINING A QUADRICUBIC CURVE

By WILLIAM P. MILNE.

[Received 15 October, 1937.—Read 18 November, 1937.]

1. *Introduction.*

Some time ago H. W. Richmond published a paper "On the tritangent planes of a quadricubic space-curve"*, and gave a canonical form for a set of four tritangent planes through whose points of contact passes a quadriquadric curve, and their three allied quadrinodal cubic surfaces. I had obtained independently another canonical form different in type from Richmond's, but had to delay publication until I had obtained direct proofs, (i) for a suitable criterion whether or not a quadriquadric should pass through the nine points of contact of three tritangent planes of a quadricubic curve†, and (ii) for the fundamental properties of the binodal plane quintic curve‡, on which two results the treatment reposes. This has now been accomplished.

2. *Relation between the quadricubic curve and the binodal plane quintic.*

Let α, β, γ, δ be four tritangent planes of the quadricubic curve Γ, contained by the quadric S and a given cubic surface, such that a quadriquadric SS' passes through their points of contact, S' being also a quadric. It is known that a unique quadrinodal cubic surface U can be found containing Γ and the twisted cubics through the points of contact of β, γ, and also of α, δ with Γ§. V and W are similarly defined with regard to γ, α (or β, δ) and α, β (or γ, δ) respectively. We proceed to find a canonical set of equations for U, V, W, and to deduce some new properties.

* H. W. Richmond, *Proc. Edinburgh Math. Soc.* (2), 4 (1935), 159–169.

† W. P. Milne, *Proc. London Math. Soc.* (2), 41 (1936), 454–461.

‡ W. P. Milne, *Proc. London Math. Soc.* (2), 44 (1938), 130–139.

§ W. P. Milne, *Proc. London Math. Soc.* (2), 21 (1922), 373–380.

There are 255 quadrinodal cubic surfaces containing a quadricubic Γ. If we project Γ on a plane ϖ from a point O on Γ itself, we obtain a binodal quintic curve Ψ, whose nodal points are the points of intersection of the generators of S at O with ϖ. The tangent at O to Γ lies in the tangent plane of S at O, and hence intersects ϖ in the point H in which the line joining the nodes of Ψ meets Ψ again. The two enveloping quadric cones from O to the quadrinodal cubic surface U intersect ϖ in two contact conics U_2, U_2' of Ψ. The polar quadric P of O with respect to U intersects S in a quadriquadric which projects into a cubic curve U_3 in ϖ passing through the nodes of Ψ and the points of contact of Ψ with U_2, U_2'. The tangent line to this quadriquadric SP at O is the line of intersection of the tangent plane to S with the tangent plane to P, *i.e.* the tangent plane to U, and hence is the tangent line at O to Γ. It follows that, in the projected figure in ϖ, U_3 passes through H. Similarly, the cubics V_3, W_3 pass through H and the nodes of Ψ, as well as through the points of contact of V_2, V_2' and W_2, W_2' respectively. The common tangent plane at O to U and P touches each of the enveloping quadric cones from O to U along one of the generating lines of P that meet in O. Hence, the tangent plane at O to U intersects ϖ in a line through H that touches each of the conics U_2, U_2' at the residual points in which they are cut by the cubic U_3, the points of contact being the points of intersection with ϖ of the generators of P at O. Similar properties hold with regard to V_3 and W_3. The tritangent conic αS to Γ projects into a conic A' passing through the nodes of Ψ and touching Ψ at three further points. Similar results hold with respect to βS, γS, δS. The quadriquadric SS' through the twelve points of contact of α, β, γ, δ with Γ projects into a quartic curve having nodes at the nodal points of Ψ, and passing through the points of contact of the conics A', B', C', D' with Ψ.

It is known that any two twisted cubics upon a quadrinodal cubic surface are the complete intersection of that quadrinodal cubic surface with a quadric passing through the four nodes of the surface and touching the cubic surface at a further point. An enveloping quadric cone from O to U touches U along a twisted cubic. Also U contains the twisted cubic through the points of contact of β, γ with Γ. These two twisted cubics therefore lie on the same quadric F. The quadriquadric FS projects from O on ϖ into a cubic curve through the nodes of Ψ and the points of contact of B', C', and U_2. Similarly, another cubic passes through the points of contact of B', C', and U_2' as well as through the nodes of Ψ. Thus we confirm by projective methods properties of the plane binodal quintic curve obtained in my previous paper*.

* See foot-note ‡ on page 345.

The following result established in my previous paper* does not appear to be easily reached by the above projective methods, but it is essential to the subsequent discussion in the present paper.

The points of contact of the conics U_2, V_2, W_2 with Ψ lie upon a cubic curve, which cuts these conics each in a residual point at which they are each touched by the same common tangent. Similar results hold with regard to the triads $U_2 V_2' W_2'$, $U_2' V_2 W_2'$, $U_2' V_2' W_2$.

The following deduction is immediate.

From every point O of the quadricubic curve Γ can be drawn four planes to touch each of the quadrinodal cubic surfaces U, V, W. Hence, U, V, W are all three inscribed in the same quartic developable Φ, which must be rational in character, since we know reciprocally that only rational twisted quartic curves exist on a Steiner surface.

3. *Steiner surfaces containing a given rational quartic.*

Let us first of all obtain the general equation to all Steiner surfaces containing the rational twisted quartic curve

$$x = a_0\theta^4 + 4a_1\theta^3\phi + 6a_2\theta^2\phi^2 + 4a_3\theta\phi^3 + a_4\phi^4, \text{ etc.} \tag{1}$$

There exists a (1, 1) correspondence between the points of (1) and those of any conic S_0. It will be convenient to take, as coordinate representation in the plane of S_0,

$$\xi = \theta^2, \quad \eta = \phi^2, \quad \zeta = 2\theta\phi, \tag{2}$$

and hence

$$S_0 \equiv \zeta^2 - 4\xi\eta = 0. \tag{3}$$

We know that we can map any Steiner surface on the plane of S_0 by means of the parametric representation,

$$x = S_1(\xi, \eta, \zeta), \text{ etc.}, \tag{4}$$

where S_1, etc. are of the second degree in ξ, η, ζ.

If this Steiner surface is to contain the curve (1), on substituting from (2) into (4), we must obtain the formulae (1). We, therefore, obtain the transformation formulae,

$$x = (a_0\xi^2 + a_4\eta^2 + a_2\zeta^2 + 2a_3\eta\zeta + 2a_1\zeta\xi + 2a_2\xi\eta) + \alpha'(\zeta^2 - 4\xi\eta), \text{ etc.}, \tag{5}$$

representing any Steiner surface containing the curve (1), α', β', γ', δ' being arbitrary constants.

* See foot-note ‡ on page 345.

The plane $lx+my+nz+pt=0$ cuts the Steiner surface in a rational plane quartic curve corresponding to the conic,

$$(L_0\xi^2+L_4\eta^2+L_2\zeta^2+2L_3\eta\zeta+2L_1\zeta\xi+2L_2\xi\eta)+\pi(\zeta^2-4\xi\eta)=0, \quad (6)$$

where
$$L_0\equiv a_0 l+b_0 m+c_0 n+d_0 p, \text{ etc.} \quad (7)$$

$$\pi\equiv \alpha' l+\beta' m+\gamma' n+\delta' p. \quad (8)$$

The plane $lx+my+nz+pt=0$ meets the Steiner surface in two conics if the conic (6) breaks up into two straight lines, *i.e.* if

$$\begin{vmatrix} L_0 & L_2-2\pi & L_1 \\ L_2-2\pi & L_4 & L_3 \\ L_1 & L_3 & L_2+\pi \end{vmatrix}=0, \quad (9)$$

i.e. if

$$4\pi^3-(L_0L_4-4L_1L_3+L_2^2)\pi$$
$$-(L_0L_2L_4+2L_1L_2L_3-L_0L_3^2-L_4L_1^2-L_2^3)=0,$$

which can be written

$$4\pi^3-I\pi-J=0, \quad (10)$$

where I and J are respectively the equianharmonic and harmonic envelopes of the rational quartic. This form resembles Euler's reducing cubic in the theory of the binary quartic and the properties are similar.

We have thus obtained in (10) the tangential equation to any Steiner surface containing the quartic curve (1).

We see immediately that, if α', β', γ', δ' are each zero, the mapping conics of (5) regarded as loci are each apolar to S_0 regarded as an envelope, and hence that the rational quartic (1) is an asymptotic curve on its harmonic envelope J. Thus, a rational twisted quartic is an asymptotic curve on one and one only of the Steiner surfaces containing it, namely its harmonic envelope.

Again, from (10), we have

$$(4\pi^3-I\pi-J)-(4\pi'^3-I\pi'-J)\equiv(\pi-\pi')\{4(\pi^2+\pi\pi'+\pi'^2)-I\}. \quad (11)$$

Thus, all tangent planes to two Steiner surfaces containing the same rational twisted quartic curve either pass through the same point $\pi-\pi'=0$ or touch the same quadric $4(\pi^2+\pi\pi'+\pi'^2)-I=0$.

4. *The quartic developable.*

Let U, V, W be three quadrinodal cubic surfaces inscribed in the same rational quartic developable Φ, which, regarded as a locus, is of the sixth order, its equation being $\Phi \equiv I^3 - 27J^2 = 0$, where I and J are the equianharmonic and harmonic surfaces of Φ respectively. The curve IJ is a rational sextic. Also, the curve IJ is a double curve on Φ, the tangent planes at every point of which to Φ are coincident and touch J along the curve IJ.

We shall use the results of (10) in their reciprocal form.

Let

$$\left.\begin{aligned} U &\equiv 4u^3 - Iu - J, \\ V &\equiv 4v^3 - Iv - J, \\ W &\equiv 4w^3 - Iw - J. \end{aligned}\right\} \tag{12}$$

V and W intersect in the cubic curve lying in the plane

$$v - w = 0 \tag{13}$$

and in a quadricubic curve lying on the quadric

$$S \equiv 4(v^2 + vw + w^2) - I = 0; \tag{14}$$

and similarly with respect to W, U and U, V. The three quadrics (14) are the same if

$$u + v + w \equiv 0. \tag{15}$$

Hence, *subject to the condition* (15), *the three quadrinodal cubic surfaces* U, V, W *inscribed in the same rational quartic developable surface* Φ *contain the same quadricubic curve* Γ, *and the quadric* S *containing* Γ *has double contact with the equianharmonic locus* I *of* Φ. U, V, W *taken two and two intersect also in residual plane cubic curves lying in the three planes* $v-w$, $w-u$, $u-v$, *having the same axis, viz. the chord of contact of* S *and* I, *and the two planes of the two conics in which* S *cuts* I *are the Hessian-planes of the triad of planes* $v-w$, $w-u$, $u-v$. *The planes* u, v, w *containing respectively the three residual plane cubics in which* J *cuts* U, V, W *also contain the chord of contact of* S *and* I, *and are the Jacobian-triad of the three planes* $v-w$, $w-u$, $u-v$.

The following result was established in my paper* on "three tritangent planes of the quadricubic curve" and will now be required.

* See footnote †, p. 345.

If α, β, γ are three tritangent planes of a quadricubic curve, such that a quadriquadric passes through their points of contact, and if U, V, W are the quadrinodal cubic surfaces containing the quadricubic and the twisted cubics through the points of contact of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ respectively, the enveloping quadric cones of V, W intersects the residual plane cubic in which they intersect in tritangent conics of this cubic belonging to the same family and similarly with regard to W, U and U, V. If, on the other hand, a quadriquadric does not pass through the points of contact of α, β, γ with the quadricubic curve, the two families of tritangent conics are different.

Using this result and reverting to (9), it is plain that $L_4(L_2+v)-L_3^2=0$ and $L_4(L_2+w)-L_3^2=0$ are each enveloping quadric cones of V and W respectively, and intersect $v-w=0$ in the same conic. Thus, since any two enveloping quadric cones of V touch V along twisted cubics lying on the same quadric and similarly with regard to W, it is evident that all enveloping cones of V and W intersect the plane $v-w=0$ containing the residual plane cubic common to V and W in tritangent conics of this cubic belonging to the same family.

The tritangent planes α, β, γ of the quadricubic curve defining U, V, W, as above explained, are such that a quadriquadric passes through their points of contact with the quadricubic curve.

A canonical form for three such quadrinodal cubic surfaces has thus been obtained in (12) and (15).

Consider now the complete intersection of U and Φ. We have:

$$I^3-27J^2\equiv 27(4u^3-Iu-J)(4u^3-Iu+J)+(I-12u^2)^2(I-3u^2). \quad (16)$$

Hence U touches Φ along a sextic curve which is the complete intersection of Φ with the quadric $I-12u^2=0$ touching I along the plane u. The residual curve of intersection of U with Φ lies on the quadric $I-3u^2=0$, which also touches I along the plane u. Similar results hold with regard to V and W.

Again, the quadric $I-12u^2=0$ meets U where it is intersected by the cubic surface $3J+2Iu=0$, which by varying u is the general cubic surface containing the curve IJ, the double curve of Φ. Hence, *every cubic surface $3J+2Iu=0$ containing the double curve IJ of Φ intersects Φ in a residual quadricubic curve, along which it is touched by a quadrinodal cubic surface U. The quadric $I-12u^2=0$ containing this contact quadricubic touches the equianharmonic locus I along the plane of the cubic curve common to J and the cubic surface $3J+2Iu=0$.*

U intersects J in a plane cubic curve lying in the plane u and in a quadricubic curve lying on the quadric $I-4u^2=0$, which touches I wherever it is met by u.

The cubic surface $U-uS\equiv 4uvw-J=0$ is plainly a cubic surface containing the quadricubic Γ and the plane cubic uU. It is obviously symmetrically related to the three surfaces U, V, W. Its equation may also be written in the form $U+V+W=0$.

To find the points in which the quadricubic Γ intersects Φ, we substitute from $I-4(v^2+vw+w^2)=0$ and $J-4uvw=0$ into $I^3-27J^2=0$. We obtain:

$$(v-w)^2(w-u)^2(u-v)^2=0. \tag{17}$$

The quadricubic curve Γ touches the developable Φ at eighteen points, which lie six by six on the three planes $v-w$, $w-u$, $u-v$.

The University,
Leeds.

UEBER DIE ENTWICKLUNG DER LÖSUNG DER TELEGRAPHENGLEICHUNG NACH BESSEL'SCHEN FUNKTIONEN

Von ARTHUR KORN.

[Received and read 18 November, 1937.]

Wir verstehen unter t die Zeit, unter x den Abstand eines Punktes einer Telegraphenleitung, welche von einem Anfangspunkte $x = 0$ gradlinig ins Unendliche läuft. Von diesem Anfangspunkte, dann gelten für Spannung und Stromstärke (V bzw. J) an der Stelle x zur Zeit t die beiden Differentialgleichungen:

$$(1) \qquad \begin{cases} -\dfrac{\partial V}{\partial x} = RJ + L\dfrac{\partial J}{\partial t}, \\ -\dfrac{\partial J}{\partial x} = C\dfrac{\partial V}{\partial t}, \end{cases}$$

in denen C, R, L drei charakteristische Konstanten der Leitung (Kapazität, Ohm'scher Widerstand, Selbstinduktion, bezogen auf die Längeneinheit) bezeichnen. Durch Elimination ergibt sich, sowohl für V als auch für J, die Differentialgleichung:

$$(2) \qquad \frac{\partial^2 W}{\partial x^2} = RC\frac{\partial W}{\partial t} + LC\frac{\partial^2 W}{\partial t^2},$$

welche als Telegraphengleichung bezeichnet wird.

Ein fundamentales Problem der Telegraphentechnik ist das Folgende: Die Leitung ist bis zum Zeitpunkte $t = 0$ in Ruhe; von diesem Momente an ist die Spannung V am Anfang der Leitung (d.h. für $x = 0$) als Funktion der Zeit gegeben:

$$(3) \qquad V = \phi(t) \quad (x = 0).$$

Wir setzen voraus, dass diese Funktion mit ihren ersten Ableitungen für ein endliches Zeitintervall

$$0 \leqslant t \leqslant \bar{t}$$

eindeutig und stetig ist und für $t = 0$ verschwindet.

Man kann—dieses grundlegende Resultat will ich hier als bekannt voraussetzen—die Stromstärke J zur Zeit t und an jeder Stelle x der Leitung in folgender Weise darstellen:

$$(4) \qquad \begin{cases} J = 0 \quad (x \geqslant vt), \\ J = Cv \int_0^{t-x/v} e^{-\rho(t-\tau)} \phi'(\tau) J_0\left(i\rho \sqrt{(t-\tau)^2 - \frac{x^2}{v^2}}\right) d\tau \quad (x \leqslant vt), \end{cases}$$

wenn wir

$$(5) \qquad v^2 = \frac{1}{LC}, \quad \rho = \frac{R}{2L}$$

setzen und unter $J_0(z)$ die Besselsche Funktion:

$$(6) \qquad J_0(z) = 1 - \left(\frac{1}{1!}\right)^2 \left(\frac{z}{2}\right)^2 + \left(\frac{1}{2!}\right)^2 \left(\frac{z}{2}\right)^4 - \ldots$$

verstehen.

Von der Lösung dieses Fundamentalproblemes kann man mit Hilfe von kombinatorischen Methoden zur Lösung des Problemes für begrenzte Leitungen übergehen. Man beherrscht mit der genannten Lösung auch die Lösung des etwas komplizierteren Problemes, welches entsteht, wenn man die Gleichungen (1) durch die Gleichungen:

$$(1') \qquad \begin{cases} -\frac{\partial V}{\partial x} = RJ + L\frac{\partial J}{\partial t}, \\ -\frac{\partial J}{\partial x} = GV + C\frac{\partial V}{\partial t} \end{cases}$$

mit 4 Konstanten C, R, G, L (G die sogenannte Ableitung) ersetzt.

So legt der fundamentale Charakter der Lösung (4) den Wunsch nahe, den analytischen Ausdruck dieser Lösung dem Verständnis näher zu bringen, im besonderen durch Reihenentwicklungen und erste Annäherungen Formeln zu schaffen, welche auch für den praktischen Elektrotechniker brauchbar sind. Annäherungen sind schon mehrfach als erste Glieder von Reihenentwicklungen berechnet worden, im allgemeinen aber ohne strenge Konvergenzuntersuchungen über die Reihen, aus denen die Annäherungen abgeleitet werden.

Anschliessend an eine Untersuchung in dieser Richtung (*Math. Zeitschrift* 35 (1932), 544–552) will ich hier eine Reihenentwicklung für die Lösung (4) ableiten, welche die frühere an Einfachheit und Eleganz übertrifft. Im übrigen wird jede der beiden Reihenentwicklungen ihr bestimmtes Anwendungsgebiet haben.

1. Wie in der zitierten Arbeit gebe ich der Lösung (4) durch die Substitution:

$$\rho(t-\tau)=\xi, \quad y=\rho\frac{x}{v}, \tag{7}$$

die Form:

$$\begin{cases} J=0 \quad (\rho t\leqslant y), \\ J=\dfrac{Cv}{\rho}\displaystyle\int_y^{\rho t}\phi'\left(t-\frac{\xi}{\rho}\right)e^{-\xi}J_0(i\sqrt{\xi^2-y^2})\,d\xi \quad (\rho t\geqslant y). \end{cases} \tag{8}$$

Indem wir noch*

$$\begin{cases} \eta=\rho t, \\ f(\xi)=\dfrac{Cv}{\rho}\phi'\left(t-\dfrac{\xi}{\rho}\right)e^{-\xi} \end{cases} \tag{9}$$

setzen, geht die zweite, uns wesentlich interessierende Formel (8) in die folgende über:

$$J=\int_y^{\eta}f(\xi)\,J_0\,(i\sqrt{\xi^2-y^2})\,d\xi \quad (y\leqslant\eta), \tag{10}$$

und es handelt sich um geeignete Reihenentwicklungen für diesen Ausdruck. Indem wir schliesslich

$$z=i\sqrt{(\eta^2-y^2)} \tag{11}$$

substituieren, wird

$$J-\psi(\eta,z)=\int_{\sqrt{(\eta^2+z^2)}}^{\eta}f(\xi)\,J_0(i\sqrt{\xi^2-\eta^2-z^2})\,d\xi, \tag{12}$$

wo

$$0\leqslant|z|\leqslant\eta, \tag{13}$$

und wir wollen jetzt eine Reihenentwicklung

$$J\equiv\psi(\eta,z)=2\sum_{j=1}^{\infty}\mu_{2j}(\eta)\,J_{2j}(i\sqrt{\eta^2-y^2}) \tag{14}$$

---

* In $f(\xi)$ spielt t die Rolle eines Parameters.

beweisen, in welcher die $\mu_{2j}(\eta)$ noch zu bestimmen sind.

Wir beschränken uns ausdrücklich auf eine Entwicklung in einem endlichen Kreisgebiete, da eine Entwicklung von der Form:

$$F(z) = 2 \sum_{j=1}^{\infty} \mu_{2j} J_{2j}(z) \quad (0 \leqslant z < \eta), \tag{15}$$

für irgend eine in dem Kreisgebiete reguläre Funktion $F(z)$, welche der Bedingung

$$F(z) \equiv F(-z)$$

genügt und für $z = 0$ verschwindet, ausser Frage steht. Wir wissen im übrigen dass die Koeffizienten der Reihe in der Form angegeben werden können:

$$\mu_{2j} = \frac{1}{2\pi i} \int F(z)\, O_{2j}(z)\, dz \qquad (j = 1, 2, \ldots), \tag{17}$$

also

$$\mu_{2j}(\eta) = \frac{1}{2\pi i} \int \psi(\eta, z)\, O_{2j}(z)\, dz \quad (j = 1, 2, \ldots), \tag{17'}$$

wobei in der komplexen z-Ebene über den Kreis mit dem Radius η zu integrieren ist und die $O_{2j}(z)$ die folgenden Funktionen bedeuten:

$$O_{2j}(z) = \frac{1}{z}\left(1 + \frac{4j^2}{z^2} + \frac{4^2 j^2 (j^2-1)}{z^4} + \frac{4^3 j^2 (j^2-1)(j^2-4)}{z^6} + \ldots\right) \tag{18}$$

$$(j = 0, 1, 2, \ldots).$$

Mit Rücksicht auf das Folgende definieren wir durch diese Formel auch $O_0(z)$ und durch die Formeln:

$$O_{2j+1}(z) = \frac{2j+1}{z^2}\left(1 + \frac{4j(j+1)}{z^2} + \frac{4^2 (j-1)(j+1)\, j(j+2)}{z^4} + \ldots\right) \tag{18'}$$

$$(j = 0, 1, 2, \ldots),$$

auch die ungraden $O_{2j+1}(z)$.

Wir können als Ergebnis dieses § 1 feststellen, dass für

$$\rho t \geqslant y, \quad \text{also} \quad x \leqslant vt,$$

J in der Form:

$$J = 2 \sum_{j=1}^{\infty} \mu_{2j}(\rho t)\, J_{2j}\left(i\rho \sqrt{t^2 - \frac{x^2}{v^2}}\right) \tag{19}$$

entwickelt werden kann, wobei die

$$\mu_{2j}$$

noch zu bestimmen sind.

2. Zur Berechnung der μ_{2j} haben die Formeln (12) und (17′) zu dienen, wobei η und $f(\xi)$ durch die Formeln (9) definiert sind.

Wir werden uns dabei eines Kunstgriffes bedienen, indem wir bemerken, dass nach (12) die Funktion $\psi(\eta, z)$ der folgenden Identität genügt:

$$z\left[\frac{\partial\psi}{\partial\eta}-f(\eta)\,J_0(z)\right] \quad \eta\,\frac{\partial\psi}{\partial z},$$

die wir zweckmässig folgendermassen schreiben:

$$\frac{\partial\psi}{\partial z}\equiv\frac{z}{\eta}\,\frac{\partial\psi}{\partial\eta}-\frac{z}{\eta}\,f(\eta)\,J_0(z), \tag{20}$$

indem wir von jetzt ab stets $\eta\neq 0$, also $t>0$, voraussetzen.

Wir machen von der folgenden rekurrenten Eigenschaft der $O_\kappa(z)$ Gebrauch:

$$O_{2j+2}(z)-O_{2j}(z)=-2O'_{2j+1}(z) \quad (j=0,\,1,\,2,\,\ldots), \tag{21}$$

nach welcher:

$$\mu_{2j+2}(\eta)-\mu_{2j}(\eta)=-\frac{1}{\pi i}\int\psi O'_{2j+1}\,dz=\frac{1}{\pi i}\int\frac{\partial\psi}{\partial z}\,O_{2j+1}\,dz, \tag{22}$$

wenn wir noch

$$\mu_0=0 \tag{23}$$

definieren, und mit Rücksicht auf (20):

$$\mu_{2j+2}(\eta)-\mu_{2j}(\eta)=\frac{2}{\eta}\,\frac{\partial B_{2j+1}}{\partial\eta}-\frac{2}{\eta}\,f(\eta)\,C_{2j+1} \quad (j=0,\,1,\,2,\,\ldots), \tag{24}$$

wenn wir

$$\begin{cases} B_{2j+1}=\dfrac{1}{2\pi i}\displaystyle\int z\psi O_{2j+1}\,dz, \\ C_{2j+1}=\dfrac{1}{2\pi i}\displaystyle\int z J_0(z)\,O_{2j+1}\,dz \end{cases} \qquad (j=0,\,1,\,2\ldots) \tag{25}$$

setzen.

Für $j=0$ wird:

$$B_1=0, \quad C_1=1, \tag{26}$$

somit:

$$\mu_2(\eta)=-\frac{2}{\eta}\,f(\eta). \tag{27}$$

Wir machen schliesslich noch von den folgenden rekurrenten Eigenschaften

der Funktionenen $O_\kappa(z)$ Gebrauch:

$$z\{(2j-1)\,O_{2j+1}+(2j+1)\,O_{2j-1}\}=2(4j^2-1)\,O_{2j} \quad (j=1,\,2,\,\ldots), \tag{28}$$

und folgern:

$$\begin{cases} (2j-1)\,B_{2j+1}+(2j+1)\,B_{2j-1}=2(4j^2-1)\,\mu_{2j}(\eta), \\ (2j-1)\,C_{2j+1}+(2j+1)\,C_{2j-1}=2(4j^2-1)\int J_0\,O_{2j}\,dz-0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (j=1,\,2,\,\ldots), \end{cases} \tag{29}$$

somit nach (24):

$$(2j-1)\{\mu_{2j+2}(\eta)-\mu_{2j}(\eta)\}+(2j+1)\{\mu_{2j}(\eta)-\mu_{2j-2}(\eta)\}=\frac{4(4j^2-1)}{\eta}\,\frac{\partial\mu_{2j}(\eta)}{\partial\eta}$$

$$(j=1,\,2,\,\ldots),$$

oder:

$$\mu_{2j+2}(\eta)=-\frac{2}{2j-1}\,\mu_{2j}(\eta)+\frac{2j+1}{2j-1}\,\mu_{2j-2}(\eta)+\frac{4(2j+1)}{\eta}\,\frac{\partial\mu_{2j}(\eta)}{\partial\eta} \tag{30}$$

$$(j=1,\,2,\,\ldots).$$

Wir haben somit eine rekurrente Formel für die Berechnung der $\mu_{2j}(\eta)$ gewonnen und können das Resultat aussprechen:

Es ist für

$$x\leqslant vt, \quad t>0 \tag{31}$$

J in der Form entwickelbar:

$$J=2\sum_{j=1}^{\infty}[\mu_{2j}(\eta)]\,J_{2j}\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right). \tag{32}$$

Dabei ist

$$\mu_2(\eta)=-\frac{2Cv}{\rho\eta}\,\phi'\left(t-\frac{\eta}{\rho}\right)e^{-\eta}, \tag{33}$$

und die übrigen μ_{2j} sind durch die Formeln zu berechnen:

$$\mu_{2j+2}(\eta)=-\frac{2}{2j-1}\,\mu_{2j}(\eta)+\frac{2j+1}{2j-1}\,\mu_{2j-2}(\eta)+\frac{4(2j+1)}{\eta}\,\frac{\partial\mu_{2j}(\eta)}{\partial\eta} \tag{34}$$

$$(j=1,\,2,\,\ldots).$$

Die Einschliessung in [—] stellt schliesslich die Substitution

$$\eta=\rho t \tag{35}$$

dar.

Wir geben die Koeffizienten von

$$J_2\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right) \quad \text{und} \quad J_4\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right)$$

in der Reihe (32). Der Koeffizient von

$$J_2\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right)$$

ist:

$$2[\mu_2(\eta)] = -\frac{4Cv}{\rho^2 t}\, e^{-\rho t}\, \phi'(0); \tag{36}$$

der Koeffizient von

$$J_4\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right)$$

ist:

$$[2\mu_4(\eta)] = \frac{4Cv}{\rho^2 t}\, e^{-\rho t}\left\{\left(1+\frac{6}{\rho t}+\frac{6}{\rho^2 t^2}\right)\phi'(0)+\frac{6}{\rho^2 t}\,\phi''(0)\right\}. \tag{37}$$

3. Ich will diesem Hauptresultat die früher von mir gegebene Reihenentwicklung gegenüberstellen; die Darstellung weicht ein wenig von der früheren ab, obwohl sie im Wesen mit derselben identisch ist.

Ich führe in der zweiten Gleichung (8) die neue Variable z durch die Substitution:

$$z = i\sqrt{(\xi^2-y^2)} \tag{38}$$

ein, so, dass

$$\begin{cases} \xi = -i\sqrt{(z^2-y^2)}, \\ d\xi = -\dfrac{iz}{\sqrt{(z^2-y^2)}}\, dz, \end{cases} \tag{39}$$

und die Formeln (8) nehmen die Gestalt an:

$$J = 0 \quad (\rho t \leqslant y),$$

$$J = \int_0^{i\rho\sqrt{(t^2-y^2/\rho^2)}} \frac{\psi(y, z)\, J_0(z)}{z}\, dz. \tag{40}$$

Der Gedankengang der früheren Entwicklung war,

$$\psi(y, z) = -\frac{Cv}{\rho}\,\frac{iz^2}{\sqrt{(z^2-y^2)}}\, e^{i\sqrt{(z^2-y^2)}}\,\phi'\left(t+i\,\frac{\sqrt{(z^2-y^2)}}{\rho}\right) \tag{41}$$

in eine Reihe von der Form:

$$\psi(y, z) = 2\sum_{j=1}^{\infty} A_{2j}(y)\, J_{2j}(z) \tag{42}$$

zu entwickeln und dann die Integrationsformeln

$$(43)\qquad \begin{cases} \int_0^z \frac{J_{2j}(z)\,J_0(z)}{z}\,dz = \frac{1}{2j}\,I_{2j}(z) & (j=1,\,2,\,\ldots), \\ I_{2j}(z) = \frac{z}{2j}\,\{J_0(z)\,J'_{2j}(z) - J_0'(z)\,J_{2j}(z)\} & (j=1,\,2,\,\ldots), \end{cases}$$

anzuwenden. Die sich so ergebende Entwicklung:

$$(44)\qquad J = \sum_{j=1}^{\infty} \frac{A_{2j}(y)}{j}\,I_{2j}\left(i\rho\,\sqrt{t^2 - \frac{x^2}{v^2}}\right)$$

ist zwar eigentlich keine Entwicklung nach den gewöhnlichen Besselschen Funktionen, sondern nach den mit ihnen eng zusammenhängenden Funktionen $I_{2j}(z)$; in manchen Fällen kann aber auch diese Entwicklung ihre Vorzüge haben.

Die Berechnung der A_{2j} erfolgt wieder mit Hilfe der Formel:

$$(45)\qquad A_{2j}(y) = \frac{1}{2\pi i}\int \psi(y,\,z)\,O_{2j}(z)\,dz,$$

wobei in der komplexen z Ebene über einen Kreis mit einem Radius

$$\bar{z} > y$$

zu integrieren ist.

Da $\psi(y,\,z)$, abgesehen von dem Faktor z^2, eine Funktion von

$$\sqrt{(z^2 - y^2)}$$

ist, besteht die Identität:

$$(46)\qquad \frac{\partial \psi}{\partial z} = \frac{2\psi}{z} - \frac{2}{y}\,\frac{\partial \psi}{\partial y},$$

wobei wir von jetzt ab stets $y > 0$, also

$$x > 0$$

voraussetzen.

Infolge der rekurrenten Formeln (21) ist:

$$(47)\qquad A_{2j+2} - A_{2j} = -\frac{1}{\pi i}\int \psi O'_{2j+1}\,dz = \frac{1}{\pi i}\int \frac{\partial \psi}{\partial z}\,O_{2j+1}\,dz \quad (j=0,\,1,\,2,\,\ldots),$$

wenn wir noch

$$(48)\qquad A_0 = 0$$

definieren, und mit Rücksicht auf (46):

$$A_{2j+2}-A_{2j}=\frac{2}{\pi i}\int \psi\frac{O_{2j+1}}{z}\,dz-\frac{1}{\pi i y}\frac{\partial}{\partial y}\int z\psi O_{2j+1}\,dz \tag{49}$$

$$(j=0,\ 1,\ 2,\ \ldots),$$

und wenn wir

$$B_{2j+1}=\frac{1}{2\pi i}\int z\psi O_{2j+1}\,dz \tag{50}$$

setzen und die rekurrenten Formeln:

$$\frac{O_{2j+1}}{z}=\frac{O_{2j+2}}{4(j+1)}+\frac{O_{2j}}{4j}-\frac{2j+1}{4j(j+1)}\frac{1}{z}\quad(j=1,\ 2,\ \ldots), \tag{51}$$

berücksichtigen:

$$A_{2j+2}-A_{2j}=\frac{A_{2j+2}}{j+1}+\frac{A_{2j}}{j}-\frac{2}{y}\frac{\partial B_{2j+1}}{\partial y}\quad(j=1,\ 2,\ \ldots),$$

oder:

$$\frac{j}{j+1}A_{2j+2}-\frac{j-1}{j}A_{2j}=-\frac{2}{y}\frac{\partial B_{2j+1}}{\partial y}\quad(j=0^*,\ 1,\ 2,\ \ldots). \tag{52}$$

Andererseits folgt aus den rekurrenten Formeln:

$$(2j-1)O_{2j+1}+(2j+1)O_{2j-1}=\frac{2(4j^2-1)}{z}O_{2j}\quad(j=1,\ 2,\ \ldots) \tag{53}$$

leicht:

$$(2j-1)B_{2j+1}+(2j+1)B_{2j-1}=2(4j^2-1)A_{2j}\quad(j=1,\ 2,\ \ldots). \tag{54}$$

Durch Elimination der B_k ergibt sich aus (52) und (54);

$$(2j-1)\left\{\frac{j}{j+1}A_{2j+2}-\frac{j+1}{j}A_{2j}\right\}+(2j+1)\left\{\frac{j-1}{j}A_{2j}-\frac{j}{j-1}A_{2j-2}\right\}$$

$$=-\frac{4(4j^2-1)}{y}\frac{\partial A_{2j}}{\partial y}\quad(j=1,\ 2,\ \ldots),$$

oder:

$$A_{2j+2}=-\frac{4}{y}\frac{(2j+1)(j+1)}{j}\frac{\partial A_{2j}}{\partial y}+\frac{2(j+1)}{j(2j-1)}A_{2j}+\frac{(2j+1)(j+1)}{(2j-1)(j-1)}A_{2j-2} \tag{55}$$

$$(j=1,\ 2,\ \ldots).$$

* Für $j=0$ ist $B_1=0$,

die Gleichung (52) besteht also auch für $j=0$ in der Form:

$$\frac{j}{j+1}A_{2j+2}=-\frac{2}{y}\frac{\partial B_{2j+1}}{\partial y}\quad(j=0). \tag{52'}$$

Wenn wir noch an Stelle der A_{2j}

$$\mathbf{A}_{2j} = (-1)^{j+1} \frac{A_{2j}}{2j} \tag{56}$$

einführen, erhalten wir das Resultat:

Die Koeffizienten der Entwicklung:

$$J = 2 \sum_{j=1}^{\infty} (-1)^{j+1} \mathbf{A}_{2j} I_{2j}\left(i\rho \sqrt{t^2 - \frac{x^2}{v^2}}\right) \tag{57}$$

erfüllen die rekurrenten Gleichungen:

$$\mathbf{A}_{2j+2} = -\frac{2}{2j-1} \mathbf{A}_{2j} + \frac{2j+1}{2j-1} \mathbf{A}_{2j-2} + \frac{4}{y} (2j+1) \frac{\partial \mathbf{A}_{2j}}{\partial y} \quad (j = 1, 2, \ldots), \tag{58}$$

aus denen, mit

$$\mathbf{A}_0 = 0 \tag{59}$$

und

$$\mathbf{A}_2 = \tfrac{1}{2} A_2 = \left| \frac{\psi(y, z)}{z^2} \right|_{z=0} = -\frac{2Cv}{\rho y} e^{-\nu} \phi'\left(t - \frac{y}{\rho}\right), \tag{60}$$

alle $\mathbf{A}_{2j}$ rekurrent zu berechnen sind.

4. Der vorangehende Paragraph gibt mein früheres Resultat* in einer etwas eleganteren Form:

Die Lösung der Telegraphengleichung kann, falls die Funktion $\phi(z)$ in einem genügend grossen Kreise der z Ehene eindeutig und regulär ist, in der folgenden Weise dargestellt werden:

$$\begin{cases} J = 0 \quad (x \geqslant vt), \\ J = 2 \sum_{j=1}^{\infty} (-1)^{j+1} \mathbf{A}_{2j} I_{2j}\left(i\rho \sqrt{t^2 - \frac{x^2}{v^2}}\right) \quad (x \leqslant vt). \end{cases} \tag{61}$$

Dabei ist

$$I_{2j}(z) = \frac{z}{2j} \{J_0(z) J'_{2j}(z) - J_0'(z) J_{2j}(z)\} \quad (j = 1, 2, \ldots), \tag{62}$$

und die Koeffizienten $\mathbf{A}_{2j}$ berechnen sich rekurrent mit Hilfe der Formeln:

$$\mathbf{A}_{2j+2} = -\frac{2}{2j-1} \mathbf{A}_{2j} + \frac{2j+1}{2j-1} \mathbf{A}_{2j-2} + \frac{4v^2}{\rho^2 x} (2j+1) \frac{\partial \mathbf{A}_{2j}}{\partial x} \quad (j = 1, 2, \ldots), \tag{63}$$

mit

$$\mathbf{A}_0 = 0 \tag{64}$$

* *Math. Zeitschrift*, 35 (1932), 552.

und

$$\mathbf{A}_2 = -\frac{2Cv^2}{\rho^2 x}\, e^{-\rho x/v}\phi'\left(t-\frac{x}{v}\right). \tag{65}$$

Neben dieses frühere Resultat stellen wir das Hauptresultat des § 2.

Die Lösung der Telegraphengleichung kann, falls die Funktion $\phi(z)$ in einem genügend grossen Kreise der z Ebene eindeutig und regulär ist, in der folgenden Weise dargestellt werden:

$$\begin{cases} J = 0 \quad (x \geqslant vt), \\ J = 2\sum_{j=1}^{\infty} \mu_{2j} J_{2j}\left(i\rho\sqrt{t^2-\frac{x^2}{v^2}}\right) \quad (x \leqslant vt). \end{cases} \tag{66}$$

Dabei sind die $J_{2j}(z)$ die gewöhnlichen Besselschen Funktionen:

$$J_{2j}(z) = \left(\frac{z}{2}\right)^{2j}\left\{\frac{1}{(2j)!} - \frac{1}{1!(2j+1)!}\left(\frac{z}{2}\right)^2 + \frac{1}{2!(2j+2)!}\left(\frac{z}{2}\right)^4 \ldots\right\} \tag{67}$$

$$(j = 1, 2, \ldots),$$

und die Koeffizienten μ_{2j} der Reihe berechnen sich rekurrent mit Hilfe der Formeln:

$$\mu_{2j+2}(\eta) = -\frac{2}{2j-1}\,\mu_{2j}(\eta) + \frac{2j+1}{2j-1}\,\mu_{2j-2}(\eta) + \frac{4(2j+1)}{\eta}\,\frac{\partial\mu_{2j}(\eta)}{\partial\eta} \tag{68}$$

$$(j = 1, 2, \ldots),$$

mit

$$\mu_0(\eta) = 0 \tag{69}$$

und

$$\mu_2(\eta) = -\frac{2Cv}{\rho\eta}\,\phi'\left(t-\frac{\eta}{\rho}\right)e^{-\eta}, \tag{70}$$

unter der Substitution:

$$\eta = \rho t. \tag{71}$$

Die Uebereinstimmung in den zur Berechnung der Koeffizienten der beiden Reihen dienenden rekurrenten Formeln ist bemerkenswert.

Berlin-Grunewald,
Wangenheimstr., 7.

FRACTIONAL INTEGRATION, AND ALMOST PERIODIC FUNCTIONS

By E. R. LOVE.

[Received 16 July, 1937.—Read 18 November, 1937.—Revised 28 February, 1938*.]

1. *Introduction.* The first definition of fractional integration suitable for use with functions defined on the whole real axis (not merely on a semi-infinite interval of it) was given by Weyl [**5**]. Any continuous periodic function of mean value zero could thereby be integrated to any real positive order, giving another such function. The function obtained by an integration of order α followed by one of order β was the same as that obtained by an integration of order $\alpha+\beta$. An integration of integral order gave the same function as the appropriate number of successive indefinite integrations in the ordinary sense.

Weyl defined the α-th integral of a periodic function f as the function having a certain trigonometric series, obtained from the Fourier series of f by a rule depending on α, as its Fourier series; and he deduced an explicit formula for the α-th integral in the case $\alpha<1$. Definition of a function by means of its Fourier series is, however, often inconvenient, and an obstacle to the admission of functions whose square is not Lebesgue integrable. Instead, Hardy and Littlewood [**2**] adopted as their formal definition Weyl's explicit formula for the integral of f of order $\alpha<1$:

$$f_\alpha(x)=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x} f(t)(x-t)^{\alpha-1}\,dt \quad (0<\alpha<1),$$

interpreting the infinite integral as the limit of L-integrals over $(-c, x)$ as $c\to\infty$, f being any periodic function L-integrable over every finite interval

* The revision consisted in making the work applicable to complex orders of integration; some minor alterations in notation were made at the same time.

and of mean value zero. This, and the equivalent formula

$$f_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x-u)\, u^{\alpha-1} du \quad (0 < \alpha < 1),$$

interpreted as the limit of L-integrals over $(0, c)$ as $c \to \infty$, will henceforth be referred to as "Weyl's formula"; the Weyl α-th integral $f_\alpha(x)$ "exists" if Weyl's formula is convergent. And it will be understood that

$$(f_\alpha)_\beta(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty f_\alpha(x-v)\, v^{\beta-1} dv \quad (0 < \beta < 1),$$

with the same mode of interpretation.

As in **3**, the range of the parameter α is extended to include complex values; $f_\alpha(x)$ is formally defined by Weyl's formula for all α such that $0 < \bar{\alpha} < 1$, where $\bar{\alpha}$ is the real part of α. *The symbols* $\bar{\alpha}$, $\bar{\beta}$, ..., *denoting the real parts of* $\alpha, \beta, \ldots$, *are used freely without further explanation* this; does not clash with a customary notation for conjugate complex numbers, because no use is made of the latter.

Weyl's formula is a linear operation with the following fundamental properties: *Suppose that f is a periodic function L-integrable over every finite interval and of mean value zero. If*

A. $0 < \bar{\alpha} < 1$, *then* $f_\alpha(x)$ *exists for almost all x, and is itself a periodic function L-integrable over every finite interval and of mean value zero.*

B. $\bar{\alpha} > 0$, $\bar{\beta} > 0$, $\bar{\alpha}+\bar{\beta} < 1$, *then* $(f_\alpha)_\beta(x)$ *exists and is equal to* $f_{\alpha+\beta}(x)$ *for all x such that the latter exists.*

C. $0 < \bar{\alpha} < 1$, *then* $(f_\alpha)_{1-\alpha}(x)$ *exists for all x and is that (ordinary) integral of f which has mean value zero.*

The purpose of the present investigation is to consider to what extent Weyl's formula possesses the above properties, or analogues of them, when applied to functions which are not restricted to be periodic; in particular, to almost periodic functions. The above properties are needed in some of Hardy and Littlewood's work on Lebesgue and Lipschitz classes of fractional integrals of periodic functions [**2**] and consequently in Love and Young's work on fractional integration by parts [**3**]. To extend these investigations to broader classes of functions, the fundamental properties of Weyl's formula must be extended; the latter task is undertaken here, as a preliminary to the former.

It was natural first to consider almost periodic functions. But it was found that the machinery used was in part applicable to larger classes of functions; so the greater part of this paper is devoted to a general discussion of Weyl's formula for functions whose indefinite integrals satisfy certain conditions of order of magnitude. In particular, if the integral of f is a bounded function, then properties analogous to A and B hold; and if, in addition, $(f_\alpha)_{1-\alpha}(x)$ exists for some one value of x, then C holds. For wider classes of functions, properties analogous to A and B can be asserted, but only in smaller ranges of the indices α and β. The concluding part of the paper discusses the application of Weyl's formula to functions which are almost periodic in Stepanoff's (second) sense [**4**]. The importance of this is that, for any such function whose integral is bounded, Weyl's formula has properties B and C, and an analogue of A differing from A in one essential respect only (see the first paragraph of § 7).

A study of the possibility of extending certain parts of **2** and **3** to almost periodic functions is being undertaken in another paper.

2. *Preliminaries.* The only functions considered are complex measurable functions of a real variable defined for almost all values of the latter. Throughout, L denotes the class of these functions which are Lebesgue-integrable on every *finite* interval.

The following elementary lemmas are often needed.

LEMMA 1. *If f is L and $\bar{\alpha} > 0$, there is a set $T_{\bar{\alpha}}(f)$ of real numbers x, whose complement has measure zero, such that, for each $c > 0$,*

$$\int_0^c f(x-u)\, u^{\alpha-1}\, du$$

exists for all x of $T_{\bar{\alpha}}(f)$ and belongs to L. If, further, $|f|^p$ is L and $1/p < \bar{\alpha} < 1$, then every real number x belongs to $T_{\bar{\alpha}}(f)$.

For, given any finite interval (a, b),

$$\int_a^b dx \int_0^1 |f(x-u)\, u^{\alpha-1}|\, du = \int_0^1 u^{\bar{\alpha}-1}\, du \int_a^b |f(x-u)|\, dx \leqslant \frac{1}{\bar{\alpha}} \int_{a-1}^b |f(t)|\, dt,$$

whence

$$\int_0^1 |f(x-u)\, u^{\alpha-1}|\, du = \int_0^1 |f(x-u)|\, u^{\bar{\alpha}-1}\, du$$

exists almost everywhere in (a, b) and is integrable on (a, b). And, for each $c > 0$,

$$\int_1^c f(x-u)\, u^{\alpha-1}\, du$$

exists and is continuous for all x (the " Faltung " of an integrable function f and a bounded function).

Also, if $p>1$, $1/p'+1/p=1$, and $\bar{\alpha}>1/p$, then $(\bar{\alpha}-1)p'+1>0$ and

$$\int_0^1 |f(x-u)u^{\alpha-1}|\,du \leqslant \left(\int_0^1 |f(x-u)|^p\,du\right)^{1/p}\left(\int_0^1 u^{(\bar{\alpha}-1)p'}\,du\right)^{1/p'}.$$

the right-hand side of which is finite for all x if $|f|^p$ is L.

LEMMA 2. *If f is L, $\bar{\alpha}>0$, $\bar{\beta}>0$, and $c>0$, then*

$$\int_0^c v^{\beta-1}\,dv\int_0^{c-v} f(x-u-v)u^{\alpha-1}\,du = \frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(a+\beta)}\int_0^c f(x-t)t^{\alpha+\beta-1}\,dt$$

for all x of $T_{\bar{\alpha}+\bar{\beta}}(f)$.

Given such an x, the right-hand side may be written

$$\int_{x-c}^x f(s)(x-s)^{\alpha+\beta-1}\,ds\int_0^1(1-\lambda)^{\alpha-1}\lambda^{\beta-1}\,d\lambda \qquad (t=x-s)$$

$$=\int_{x-c}^x f(s)\,ds\int_s^x (r-s)^{\alpha-1}(x-r)^{\beta-1}\,dr \quad \left(\lambda=\frac{x-r}{x-s}\right)$$

$$=\int_{x-c}^x (x-r)^{\beta-1}\,dr\int_{x-c}^r f(s)(r-s)^{\alpha-1}\,ds,$$

by the absolute convergence of the former repeated integral (the right-hand side of the equation to be proved being absolutely convergent). The proof is completed by making the substitutions $r=x-v$, $s=x-u-v$.

LEMMA 3. *If f is L, $\bar{\gamma}>0$, and ω is an increasing function such that, for $c>0$ and all x,*

$$\left|\int_x^{x+c} f(t)\,dt\right| \leqslant \omega(c),$$

then, for $b>a>0$ and all x,

$$\left|\int_a^b f(x-u)u^{-\gamma}\,du\right| \leqslant \frac{\lfloor\gamma\rfloor}{\bar{\gamma}}\left(\int_0^\infty (a+t)^{-\bar{\gamma}}\,d\omega(t)+a^{-\bar{\gamma}}\omega(0)\right).$$

Given $b>a>0$ and x, define

$$F(u)=\int_{x-u}^{x-a} f(t)\,dt.$$

Then

$$\int_a^b f(x-u)\,u^{-\gamma}\,du = F(b)\,b^{-\gamma} - F(a)\,a^{-\gamma} + \gamma\int_a^b F(u)\,u^{-\gamma-1}\,du,$$

$$\left|\int_a^b f(x-u)\,u^{-\gamma}\,du\right| \leqslant \omega(b-a)\,b^{-\bar{\gamma}} + |\gamma|\int_a^b \omega(u-a)u^{-\bar{\gamma}-1}\,du$$

$$\leqslant \frac{|\gamma|}{\bar{\gamma}}\left(\omega(b-a)\,b^{-\bar{\gamma}} + \int_a^b \omega(u-a)\,d(-u^{-\bar{\gamma}})\right)$$

$$= \frac{|\gamma|}{\gamma}\left(\omega(0)\,a^{-\bar{\gamma}} + \int_a^b u^{-\bar{\gamma}}\,d\omega(u-a)\right),$$

these Stieltjes integrals existing since $u^{-\bar{\gamma}}$ is continuous and $\omega(u-a)$ increasing in (a, b). The required inequality follows by the substitution $u = a+t$, since ω is increasing.

LEMMA 4. *If $\bar{\gamma} > 0$, and the integral of f exists and is bounded, so that, for all x and y,*

$$\left|\int_x^y f(t)\,dt\right| \leqslant \omega_0,$$

then, for $b > a > 0$ and all x,

$$\left|\int_a^b f(x-u)\,u^{-\gamma}\,du\right| \leqslant \frac{|\gamma|}{\bar{\gamma}}\,\omega_0 a^{-\bar{\gamma}}.$$

For we may put $\omega(c) = \omega_0$ $(c \geqslant 0)$ in Lemma 3.

LEMMA 5. *If $\delta > 0$, ω is an increasing function, and*

$$\int_1^\infty t^{-\delta}\,d\omega(t)$$

is convergent, then

$$\int_0^\infty (c+t)^{-\delta}\,d\omega(t) \to 0 \quad (c \to \infty).$$

For, given $\epsilon > 0$, there is an $h > 1$ such that

$$\int_h^\infty t^{-\delta}\,d\omega(t) < \tfrac{1}{2}\epsilon;$$

and

$$\int_0^\infty (c+t)^{-\delta}\,d\omega(t) \leqslant c^{-\delta}\int_0^h d\omega(t) + \int_h^\infty t^{-\delta}\,d\omega(t) < \epsilon,$$

if $c^\delta > 2\big(\omega(h)-\omega(0)\big)/\epsilon$.

3. *Classes I_λ of functions of mean value zero.* These classes are defined, for $0 < \lambda \leqslant 1$, as follows :

A function f belongs to the class I_λ if f is L and there is an increasing function ω such that, for all $c > 0$ and all x,

$$\left| \int_x^{x+c} f(t)\,dt \right| \leqslant \omega(c)$$

and

$$\int_1^\infty t^{\lambda-1}\,d\omega(t)$$

is convergent.

It may, and will, always be supposed that $\omega(0) = 0$; but it is not required that $\omega(+0) = 0$.

The I_λ-condition is essentially a uniform condition on the order of magnitude of the differences of the integral of f for large increments of the variable; for the convergence condition restricts the order of magnitude of $\omega(c)$ for large (but only large) c. In this respect the I_λ-condition resembles a Lipschitz condition, but for large increments instead of small, on the integral of f. A condition of uniformity for all other increments is also implied.

It is easy to verify that:

Every function belonging to some class I_λ has mean value zero; in fact, for large c,

$$\omega(c) = o(c^{1-\lambda}) \ \ (0 < \lambda < 1), \quad \omega(c) = O(1) \ \ (\lambda = 1).$$

For
$$\omega(c) - \omega(1) \leqslant \sqrt{c}^{1-\lambda} \int_1^{\sqrt{c}} t^{\lambda-1}\,d\omega(t) + c^{1-\lambda} \int_{\sqrt{c}}^{c} t^{\lambda-1}\,d\omega(t)$$

$$= o(c^{1-\lambda}) \quad (0 < \lambda < 1).$$

The class I_λ contracts as λ increases.

The use of the (improper) Stieltjes integral in the definition of I_λ is largely a matter of formal convenience; the work can be done without it, but is more cumbersome. I am indebted to Mr. L. C. Young for suggesting this simplification.

The meaning of the I_λ-condition may become clearer if we notice that

$$\int_1^\infty t^{\lambda-1}\,d\omega(t)$$

is convergent for all $\lambda < \mu\ (<1)$, but for no $\lambda \geqslant \mu$, if $\omega(t)$ behaves like $t^{1-\mu}$ for large t; and is convergent for all $\lambda \leqslant \mu\ (<1)$, but for no $\lambda > \mu$, if $\omega(t)$ behaves like $t^{1-\mu}/(\log t)^2$ for large t. Thus a function f of L such that, for all $c > 0$ and all x,

$$\left| \int_x^{x+c} f(t)\,dt \right| \leqslant 1 + \frac{c^{1-\lambda}}{\left(\log(e+c)\right)^2}$$

belongs to I_λ.

Some properties of Weyl's formula for functions belonging to the classes I_λ are given by the following two theorems:

THEOREM 1. *If f is I_λ and $0 < \bar{\alpha} \leqslant \lambda < 1$, then $f_\alpha(x)$ exists for all x of $T_{\bar{\alpha}}(f)$, that is, for almost all x; f_α is L and has mean value zero, and is $I_{\lambda-\bar{\alpha}}$ if $\bar{\alpha} < \lambda$; and*

$$\frac{1}{\Gamma(\alpha)} \int_0^c f(x-u)\,u^{\alpha-1}\,du \to f_\alpha(x) \quad (c \to \infty)$$

uniformly for all x of $T_{\bar{\alpha}}(f)$.

Complementary to this, there is a function f of I_λ for which $f_\alpha(x)$ exists for no value of x at all if $0 < \lambda < \bar{\alpha} < 1$ [see § 5 (ii)].

THEOREM 2. *If f is I_λ, and $\bar{\alpha} > 0$, $\bar{\beta} > 0$, $\bar{\alpha}+\bar{\beta} \leqslant \lambda < 1$, then $(f_\alpha)_\beta(x)$ exists and is equal to $f_{\alpha+\beta}(x)$ for all x of $T_{\bar{\alpha}+\bar{\beta}}(f)$, that is, for almost all x; and*

$$\frac{1}{\Gamma(\beta)} \int_0^c f_\alpha(x-v)\,v^{\beta-1}\,dv \to f_{\alpha+\beta}(x) \quad (c \to \infty)$$

uniformly for all x of $T_{\bar{\alpha}+\bar{\beta}}(f)$.

The proof of Theorem 1 is as follows.

Given x of $T_{\bar{\alpha}}(f)$, and $b > a > 0$, Lemmas 1, 3 and 5 yield

$$\left| \int_0^b f(x-u)\,u^{\alpha-1}\,du - \int_0^a f(x-u)\,u^{\alpha-1}\,du \right| = \left| \int_a^b f(x-u)\,u^{\alpha-1}\,du \right|$$

$$= O\left(\int_0^\infty (a+t)^{\bar{\alpha}-1}\,d\omega(t) \right)$$

$$= O\left(a^{\bar{\alpha}-\lambda} \int_0^\infty (a+t)^{\lambda-1}\,d\omega(t) \right)$$

$$= o(a^{\bar{\alpha}-\lambda})$$

for large a, uniformly in x and b. Thus

$$\frac{1}{\Gamma(a)}\int_0^c f(x-u)\,u^{a-1}\,du$$

is convergent, uniformly throughout $T_{\bar{a}}(f)$, as $c\to\infty$. And this function is L for each $c>0$ (Lemma 1), hence its (almost everywhere uniform) limit $f_a(x)$ is also L.

Given $T>0$, define

$$F_T(x)=\frac{1}{T}\int_0^T f(x+s)\,ds.$$

Then
$$|F_T(x)|\leqslant\omega(T)/T,$$

and
$$\left|\int_x^{x+c}F_T(t)\,dt\right|=\left|\frac{1}{T}\int_x^{x+c}dt\int_0^T f(t+s)\,ds\right|$$

$$=\left|\frac{1}{T}\int_0^T ds\int_x^{x+c}f(s+t)\,dt\right|\leqslant\omega(c)$$

for $c>0$ and all x.

By the almost everywhere uniform convergence just established

$$\frac{\Gamma(a)}{T}\int_X^{X+T}f_a(x)\,dx=\lim_{c\to\infty}\frac{1}{T}\int_X^{X+T}dx\int_0^c f(x-u)\,u^{a-1}\,du,$$

$$=\frac{1}{T}\int_0^\infty u^{a-1}\,du\int_X^{X+T}f(x-u)\,dx$$

by the absolute convergence of this integral when the terminal ∞ is replaced by c. Therefore, using Lemmas 3 and 5,

$$\left|\frac{\Gamma(a)}{T}\int_X^{X+T}f_a(x)\,dx\right|=\left|\int_0^T F_T(X-u)\,u^{a-1}\,du+\int_T^\infty F_T(X-u)\,u^{a-1}\,du\right|$$

$$\leqslant\frac{\omega(T)}{T}\int_0^T u^{\bar{a}-1}\,du+\frac{|1-a|}{1-\bar{a}}\int_0^\infty(T+t)^{\bar{a}-1}\,d\omega(t)$$

$$\leqslant\frac{\omega(T)}{T}\,\frac{T^{\bar{a}}}{\bar{a}}+\frac{|1-a|}{1-\bar{a}}\,T^{\bar{a}-\lambda}\int_0^\infty(T+t)^{\lambda-1}\,d\omega(t)$$

$$=o(T^{\bar{a}-\lambda})+o(T^{\bar{a}-\lambda})$$

for large T, uniformly in X. Thus f_a has mean value zero,

It remains to show that f_a is $I_{\lambda-\bar{a}}$ when $\bar{a}<\lambda$. For this, replace X by x, T by c, and x by t in the inequality five lines above; whence, for $c>0$,

$$\left|\Gamma(a)\int_x^{x+c} f_a(t)\,dt\right| \leqslant \frac{c^{\bar{a}}}{\bar{a}}\,\omega(c)+\frac{|1-a|}{1-\bar{a}}\,c\int_0^\infty (c+t)^{\bar{a}-1}\,d\omega(t).$$

Let
$$\Omega(s)=\frac{s^{\bar{a}}}{\bar{a}}\,\omega(s)+s^{\bar{a}}\,\omega(1)+s\int_1^\infty (s+t)^{\bar{a}-1}\,d\omega(t) \qquad (s>0);$$

then
$$\left|\Gamma(a)\int_x^{x+c} f_a(t)\,dt\right| \leqslant \frac{|1-a|}{1-\bar{a}}\,\Omega(c) \qquad (c>0,\ \text{all } x),$$

and Ω is increasing (it is easily verified that the three terms of Ω are each increasing) throughout $s>0$. Therefore it is now sufficient to prove the boundedness for large X (from which follows the convergence as $X\to\infty$) of

$$\int_1^X s^{\lambda-\bar{a}-1}\,d\Omega(s)=J_1(X)+J_2(X)+J_3(X),$$

the three integrals on the right corresponding to the three terms of Ω.

To this end, we have immediately, for $X>1$,

$$\begin{aligned}
\bar{a}\,J_1(X) &= \int_1^X s^{\lambda-\bar{a}-1}\,d\left(s^{\bar{a}}\omega(s)\right)\\
&= \left[s^{\lambda-1}\omega(s)\right]_1^X+(1-\lambda+\bar{a})\int_1^X s^{\lambda-2}\omega(s)\,ds\\
&= \left(1-\frac{1-\lambda+\bar{a}}{1-\lambda}\right)\left[s^{\lambda-1}\omega(s)\right]_1^X+\frac{1-\lambda+\bar{a}}{1-\lambda}\int_1^X s^{\lambda-1}\,d\omega(s)\\
&= \frac{\bar{a}}{1-\lambda}\left(\omega(1)-X^{\lambda-1}\omega(X)\right)+\left(1+\frac{\bar{a}}{1-\lambda}\right)\int_1^X s^{\lambda-1}\,d\omega(s),
\end{aligned}$$

and
$$\frac{J_2(X)}{\omega(1)}=\int_1^X s^{\lambda-\bar{a}-1}\,d(s^{\bar{a}})=\bar{a}\int_1^X s^{\lambda-2}\,ds,$$

are bounded in $X>1$. Further,

$$\begin{aligned}
J_3(X) &= \int_1^X s^{\lambda-\bar{a}-1}\,d\left(s\int_1^\infty (s+t)^{\bar{a}-1}\,d\omega(t)\right)\\
&= \left[s^{\lambda-\bar{a}}\int_1^\infty (s+t)^{\bar{a}-1}\,d\omega(t)\right]_1^X+(1-\lambda+\bar{a})\int_1^X s^{\lambda-\bar{a}-1}\,ds\int_1^\infty (s+t)^{\bar{a}-1}\,d\omega(t),
\end{aligned}$$

of which the first term is a bounded function of X since

$$X^{\lambda-\bar{\alpha}}\int_1^\infty (X+t)^{\bar{\alpha}-1}d\omega(t) \leqslant \int_1^\infty (X+t)^{\lambda-1}d\omega(t) \leqslant \int_1^\infty t^{\lambda-1}d\omega(t) \quad (X>0).$$

By a Stieltjes inversion theorem on a finite rectangle, and monotonic convergence,

$$\begin{aligned}\int_1^X s^{\lambda-\bar{\alpha}-1}ds\int_1^\infty (s+t)^{\bar{\alpha}-1}d\omega(t) &= \int_1^\infty d\omega(t)\int_1^X s^{\lambda-\bar{\alpha}-1}(s+t)^{\bar{\alpha}-1}ds \qquad (X>1)\\ &\leqslant \int_1^\infty d\omega(t)\left(\int_0^t s^{\lambda-\bar{\alpha}-1}t^{\bar{\alpha}-1}ds+\int_t^\infty s^{\lambda-\bar{\alpha}-1}s^{\bar{\alpha}-1}ds\right)\\ &= \int_1^\infty d\omega(t)\left(\frac{t^{\lambda-\bar{\alpha}}}{\lambda-\bar{\alpha}}t^{\bar{\alpha}-1}+\frac{t^{\lambda-1}}{1-\lambda}\right)\\ &= \left(\frac{1}{\lambda-\bar{\alpha}}+\frac{1}{1-\lambda}\right)\int_1^\infty t^{\lambda-1}d\omega(t).\end{aligned}$$

Therefore J_1, J_2 and J_3 are bounded in $X>1$; and so the proof is complete.

To prove Theorem 2, consider the two integrals

$$\int_0^c v^{\beta-1}dv\int_0^{c-v} f(x-u-v)\,u^{\alpha-1}du, \qquad \int_0^c v^{\beta-1}dv\int_{c-v}^\infty f(x-u-v)\,u^{\alpha-1}du.$$

For $c>0$ and x of $T_{\bar{\alpha}+\bar{\beta}}(f)$, the former is convergent and equal to

$$\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}\int_0^c f(x-t)\,t^{\alpha+\beta-1}dt \qquad \text{(Lemma 2)}.$$

Also $f_{\alpha+\beta}(x)$ exists (Theorem 1); and it is easily shown, as in Theorem 1, that

$$\int_c^\infty f(x-t)\,t^{\alpha+\beta-1}dt = o(c^{\bar{\alpha}+\bar{\beta}-\lambda})$$

for large c, uniformly in x.

Again, for $c>0$, $v<c$, and any x, it can be shown as in Theorem 1 that

$$\int_{c-v}^\infty f(x-u-v)\,u^{\alpha-1}du$$

is convergent. By Lemma 3, the modulus of this integral does not exceed

$$\frac{|1-\alpha|}{1-\bar{\alpha}}\int_0^\infty (c-v+t)^{\bar{\alpha}-1}d\omega(t);$$

so, using a Stieltjes inversion theorem on a finite rectangle, and monotonic convergence, we have

$$\int_0^c v^{\bar\beta-1}dv\left|\int_{c-v}^{\infty} f(x-u-v)\,u^{\alpha-1}du\right| \leqslant \frac{|1-a|}{1-\bar a}\int_0^c v^{\bar\beta-1}dv\int_0^\infty (c-v+t)^{\bar\alpha-1}d\omega(t)$$

$$= \frac{|1-a|}{1-\bar a}\int_0^\infty d\omega(t)\int_0^c (c-v+t)^{\bar\alpha-1}v^{\bar\beta-1}dv$$

$$= O\left(\int_0^\infty (c+t)^{\bar\alpha-\bar\beta-1}d\omega(t)\int_0^1 (1-r)^{\bar\alpha-1}r^{\bar\beta-1}dr\right)$$

(substituting $v=(c+t)r$ in the inner integral and increasing its upper terminal)

$$= O\left(c^{\bar\alpha+\bar\beta-\lambda}\int_0^\infty (c+t)^{\lambda-1}d\omega(t)\right)$$

$$= o(c^{\bar\alpha+\bar\beta-\lambda})$$

for large c, uniformly in x, by Lemma 5. In particular, it follows that

$$\int_0^c v^{\beta-1}dv\int_{c-v}^{\infty} f(x-u-v)\,u^{\alpha-1}du$$

is convergent for all $c>0$ and all x, and is $o(c^{\bar\alpha+\bar\beta-\lambda})$ for large c uniformly in x.

Therefore, given $c>0$ and x of $T_{\bar\alpha+\bar\beta}(f)$,

$$\Gamma(a)\int_0^c f_a(x-v)\,v^{\beta-1}dv = \int_0^c v^{\beta-1}dv\int_0^\infty f(x-u-v)u^{\alpha-1}du$$

is convergent, and

$$\Gamma(\alpha+\beta)\left(f_{\alpha+\beta}(x)-\frac{1}{\Gamma(\beta)}\int_0^c f_\alpha(x-v)\,v^{\beta-1}dv\right)$$

$$= \int_c^\infty f(x-t)\,t^{\alpha+\beta-1}dt - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^c v^{\beta-1}dv\int_{c-v}^\infty f(x-u-v)\,u^{\alpha-1}du$$

$$= o(c^{\bar\alpha+\bar\beta-\lambda})$$

for large c, uniformly in x.

4. *Functions whose indefinite integral is bounded.* An interesting application of the preceding work is to the class I_1; this is the class of functions whose indefinite integral exists and is bounded. Since it is contained

in every class I_λ such that $\lambda < 1$, we deduce immediately from Theorems 1 and 2:

THEOREM 3. *If the integral of f exists and is bounded, and $0 < \bar{\alpha} < 1$, then $f_\alpha(x)$ exists for all x of $T_{\bar{\alpha}}(f)$, that is, for almost all x, and f_α is L. Also*

$$\frac{1}{\Gamma(\alpha)} \int_0^c f(x-u)\, u^{\alpha-1}\, du \to f_\alpha(x) \quad (c \to \infty)$$

uniformly for all x of $T_{\bar{\alpha}}(f)$. Further, f_α is I_μ for all $\mu < 1-\bar{\alpha}$.

But f_α may not be $I_{1-\bar{\alpha}}$; for otherwise $(f_\alpha)_{1-\alpha}(x)$ exists almost everywhere (Theorem 1), which is not generally true (§6).

THEOREM 4. *If the integral of f exists and is bounded, and $\bar{\alpha} > 0$, $\bar{\beta} > 0$, $\bar{\alpha}+\bar{\beta} < 1$, then $(f_\alpha)_\beta(x)$ exists and is equal to $f_{\alpha+\beta}(x)$ for all x of $T_{\bar{\alpha}+\bar{\beta}}(f)$, that is, for almost all x; and*

$$\frac{1}{\Gamma(\beta)} \int_0^c f_\alpha(x-v)\, v^{\beta-1}\, dv \to f_{\alpha+\beta}(x) \quad (c \to \infty)$$

uniformly for all x of $T_{\bar{\alpha}+\bar{\beta}}(f)$.

These theorems assert properties of Weyl's formula analogous to A and B; the question arises whether there is also an analogue of C. The methods of the preceding work fail to establish such an analogue, but this is only to be expected in view of the divergence of Weyl's formula for the index 1. The question is reduced to that of the existence of $(f_\alpha)_{1-\alpha}(x)$ for one single value of x by the following theorem:

THEOREM 5. *If the integral of f exists and is bounded, $0 < \bar{\alpha} < 1$, and there is a ξ such that $(f_\alpha)_{1-\alpha}(\xi)$ exists, then $(f_\alpha)_{1-\alpha}(x)$ exists for all x and*

$$(f_\alpha)_{1-\alpha}(x) - (f_\alpha)_{1-\alpha}(\xi) = \int_\xi^x f(t)\, dt.$$

However, mere existence and boundedness of the integral of f are not sufficient to establish the existence of $(f_\alpha)_{1-\alpha}$ (see §6). On the other hand, it is sufficient for the existence of $(f_\alpha)_{1-\alpha}$ that the integral of f should exist and be uniformly almost periodic: this will be proved in Theorem 8, independently of Theorem 5.

To prove Theorem 5, choose x and ξ and henceforth suppose them fixed. It will be sufficient to prove the existence of

$$\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)} \int_0^c v^{-\alpha}\, dv \int_{c-v}^\infty \Big(f(x-u-v) - f(\xi-u-v) \Big)\, u^{\alpha-1}\, du - \int_{\xi-c}^{x-c} f(s)\, ds$$

for positive c and that this tends to zero as c tends to infinity. For then we can argue as follows (asserting the existence of all integrals mentioned):

$$\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^c v^{-\alpha}dv\int_0^{c-v}\Big(f(x-u-v)-f(\xi-u-v)\Big)u^{\alpha-1}du$$

$$=\int_0^c\Big(f(x-t)-f(\xi-t)\Big)dt$$

$$=\int_{x-c}^{x}f(s)\,ds-\int_{\xi-c}^{\xi}f(s)\,ds$$

$$=\int_{\xi}^{x}f(s)\,ds-\int_{\xi-c}^{x-c}f(s)\,ds,$$

so

$$\frac{1}{\Gamma(1-\alpha)}\int_0^c\Big(f_\alpha(x-v)-f_\alpha(\xi-v)\Big)v^{-\alpha}dv-\int_{\xi}^{x}f(t)\,dt$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^c v^{-\alpha}dv\int_0^{\infty}\Big(f(x-v-u)-f(\xi-v-u)\Big)u^{\alpha-1}du-\int_{\xi}^{x}f(t)\,dt$$

$$=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^c v^{-\alpha}dv\int_{c-v}^{\infty}\Big(f(x-u-v)-f(\xi-u-v)\Big)u^{\alpha-1}du-\int_{\xi-c}^{x-c}f(s)\,ds,$$

which tends to zero as c tends to infinity; at the same time

$$\frac{1}{\Gamma(1-\alpha)}\int_0^c f_\alpha(\xi-v)\,v^{-\alpha}dv\to(f_\alpha)_{1-\alpha}(\xi).$$

Thus $(f_\alpha)_{1-\alpha}(x)$ exists and satisfies the equation stated.

We prove first the existence, and convergence to zero as c tends to infinity, of

$$\int_0^c v^{-\alpha}dv\int_c^{\infty}\Big(f(x-u-v)-f(\xi-u-v)\Big)u^{\alpha-1}du.$$

The integral obtained by replacing ∞ by a finite X $(>c)$ here is absolutely convergent, and the inner integral is convergent uniformly in v as $X\to\infty$ (Lemma 4); hence the repeated integral (with terminal ∞) exists and is equal to

$$\lim_{X\to\infty}\int_0^c v^{-\alpha}dv\int_c^{X}\Big(f(x-u-v)-f(\xi-u-v)\Big)u^{\alpha-1}du$$

$$=\int_c^{\infty}u^{\alpha-1}du\int_0^c\Big(f(x-u-v)-f(\xi-u-v)\Big)v^{-\alpha}dv.$$

Now

$$\int_1^Y dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du$$

$$= \int_1^Y dv \left(\int_{x-X}^{x} f(t-v)\, dt - \int_{\xi-X}^{\xi} f(t-v)\, dt \right)$$

$$= \int_1^Y dv \left(\int_{\xi}^{x} f(t-v)\, dt - \int_{\xi-X}^{x-X} f(t-v)\, dt \right)$$

$$= \int_{\xi}^{x} dt \int_1^Y f(t-v)\, dv - \int_{\xi-X}^{x-X} dt \int_1^Y f(t-v)\, dv,$$

which is a bounded function of X and Y. By partial integration, then,

$$\int_1^c v^{-\alpha} dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du$$

$$= c^{-\alpha} \int_1^c dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du$$

$$+ \alpha \int_1^c t^{-\alpha-1} dt \int_1^t dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du$$

$$= O\left(c^{-\bar{\alpha}} + |\alpha| \int_1^c t^{-\bar{\alpha}-1} dt \right) = O(1)$$

for all X, uniformly in $c > 1$. Also, for all X, independently of c,

$$\int_0^1 v^{-\alpha} dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du = O\left(\int_0^1 v^{-\bar{\alpha}} dv \right) = O(1).$$

Thus, by the absolute convergence of the second repeated integral below,

$$\int_0^X du \int_0^c \Big(f(x-u-v)-f(\xi-u-v)\Big)\, v^{-\alpha} dv$$

$$= \int_0^c v^{-\alpha} dv \int_0^X \Big(f(x-u-v)-f(\xi-u-v)\Big)\, du = O(1)$$

for all X, uniformly in $c > 1$.

Therefore we may integrate by parts, for $c > 1$, the integral

$$\int_c^\infty u^{\alpha-1} du \int_0^c \Big(f(x-u-v)-f(\xi-u-v)\Big)\, v^{-\alpha} dv$$

$$= -c^{\alpha-1} \int_0^c du \int_0^c \Big(f(x-u-v)-f(\xi-u-v)\Big)\, v^{-\alpha} dv$$

$$- (\alpha-1) \int_c^\infty t^{\alpha-2} dt \int_0^t du \int_0^c \Big(f(x-u-v)-f(\xi-u-v)\Big)\, v^{-\alpha} dv$$

$$= O\left(c^{\bar{\alpha}-1} + |1-\alpha| \int_c^\infty t^{\bar{\alpha}-2} dt \right) = O(c^{\bar{\alpha}-1}) \quad (c > 1).$$

Hence, for large c,

$$\int_0^c v^{-\alpha}\,dv\int_c^\infty \Big(f(x-u-v)-f(\xi-u-v)\Big)\,u^{\alpha-1}\,du = O(c^{\bar\alpha-1}).$$

It now remains to prove the existence, and convergence to zero as c tends to infinity, of

$$\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^c v^{-\alpha}\,dv\int_{c-v}^c \Big(f(x-u-v)-f(\xi-u-v)\Big)\,u^{\alpha-1}\,du-\int_{\xi-c}^{x-c} f(s)\,ds.$$

The double integral exists, by Lemma 4 applied to the inner integral, because of the convergence of

$$\int_0^c v^{-\bar\alpha}(c-v)^{\bar\alpha-1}\,dv.$$

By the substitution $u=st$, $v=s(1-t)$ the double integral becomes

$$\int_c^{2c}\Big(f(x-s)-f(\xi-s)\Big)\,ds\int_{1-c/s}^{c/s} t^{\alpha-1}(1-t)^{-\alpha}\,dt,$$

$$=\Big[\int_{x-s}^{\xi-s} f(r)\,dr\int_{1-c/s}^{c/s} t^{\alpha-1}(1-t)^{-\alpha}\,dt\Big]_c^{2c}$$

$$+\int_c^{2c}\left\{\left(\frac{s}{c}-1\right)^{-\alpha}+\left(\frac{s}{c}-1\right)^{\alpha-1}\right\}\frac{ds}{s}\int_{x-s}^{\xi-s} f(r)\,dr$$

by partial integration in which $\Big(f(x-s)-f(\xi-s)\Big)$ is the factor integrated. Thus

$$\int_0^c v^{-\alpha}\,dv\int_{c-v}^c\Big(f(x-u-v)-f(\xi-u-v)\Big)\,u^{\alpha-1}\,du-\Gamma(\alpha)\,\Gamma(1-\alpha)\int_{\xi-c}^{x-c} f(s)\,ds$$

$$=\int_c^{2c}\left(\left(\frac{s}{c}-1\right)^{-\alpha}+\left(\frac{s}{c}-1\right)^{\alpha-1}\right)\frac{ds}{s}\int_{x-s}^{\xi-s} f(r)\,dr$$

$$=\int_0^c\left(\left(\frac{t}{c}\right)^{-\alpha}+\left(\frac{t}{c}\right)^{\alpha-1}\right)\frac{dt}{c+t}\int_{x-c-t}^{\xi-c-t} f(r)\,dr.$$

Therefore it is enough to prove that, for the values α and $1-\alpha$ of β,

$$\int_0^c\left(\frac{c}{t}\right)^{\beta}\frac{dt}{c+t}\int_x^\xi f(r-c-t)\,dr\to 0\quad(c\to\infty).$$

By partial integration,

$$\int_1^c \left(\frac{c}{t}\right)^\beta \frac{dt}{c+t}\int_x^\xi f(r-c-t)\,dr$$

$$= \frac{1}{2c}\int_1^c ds\int_x^\xi f(r-c-s)\,dr + c^\beta \int_1^c \left(\frac{\beta t^{-\beta-1}}{c+t} + \frac{t^{-\beta}}{(c+t)^2}\right) dt \int_1^t ds \int_x^\xi f(r-c-s)\,dr.$$

But $$\int_1^t ds\int_x^\xi f(r-c-s)\,dr = \int_x^\xi dr \int_1^t f(r-c-s)\,ds$$

is a bounded function of t and c. Therefore, for large c,

$$\int_0^c \left(\frac{c}{t}\right)^\beta \frac{dt}{c+t}\int_x^\xi f(r-c-t)\,dr$$

$$= O\left(\int_0^1 \left(\frac{c}{t}\right)^{\bar\beta}\frac{dt}{c}\right) + O\left(\frac{1}{c} + c^{\bar\beta-2}\int_1^c (c|\beta|\,t^{-\bar\beta-1} + t^{-\bar\beta})\,dt\right)$$

$$= O\left(c^{\bar\beta-1} + c^{-1} + c^{\bar\beta-2}(c+c^{1-\bar\beta})\right)$$

$$= O(c^{\bar\beta-1}).$$

5. *Illustration of the preceding work.* The necessity of conditions such as those imposed in Theorems 1 and 2 is exhibited by the example discussed below, a real uniformly almost periodic function f of mean value zero depending on a parameter λ. Its properties are:

(i) *If $\lambda = 0$, then $f_\alpha(x)$ does not exist for any value of x or any value of α satisfying $0 < \bar\alpha < 1$.*

(ii) *If $0 < \lambda < 1$, then f is I_λ and consequently has the properties proved in Theorems 1 and 2; but $f_\alpha(x)$ does not exist for any value of x if $\lambda < \bar\alpha < 1$.*

(iii) *If $\lambda = 1$, then the integral of f is not only bounded but also uniformly almost periodic, so that f has the properties proved in Theorems 3, 4, and 5.*

This function f is defined by

$$f(x) = \sum_{m=1}^\infty \frac{1}{m^3}\sum_{n=1}^\infty \frac{\cos(x/n)}{n^{1+\lambda+1/m}}.$$

This double series is absolutely and uniformly convergent, for

$$\frac{1}{m^3}\sum_{n=1}^\infty \frac{1}{n^{1+\lambda+1/m}} \leqslant \frac{1}{m^3}\left(1+\int_1^\infty \frac{dt}{t^{1+\lambda+1/m}}\right) = \frac{1}{m^3} + \frac{1}{\lambda m^3 + m^2} \leqslant \frac{2}{m^2}.$$

Thus f is uniformly almost periodic. Further,

$$\int_0^x f(t)\,dt = \sum_{m=1}^\infty \frac{1}{m^3}\sum_{n=1}^\infty \frac{\sin(x/n)}{n^{\lambda+1/m}}$$

which is bounded if $\lambda = 1$ (consider this series with $\sin(x/n)$ replaced by 1); this gives (iii).

To establish (i) and (ii) it will be sufficient to prove that

(*a*) if $0 < \lambda < 1$, then f is I_λ,

(*b*) if $0 \leqslant \lambda < \bar{a} < 1$, then $f_a(x)$ exists for no value of x.

For (*a*) we require a lemma:

LEMMA 6. *If $\delta > 0$, there are positive numbers U_δ and V_δ such that, for any positive integer k and all $c \geqslant e^{(k+1)(1+\delta)}$,*

$$\frac{U_\delta}{(\log c)^\delta} < \sum_{n=k}^{\infty} \frac{1}{n^{1+\delta} c^{1/n}} < \frac{V_\delta}{(\log c)^\delta}.$$

Let $x = (1+\delta)^{-1} \log c \geqslant k+1$; then $n^{1+\delta} c^{1/n} = (ne^{x/n})^{1+\delta}$. For fixed x, $te^{x/t}$ is decreasing if $0 < t < x$ and increasing if $t > x$, with minimum value xe. Thus

$$\sum_{n \geqslant k} \frac{1}{n^{1+\delta} c^{1/n}} \geqslant \sum_{n \geqslant x-1} \frac{1}{(ne^{x/n})^{1+\delta}} > \int_x^\infty \frac{dt}{(te^{x/t})^{1+\delta}},$$

$$\sum_{n \geqslant k} \frac{1}{n^{1+\delta} c^{1/n}} \leqslant \sum_{n \leqslant x+1} \frac{1}{(ne^{x/n})^{1+\delta}} + \sum_{n > x+1} \frac{1}{(ne^{x/n})^{1+\delta}}$$

$$< \frac{2x}{(xe)^{1+\delta}} + \int_x^\infty \frac{dt}{(te^{x/t})^{1+\delta}}.$$

The inequalities to be proved follow from these by the substitution $t = (1+\delta)x/s$, which gives

$$\int_x^\infty \frac{dt}{(te^{x/t})^{1+\delta}} = \frac{1}{x^\delta (1+\delta)^\delta} \int_0^{1+\delta} s^{\delta-1} e^{-s}\, ds.$$

The proof of (*a*) is as follows. Suppose $0 < \lambda < 1$, and c large.

$$\left| \int_0^c f(x+t)\, dt \right| = \left| \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{1}{n^{\lambda+1/m}} \left(\sin \frac{x+c}{n} - \sin \frac{x}{n} \right) \right|$$

$$\leqslant \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{2}{n^{\lambda+1/m}} \left| \sin \frac{c}{2n} \right|$$

$$\leqslant \sum_{m=1}^{m'} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{2}{n^{\lambda+1/m}} + (1-\lambda)^3 \left(\sum_{n \leqslant c} \frac{2}{n} + \sum_{n > c} \frac{c}{n^2} \right)$$

$$+ \sum_{m=m''}^{\infty} \frac{1}{m^3} \left(\sum_{n \leqslant c} \frac{2}{n^{\lambda+1/m}} + \sum_{n > c} \frac{c}{n^{1+\lambda+1/m}} \right),$$

where m' and m'' are integers satisfying $m' < (1-\lambda)^{-1} < m''$, consecutive unless $(1-\lambda)^{-1}$ is integral in which case they differ from it by 1. The summation with respect to m is thus divided into three ranges, in which $\lambda+1/m$ is greater than, equal to, and less than 1 (the second range may not occur). In the first range

$$\sum_{n=1}^{\infty} \frac{2}{n^{\lambda+1/m}} \leqslant \sum_{n=1}^{\infty} \frac{2}{n^{\lambda+1/m'}} < \infty,$$

and in the third

$$\sum_{n \leqslant c} \frac{2}{n^{\lambda+1/m}} \leqslant \frac{2(c+1)^{1-\lambda-1/m}}{1-\lambda-1/m} \leqslant \frac{4c^{1-\lambda-1/m}}{1-\lambda-1/m''},$$

$$\sum_{n>c} \frac{c}{n^{1+\lambda+1/m}} \leqslant \frac{c(c-1)^{-\lambda-1/m}}{\lambda+1/m} \leqslant \frac{2c^{1-\lambda-1/m}}{\lambda}.$$

Hence, using Lemma 6, we have

$$\int_0^c f(x+t)\,dt = O(1)+O(\log c)+O\left(c^{1-\lambda} \sum_{m=m''}^{\infty} \frac{1}{m^3 c^{1/m}}\right) = O\left(\frac{c^{1-\lambda}}{(\log c)^2}\right)$$

uniformly in x. Thus there are positive c_0 and k, independent of x, such that

$$\left|\int_0^c f(x+t)\,dt\right| < k\,\frac{c^{1-\lambda}}{(\log c)^2} \quad (c \geqslant c_0).$$

Define an increasing function ω as follows: choose the above c_0 so large that $t^{1-\lambda}/(\log t)^2$ is increasing throughout $t \geqslant c_0$, and write

$$\omega(t) = h + kc_0^{1-\lambda}/(\log c_0)^2 \quad (0 < t < c_0),$$

$$\omega(t) = h + kt^{1-\lambda}/(\log t)^2 \quad (t \geqslant c_0),$$

where

$$h = c_0 \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda+1/m}}.$$

Then

$$\left|\int_x^{x+c} f(s)\,ds\right| = \left|\int_0^c f(x+t)\,dt\right| < \omega(c) \quad (c > 0);$$

and

$$\int_1^T t^{\lambda-1}\,d\omega(t) = \left[ht^{\lambda-1} + \frac{k}{(\log t)^2}\right]_{c_0}^{T} + (1-\lambda)\int_{c_0}^{T} \left(ht^{\lambda-2} + \frac{k}{t(\log t)^2}\right) dt$$

is bounded for large T. Thus f is I_λ.

For (*b*) several lemmas are needed.

LEMMA 7. *If f is bounded and is the integral of a function g of L, $0<\bar{\alpha}<1$, and there is a ξ such that $f_\alpha(\xi)$ exists, then $f_\alpha(x)$ exists for all x, and*

$$f_\alpha(x)-f_\alpha(\xi)=\int_\xi^x g_\alpha(t)\,dt.$$

For
$$\int_0^c \Big(f(x-u)-f(\xi-u)\Big)\,u^{\alpha-1}\,du=\int_0^c u^{\alpha-1}\,du\int_\xi^x g(t-u)\,dt$$
$$=\int_\xi^x dt\int_0^c g(t-u)\,u^{\alpha-1}\,du$$
$$\to\Gamma(\alpha)\int_\xi^x g_\alpha(t)\,dt\quad(c\to\infty),$$

since the inner integral is uniformly convergent almost everywhere to $\Gamma(\alpha)g_\alpha(t)$ (Theorem 3).

LEMMA 8. *If $0<\bar{\alpha}<1$, then*

$$\int_0^\infty t^{\alpha-1}\cos t\,dt=\Gamma(\alpha)\cos\tfrac{1}{2}\pi\alpha;$$

consequently there is a finite bound H_α such that, for all positive a and b,

$$\left|\int_a^b t^{\alpha-1}\cos t\,dt\right|\leqslant H_\alpha.$$

Proof by contour integration.

LEMMA 9. *If $0<\bar{\alpha}<1$ and $\bar{\gamma}>0$, then, for all $c>0$,*

$$\left|\sum_{n=1}^{\infty}(n+1)^{-\gamma}\int_{(c/n+1)}^{c/n} t^{\alpha-1}\cos t\,dt\right|\leqslant\left(1+\frac{|\gamma|}{\bar{\gamma}}\right)H_\alpha.$$

For, given integers M and N such that $0<M\leqslant N$,

$$\left|\sum_{n=M}^{N}(n+1)^{-\gamma}\int_{c/n+1}^{c/n} t^{\alpha-1}\cos t\,dt\right|$$
$$=\left|M^{-\gamma}\int_{(c/N+1)}^{c/M} t^{\alpha-1}\cos t\,dt+\sum_{n=M}^{N}\Big((n+1^{-\gamma}-n^{-\gamma})\Big)\int_{(c/N+1)}^{c/n} t^{\alpha-1}\cos t\,dt\right|$$
$$\leqslant H_\alpha\left(M^{-\bar{\gamma}}+\sum_{n=M}^{N}\left|\gamma\int_n^{n+1} t^{-\gamma-1}\,dt\right|\right)$$
$$\leqslant H_\alpha\left(M^{-\bar{\gamma}}+|\gamma|\int_M^\infty t^{-\bar{\gamma}-1}\,dt\right)$$
$$=H_\alpha\left(1+\frac{|\gamma|}{\bar{\gamma}}\right)M^{-\bar{\gamma}}.$$

LEMMA 10. *If* $\bar{\mu} \leqslant 1$ *and* $\mu \neq 0$, *then, for all* n,

$$\sum_{\nu=1}^{n} \nu^{\mu-1} = \frac{(n+1)^{\mu}-1}{\mu} - \frac{(n+1)^{\mu-1}-1}{2} + (\mu-1)(\mu-2)\left(h(\mu)+k_n(\mu)\, n^{\bar{\mu}-2}\right),$$

where $h(\mu)$ *and* $k_n(\mu)$ *have absolute values less than* 1.

We have identically

$$\sum_{\nu=1}^{n} \nu^{\mu-1} = \frac{(n+1)^{\mu}-1}{\mu} - \frac{(n+1)^{\mu-1}-1}{2} - \sum_{\nu=1}^{n} \left(\rho_\nu(\mu)-\sigma_\nu(\mu)\right),$$

where
$$\rho_\nu(\mu) = \frac{\nu^{\mu}}{\mu}\left\{\left(1+\frac{1}{\nu}\right)^{\mu} - 1 - \frac{\mu}{\nu} - \frac{\mu(\mu-1)}{2\nu^2}\right\}$$

and
$$\sigma_\nu(\mu) = \frac{\nu^{\mu-1}}{2}\left\{\left(1+\frac{1}{\nu}\right)^{\mu-1} - 1 - \frac{\mu-1}{\nu}\right\}.$$

The remainder after r terms of Taylor's series for $(1+x)^z$ in powers of x, where $x > 0$ and z is complex, is

$$\frac{z(z-1)\ldots(z-r+1)}{(r-1)!}\, x^r \int_0^1 (1-t)^{r-1}(1+xt)^{z-r}\, dt,$$

of which the modulus does not exceed $|z(z-1)\ldots(z-r+1)|\, x^r/r!$ if $\bar{z} \leqslant r$. Accordingly, since $\bar{\mu} \leqslant 3$ and $\bar{\mu}-1 \leqslant 2$,

$$|\rho_\nu(\mu)| \leqslant \tfrac{1}{6}|(\mu-1)(\mu-2)|\, \nu^{\bar{\mu}-3}, \quad |\sigma_\nu(\mu)| \leqslant \tfrac{1}{4}|(\mu-1)(\mu-2)|\, \nu^{\bar{\mu}-3},$$

and so
$$\sigma_\nu(\mu)-\rho_\nu(\mu) = (\mu-1)(\mu-2)\,\nu^{\mu-3}\tau_\nu(\mu),$$

where
$$|\tau_\nu(\mu)| \leqslant \tfrac{1}{6}+\tfrac{1}{4} < \tfrac{1}{2}.$$

Thus

$$\sum_{\nu=1}^{n} \left(\sigma_\nu(\mu)-\rho_\nu(\mu)\right) = (\mu-1)(\mu-2)\left(\sum_{\nu=1}^{\infty} \nu^{\mu-3}\tau_\nu(\mu) - \sum_{\nu=n+1}^{\infty} \nu^{\mu-3}\tau_\nu(\mu)\right);$$

this completes the lemma, for, since $\bar{\mu} \leqslant 1$, we have the following absolute bounds:

$$\left|\sum_{\nu=1}^{\infty} \nu^{\mu-3}\tau_\nu(\mu)\right| < \tfrac{1}{2}\sum_{\nu=1}^{\infty} \nu^{\bar{\mu}-3} \leqslant \tfrac{1}{2}\sum_{\nu=1}^{\infty} \nu^{-2} < 1,$$

$$n^{2-\bar{\mu}}\left|\sum_{\nu=n+1}^{\infty} \nu^{\mu-3}\tau_\nu(\mu)\right| < \tfrac{1}{2}n^{2-\bar{\mu}}\sum_{\nu=n+1}^{\infty} \nu^{\bar{\mu}-3} \leqslant \tfrac{1}{2}.$$

LEMMA 11. *If* $0<\eta<\bar{\alpha}<1$, *then, for all* $c>1$ *and uniformly in* η,

$$\sum_{n=1}^{\infty}\frac{(n+1)^{\alpha-\eta}}{\alpha-\eta}\int_{c/\overline{n+1}}^{c/n}t^{\alpha-1}\cos t\,dt=\frac{c^{\alpha-\eta}}{\alpha-\eta}\int_0^c t^{\eta-1}\cos t\,dt+O\left(\frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\eta}}\right).$$

Note first that if $b>a>0$ and $\bar{\mu}\leqslant 1$, then $|b^{\mu}-a^{\mu}|\leqslant(b-a)|\mu|a^{\bar{\mu}-1}$. Let

$$\begin{aligned}(\alpha-\eta)J_n&=(n+1)^{\alpha-\eta}\int_{c/\overline{n+1}}^{c/n}t^{\alpha-1}\cos t\,dt-c^{\alpha-\eta}\int_{c/\overline{n+1}}^{c/n}t^{\eta-1}\cos t\,dt\\&=c^{\alpha-\eta}\int_{c/\overline{n+1}}^{c/n}\left\{\left(\frac{c}{n+1}\right)^{\eta-\alpha}-t^{\eta-\alpha}\right\}t^{\alpha-1}\cos t\,dt,\\&=c^{\alpha-\eta}\left\{\left(\frac{c}{n+1}\right)^{\eta-\alpha}-\left(\frac{c}{n}\right)^{\eta-\alpha}\right\}\left(\frac{c}{n}\right)^{\alpha-1}\sin\frac{c}{n}\\&\quad+c^{\alpha-\eta}\int_{c/\overline{n+1}}^{c/n}\left[(1-\alpha)\left\{\left(\frac{c}{n+1}\right)^{\eta-\alpha}-t^{\eta-\alpha}\right\}+(\eta-\alpha)t^{\eta-\alpha}\right]t^{\alpha-2}\sin t\,dt\end{aligned}$$

by partial integration. From the not partially integrated form of J_n,

$$\begin{aligned}|(\alpha-\eta)J_n|&\leqslant c^{\bar{\alpha}-\eta}\left(\frac{c}{n}-\frac{c}{n+1}\right)|\eta-\alpha|\left(\frac{c}{n+1}\right)^{\eta-\bar{\alpha}-1}\left(\frac{c}{n+1}\right)^{\bar{\alpha}-1}\left(\frac{c}{n}-\frac{c}{n+1}\right)\\&=|\alpha-\eta|\frac{c^{\bar{\alpha}}}{n^2(n+1)^{\eta}};\end{aligned}$$

and, from the partially integrated form of J_n,

$$\begin{aligned}|(\alpha-\eta)J_n|&\leqslant c^{\bar{\alpha}-\eta}\left(\frac{c}{n}-\frac{c}{n+1}\right)|\eta-\alpha|\left(\frac{c}{n+1}\right)^{\eta-\bar{\alpha}-1}\\&\quad\times\left[\left(\frac{c}{n}\right)^{\bar{\alpha}-1}+|1-\alpha|\left(\frac{c}{n+1}\right)^{\bar{\alpha}-2}\left(\frac{c}{n}-\frac{c}{n+1}\right)\right]\\&\quad+c^{\bar{\alpha}-\eta}|\eta-\alpha|\left(\frac{c}{n+1}\right)^{\eta-2}\left(\frac{c}{n}-\frac{c}{n+1}\right)\\&\leqslant|\alpha-\eta|c^{\bar{\alpha}-1}\frac{(n+1)^{1-\eta}}{n}\left(1+\frac{|1-\alpha|}{n}+1\right)\\&\leqslant|\alpha-\eta|(2+|1-\alpha|)c^{\bar{\alpha}-1}\frac{(n+1)^{1-\eta}}{n}.\end{aligned}$$

Now, since

$$\sum_{n=1}^{N}\frac{(n+1)^{1-\eta}}{n}<\frac{4N^{1-\eta}}{1-\eta},\quad\sum_{n=N}^{\infty}\frac{1}{n^2(n+1)^{\eta}}<\frac{4}{N^{1+\eta}},$$

we have, for $c > 1$,

$$\left| \sum_{n=1}^{\infty} \frac{(n+1)^{\alpha-\eta}}{\alpha-\eta} \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt - \frac{c^{\alpha-\eta}}{\alpha-\eta} \int_0^c t^{\eta-1} \cos t\, dt \right|$$

$$= \left| \sum_{n=1}^{\infty} J_n \right| \leqslant \sum_{n<c/\log c} |J_n| + \sum_{n\geqslant c/\log c} |J_n|$$

$$< (2+|1-\alpha|)\, c^{\bar{\alpha}-1} \frac{4}{1-\eta} \left(\frac{c}{\log c}\right)^{1-\eta} + 4c^{\bar{\alpha}} \left(\frac{\log c}{c}\right)^{1+\eta}$$

$$< \left\{ \frac{4}{1-\bar{\alpha}} (2+|1-\alpha|) + 4 \frac{(\log c)^2}{c} \right\} \frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\eta}},$$

which proves the lemma, since $(\log c)^2/c$ is bounded for $c > 1$.

The proof of (b) may now be given. It is sufficient, by Lemma 7, to show that $f_\alpha(0)$ does not exist if $0 \leqslant \lambda < \bar{\alpha} < 1$; for f is bounded and is the integral of the bounded function

$$f'(x) = -\sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{\sin(x/n)}{n^{2+\lambda+1/m}}.$$

Therefore consider, for fixed λ and α such that $0 \leqslant \lambda < \bar{\alpha} < 1$, and $c > e$,

$$\int_0^c f(-u)\, u^{\alpha-1}\, du = \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda+1/m}} \int_0^c u^{\alpha-1} \cos\frac{u}{n}\, du = \sum_{m=1}^{\infty} \frac{1}{m^3} J_{\lambda+1/m}(c),$$

where

$$J_\eta(c) = \sum_{n=1}^{\infty} \frac{1}{n^{1+\eta-\alpha}} \int_0^{c/n} t^{\alpha-1} \cos t\, dt.$$

Let M be the least integer m such that $\lambda + 1/m < \bar{\alpha}$. By Lemma 8,

$$\left| \sum_{m=1}^{M-1} \frac{1}{m^3} J_{\lambda+1/m}(c) \right| \leqslant \sum_{m=1}^{M-1} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda+1/m-\bar{\alpha}}} \left| \int_0^{c/n} t^{\alpha-1} \cos t\, dt \right|$$

$$< M \sum_{n=1}^{\infty} \frac{1}{n} \left| \int_0^{c/n} t^{\alpha-1} \cos t\, dt \right|$$

$$= O\Big(\sum_{n<c} \frac{1}{n} \Big) + O\Big(\sum_{n\geqslant c} \frac{1}{n} \int_0^{c/n} t^{\bar{\alpha}-1} dt \Big)$$

$$= O\Big(\log c + c^{\bar{\alpha}} \sum_{n\geqslant c} n^{-1-\bar{\alpha}} \Big)$$

$$= O(\log c).$$

Now consider values of $m \geqslant M$, and put $\lambda+1/m=\eta$; then

$$0 \leqslant \lambda < \eta \leqslant \lambda+1/M < \bar{\alpha} < 1, \quad 0 < |\alpha-\lambda-1/M| \leqslant |\alpha-\eta| < |\alpha-\lambda| \leqslant |\alpha|.$$

By partial summation,

$$\sum_{n=1}^{N} n^{\alpha-\eta-1} \int_0^{c/n} t^{\alpha-1} \cos t\, dt$$

$$= \sum_{n=1}^{N} n^{\alpha-\eta-1} \int_0^{c/\overline{N+1}} t^{\alpha-1} \cos t\, dt + \sum_{\nu=1}^{N} \left(\sum_{n=1}^{\nu} n^{\alpha-\eta-1} \right) \int_{c/\overline{\nu+1}}^{c/\nu} t^{\alpha-1} \cos t\, dt.$$

For fixed c, the modulus of the former sum on the right does not exceed

$$\sum_{n=1}^{N} n^{\bar{\alpha}-\eta-1} \int_0^{c/N} t^{\bar{\alpha}-1}\, dt = O(N^{\bar{\alpha}-\eta} \cdot c^{\bar{\alpha}} N^{-\bar{\alpha}}) = O(N^{-\eta});$$

therefore

$$J_\eta(c) = \sum_{n=1}^{\infty} \left(\sum_{\nu=1}^{n} \nu^{\alpha-\eta-1} \right) \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt,$$

$$= \sum_{n=1}^{\infty} \left\{ \frac{(n+1)^{\alpha-\eta}-1}{\alpha-\eta} + \frac{1-(n+1)^{\alpha-\eta-1}}{2} \right.$$

$$\left. +(\alpha-\eta-1)(\alpha-\eta-2)\left(h(\alpha-\eta)+k_n(\alpha-\eta)\, n^{\bar{\alpha}-\eta-2}\right) \right\} \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt,$$

where $|h(\alpha-\eta)| < 1$, $|k_n(\alpha-\eta)| < 1$ (Lemma 10). Now $|\alpha-\eta|$ has positive bounds independent of η, and $\eta > 0$; so, by Lemmas 8 and 9, for all c and *uniformly in* η,

$$J_\eta(c) - \sum_{n=1}^{\infty} \frac{(n+1)^{\alpha-\eta}}{\alpha-\eta} \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt$$

$$= \left(\frac{-1}{\alpha-\eta} + \tfrac{1}{2} + (\alpha-\eta-1)(\alpha-\eta-2)\, h(\alpha-\eta) \right) \int_0^c t^{\alpha-1} \cos t\, dt$$

$$-\tfrac{1}{2} \sum_{n=1}^{\infty} (n+1)^{\alpha-\eta-1} \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt$$

$$+(\alpha-\eta-1)(\alpha-\eta-2) \sum_{n=1}^{\infty} k_n(\alpha-\eta)\, n^{\bar{\alpha}-\eta-2} \int_{c/\overline{n+1}}^{c/n} t^{\alpha-1} \cos t\, dt$$

$$= O\left(\frac{1}{|\alpha-\eta|}\right) + O\left(\frac{1}{1-\bar{\alpha}+\eta}\right) + O\left(\sum_{n=1}^{\infty} n^{\bar{\alpha}-\eta-2}\right) = O(1).$$

Therefore, by Lemma 11, for $c \geqslant e$ and uniformly in η,

$$J_\eta(c) = \frac{c^{\alpha-\eta}}{\alpha-\eta}\int_0^c t^{\eta-1}\cos t\,dt + O\left(\frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\eta}}\right) + O(1)$$

$$= \frac{c^{\alpha-\eta}}{\alpha-\eta}\,\Gamma(\eta)\cos\tfrac{1}{2}\pi\eta - \frac{c^{\alpha-\eta}}{\alpha-\eta}\int_c^\infty t^{\eta-1}\cos t\,dt + O\left(\frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\eta}}\right)$$

$$= \frac{c^{\alpha-\eta}}{\alpha-\eta}\,\Gamma(\eta)\cos\tfrac{1}{2}\pi\eta + O\left(\frac{c^{\bar{\alpha}-\eta+\eta-1}}{|\alpha-\eta|}\right) + O\left(\frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\eta}}\right)$$

$$= \frac{c^{\alpha-\eta}}{\alpha-\eta}\,\Gamma(\eta)\cos\tfrac{1}{2}\pi\eta + O\left(\frac{c^{\bar{\alpha}-\eta}}{(\log c)^{1-\bar{\alpha}}}\right),$$

by Lemmas 8 and 4.

If $\lambda > 0$, $\Gamma(x)\cos\frac{1}{2}\pi x/(\alpha-x)$ has a continuous derivative in $\lambda \leqslant x \leqslant \lambda+1/M$, and the bounds of this derivative in this interval depend only on α and λ; therefore

$$\frac{\Gamma(\lambda+1/m)}{\alpha-\lambda-1/m}\cos\tfrac{1}{2}\pi(\lambda+1/m) = \frac{\Gamma(\lambda)}{\alpha-\lambda}\cos\tfrac{1}{2}\pi\lambda + O\left(\frac{1}{m}\right),$$

$$\sum_{m=M}^{\infty}\frac{1}{m^3}J_{\lambda+1/m}(c)$$

$$= \frac{\Gamma(\lambda)}{\alpha-\lambda}\cos\tfrac{1}{2}\pi\lambda\sum_{m=M}^{\infty}\frac{c^{\alpha-\lambda}}{m^3 c^{1/m}} + \sum_{m=M}^{\infty}O\left(\frac{c^{\bar{\alpha}-\lambda}}{m^4 c^{1/m}}\right) + \sum_{m=M}^{\infty}O\left(\frac{c^{\bar{\alpha}-\lambda}(\log c)^{\bar{\alpha}-1}}{m^3 c^{1/m}}\right),$$

where O is independent of m and c; and so, by Lemma 6,

$$\int_0^c f(-u)\,u^{\alpha-1}\,du$$

$$= \frac{\Gamma(\lambda)}{\alpha-\lambda}\,c^{\alpha-\lambda}\cos\tfrac{1}{2}\pi\lambda\sum_{m=M}^{\infty}\frac{1}{m^3 c^{1/m}} + O\left(\frac{c^{\bar{\alpha}-\lambda}}{(\log c)^3}\right) + O\left(\frac{c^{\bar{\alpha}-\lambda}}{(\log c)^{3-\bar{\alpha}}}\right) + O(\log c)$$

for large c. In this expression the O-terms are of lower order than the other term, by Lemma 6; therefore the integral is divergent as $c \to \infty$.

If $\lambda = 0$, $\Gamma(1+x)\cos\frac{1}{2}\pi x/(\alpha-x)$ has a continuous derivative in $0 \leqslant x \leqslant 1/M$, the bounds of which depend only on α, and so

$$\frac{\Gamma(1/m)\cos\frac{1}{2}\pi/m}{\alpha-1/m} = \frac{\Gamma(1+1/m)\cos\frac{1}{2}\pi/m}{(1/m)(\alpha-1/m)} = m\left(\frac{\Gamma(1)}{\alpha} + O\left(\frac{1}{m}\right)\right),$$

$$\sum_{m=M}^{\infty}\frac{1}{m^3}J_{1/m}(c) = \frac{c^\alpha}{\alpha}\sum_{m=M}^{\infty}\frac{1}{m^2 c^{1/m}} + \sum_{m=M}^{\infty}O\left(\frac{c^{\bar{\alpha}}}{m^3 c^{1/m}}\right) + \sum_{m=M}^{\infty}O\left(\frac{c^{\bar{\alpha}}(\log c)^{\bar{\alpha}-1}}{m^3 c^{1/m}}\right),$$

where O is independent of m and c; and so, by Lemma 6 as before,

$$\int_0^c f(-u)\,u^{\alpha-1}du = \frac{c^\alpha}{\alpha}\sum_{m=M}^{\infty}\frac{1}{m^2 c^{1/m}} + O\left(\frac{c^{\bar\alpha}}{(\log c)^2}\right) + O\left(\frac{c^{\bar\alpha}}{(\log c)^{3-\bar\alpha}}\right) + O(\log c),$$

in which the O-terms are of lower order than the other (again by Lemma 6), and so the integral is divergent as $c\to\infty$.

6. "*Gegenbeispiel*" *for Theorem* 5. The example given below shows the necessity of the clause "*there is a* ξ *such that* $(f_\alpha)_{1-\alpha}(\xi)$ *exists*" in the hypothesis of Theorem 5. It is the real function

$$f(t) = \cos(\log|t|)/t \quad (|t|>1), \qquad f(t) = 0 \quad (|t|\leqslant 1),$$

of which the integral

$$F(x) = \int_0^x f(t)\,dt$$

is the bounded function

$$F(x) = \sin(\log|x|) \quad (|x|>1), \qquad F(x) = 0 \quad (|x|\leqslant 1).$$

To prove that $(f_\alpha)_{1-\alpha}(x)$ exists for no value of x it is sufficient (in view of Theorem 5) to prove that $(f_\alpha)_{1-\alpha}(0)$ does not exist.

First, $f_\alpha(x)$ exists everywhere and is continuous, being the uniform limit as $c\to\infty$ of

$$\frac{1}{\Gamma(\alpha)}\int_0^c f(x-u)\,u^{\alpha-1}\,du,$$

which is continuous (the "Faltung" of a bounded function f and an integrable function); the limit is uniform, by Theorem 1, since every x belongs to $T_{\bar\alpha}(f)$ (Lemma 1). Therefore

$$\int_0^c f_\alpha(-v)\,v^{-\alpha}\,dv$$

exists, for each $c>0$; our task is to prove that it is not convergent as $c\to\infty$.

The following decomposition is valid, by Lemma 2:

$$\int_0^c v^{-\alpha}\,dv\int_0^\infty f(-v-u)\,u^{\alpha-1}\,du = \Gamma(\alpha)\Gamma(1-\alpha)\int_0^c f(-t)\,dt + \int_0^c v^{-\alpha}\,dv\int_{c-v}^\infty f(-v-u)\,u^{\alpha-1}\,du.$$

Also, if $v<c$,

$$\int_{c-v}^\infty f(-v-u)\,u^{\alpha-1}\,du = \Big[-F(-v-u)\,u^{\alpha-1}\Big]_{c-v}^{\infty} - (1-\alpha)\int_{c-v}^\infty F(-v-u)\,u^{\alpha-2}\,du,$$

so

$$\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty} f(-v-u)\,u^{\alpha-1}\,du$$

$$= F(-c)\int_0^c v^{-\alpha}(c-v)^{\alpha-1}\,dv-(1-\alpha)\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty} F(-v-u)\,u^{\alpha-2}\,du,$$

the integral on the left and that first on the right being known to exist. Therefore

$$\int_0^c v^{-\alpha}\,dv\int_0^{\infty} f(-v-u)\,u^{\alpha-1}\,du$$

$$= \Gamma(\alpha)\,\Gamma(1-\alpha)\Big(-F(-c)+F(-c)\Big)-(1-\alpha)\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty} F(-v-u)\,u^{\alpha-2}\,du.$$

Hence it is sufficient to prove the divergence as $c\to\infty$ of

$$\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty} F(-u-v)\,u^{\alpha-2}\,du.$$

For fixed positive c, this integral is the limit as $k\to\infty$ of

$$\int_0^c v^{-\alpha}\,dv\int_{c-v}^{k-v} F(-u-v)\,u^{\alpha-2}\,du,$$

which is itself the limit as $h\to+0$ of

$$\int_0^c v^{-\alpha}\,dv\int_{c+h-v}^{k-v} F(-u-v)\,u^{\alpha-2}\,du;$$

for, for large k and small positive h respectively, and fixed c,

$$\int_0^c v^{-\bar\alpha}\,dv\int_{k-v}^{\infty} |F(-u-v)|\,u^{\bar\alpha-2}\,du = O\left(\int_0^c v^{-\bar\alpha}(k-v)^{\bar\alpha-1}\,dv\right) = O\Big((k-c)^{\bar\alpha-1}\Big)$$

$$= O(k^{\bar\alpha-1}),$$

$$\int_0^c v^{-\bar\alpha}\,dv\int_{c-v}^{c+h-v} |F(-u-v)|\,u^{\bar\alpha-2}\,du$$

$$= O\left(\int_0^c v^{-\bar\alpha}\Big((c-v)^{\bar\alpha-1}-(c+h-v)^{\bar\alpha-1}\Big)\,dv\right)$$

$$= O\left(\int_0^c v^{-\bar\alpha}(c-v)^{\bar\alpha-1}\,dv-\int_0^{c+h} v^{-\bar\alpha}(c+h-v)^{\bar\alpha-1}\,dv+\int_c^{c+h} v^{-\bar\alpha}(c+h-v)^{\bar\alpha-1}\,dv\right)$$

$$= O\left(\int_c^{c+h}(c+h-v)^{\bar\alpha-1}\,dv\right) = O(h^{\bar\alpha}).$$

Now, by the substitution $u = r(1-s)$, $v = rs$, we have, for fixed $c > 1$,

$$\int_0^c v^{-\alpha}\,dv \int_{c+h-v}^{k-v} F(-u-v)\,u^{\alpha-2}\,du = \int_{c+h}^{k} F(-r)\,\frac{dr}{r}\int_0^{c/r}(1-s)^{\alpha-2}s^{-\alpha}\,ds$$

$$= \int_{c+h}^{k} \frac{\sin(\log r)}{r}\,dr \int_0^{c/r}(1-s)^{\alpha-2}s^{-\alpha}\,ds,$$

which can be integrated by parts, the inner integral being differentiable in $c+h \leqslant r \leqslant k$, giving

$$\Big[(\cos\log c - \cos\log r)\int_0^{c/r}(1-s)^{\alpha-2}s^{-\alpha}\,ds\Big]_{c+h}^{k}$$

$$+ c^{1-\alpha}\int_{c+h}^{k}(\cos\log c - \cos\log r)(r-c)^{\alpha-2}\,dr.$$

But, if $0 < h < c$,

$$|\cos\log c - \cos\log(c+h)| \leqslant \log(1+h/c) = O(h),$$

$$\left|\int_0^{c/(c+h)}(1-s)^{\alpha-2}s^{-\alpha}\,ds\right| \leqslant \int_0^{\frac{1}{2}}(1-\tfrac{1}{2})^{\bar{\alpha}-2}s^{-\bar{\alpha}}\,ds + \int_{\frac{1}{2}}^{c/(c+h)}(1-s)^{\bar{\alpha}-2}(\tfrac{1}{2})^{-\bar{\alpha}}\,ds = O(h^{\bar{\alpha}-1}),$$

$$\Big(\cos\log c - \cos\log(c+h)\Big)\int_0^{c/(c+h)}(1-s)^{\alpha-2}s^{-\alpha}\,ds = O(h^{\bar{\alpha}}).$$

Let $h \to 0$, and then let $k \to \infty$:

$$\int_0^c v^{-\alpha}\,dv \int_{c-v}^{k-v} F(-u-v)\,u^{\alpha-2}\,du$$

$$= (\cos\log c - \cos\log k)\int_0^{c/k}(1-s)^{\alpha-2}s^{-\alpha}\,ds$$

$$+ c^{1-\alpha}\int_c^k (\cos\log c - \cos\log r)(r-c)^{\alpha-2}\,dr,$$

$$\int_0^c v^{-\alpha}\,dv \int_{c-v}^{\infty} F(-u-v)\,u^{\alpha-2}\,du$$

$$= c^{1-\alpha}\int_c^{\infty}(\cos\log c - \cos\log r)(r-c)^{\alpha-2}\,dr$$

$$= \int_0^{\infty}\Big(\cos\log c - \cos\log\Big(c(1+t)\Big)\Big)\,t^{\alpha-2}\,dt$$

$$= \cos\log c\int_0^{\infty}\Big(1-\cos\log(1+t)\Big)\,t^{\alpha-2}\,dt + \sin\log c\int_0^{\infty}\sin\log(1+t)\,t^{\alpha-2}\,dt.$$

This oscillates finitely as $c \to \infty$.

7. *Almost periodic functions.* For the application of Weyl's formula to functions almost periodic in Stepanoff's (second) sense [**4**; and **1**, ch. II, §2 ("S.a.p.")] further conclusions can be added to Theorems 1, 3, and 5, thus establishing properties in closer analogy with A, B, and C (§ 1). In fact both B and C persist unchanged under the more general hypothesis that f is S-almost periodic and its integral is bounded. The analogue of A shows, however, one essential difference: for periodic functions, A asserts that f_α has all the properties that are postulated of f, so that Weyl's formula can be applied to f_α, and so on repeatedly without restriction (other than that each α should satisfy $0 < \bar{\alpha} < 1$); but this is in general not true for almost periodic functions. For instance, if f is the function defined in the remark which follows the enunciation of Theorem 7 below, and $0 < \bar{\beta} < \bar{\alpha} < 1$, then the integral of f_α is not bounded, and $\big((f_\alpha)_{1-\alpha}\big)_\beta(x)$ exists for no x.

The notation of **1** is used. In particular, the distance-functions associated with the classes *Uap* (uniformly almost periodic functions) and *Sap* (Stepanoff's almost periodic functions) are

$$D_U(f) = \overline{\operatorname{bnd}}_{(x)} |f(x)|, \qquad D_S(f) = \overline{\operatorname{bnd}}_{(x)} \int_x^{x+1} |f(t)|\, dt,$$

and the mean value of an *Sap*-function f is

$$M(f) = M\big(f(t)\big) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt.$$

The specialised forms of Theorems 1 and 3 are as follows:

THEOREM 6. *If f is Sap and I_λ, and $0 < \bar{\alpha} \leqslant \lambda < 1$, then f_α exists almost everywhere and is Sap; also f_α is $I_{\lambda-\bar{\alpha}}$ if $\bar{\alpha} < \lambda$.*

But there is a *Uap* (and therefore *Sap*) function f of I_λ for which $f_\alpha(x)$ exists for no value of x at all if $0 < \lambda < \bar{\alpha} < 1$ [§5 (ii)].

THEOREM 7. *If f is Sap and its integral is bounded, and $0 < \bar{\alpha} < 1$, then f_α exists almost everywhere and is Sap and of mean value zero.*

But there is a *Uap* function

$$f(x) = -\sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{\sin(x/n)}{n^{2+1/m}},$$

whose integral

$$F(x) = \sum_{m=1}^{\infty} \frac{1}{m^3} \sum_{n=1}^{\infty} \frac{\cos(x/n)}{n^{1+1/m}}$$

is bounded, such that the integral of f_α is bounded for no α satisfying $0<\bar{\alpha}<1$. For otherwise, if $0<\bar{\beta}<\bar{\alpha}<1$ and the integral of f_α is bounded, then $\left((f_\alpha)_{1-\alpha}\right)_\beta$ exists almost everywhere (Theorem 4); which is false, for $(f_\alpha)_{1-\alpha}=F$ (Theorem 8) and $F_\beta(x)$ exists for no x [§ 5 (i)].

The proof of Theorem 6 is as follows. Given $c>0$, x, and τ,

$$\int_x^{x+1}\left|\int_0^c f(t+\tau-u)\,u^{\alpha-1}\,du-\int_0^c f(t-u)\,u^{\alpha-1}\,du\right|dt$$

$$\leqslant\int_x^{x+1}dt\int_0^c|f(t+\tau-u)-f(t-u)|\,u^{\bar{\alpha}-1}\,du$$

$$=\int_0^c u^{\bar{\alpha}-1}\,du\int_x^{x+1}|f(t+\tau-u)-f(t-u)|\,dt$$

$$\leqslant\frac{c^{\bar{\alpha}}}{\bar{\alpha}}\,D_S\Big(f(s+\tau)-f(s)\Big),$$

so

$$D_S\left(\int_0^c f(t+\tau-u)\,u^{\alpha-1}\,du-\int_0^c f(t-u)\,u^{\alpha-1}\,du\right)\leqslant\frac{c^{\bar{\alpha}}}{\bar{\alpha}}\,D_S\Big(f(t+\tau)-f(t)\Big).$$

Therefore $$\int_0^c f(x-u)\,u^{\alpha-1}\,du$$

is *Sap* for each $c>0$. Further

$$D_S\Big(f_\alpha(t)-\frac{1}{\Gamma(\alpha)}\int_0^c f(t-u)\,u^{\alpha-1}\,du\Big)\to 0\quad(c\to\infty),$$

on account of the uniform convergence almost everywhere to f_α (Theorem 1). Therefore f_α is *Sap*. This proves Theorem 6; and Theorem 7 follows at once, if we remember that f_α is I_μ for some μ (Theorem 3) and so has mean value zero.

8. *Property* C *for almost periodic functions.* Here the clause "*there is a* ξ *such that* $(f_\alpha)_{1-\alpha}(\xi)$ *exists*" in the hypothesis of Theorem 5 can be dispensed with. But the proof given below is independent of Theorem 5, and yields in addition a uniform convergence property of $(f_\alpha)_{1-\alpha}$ which the proof of Theorem 5 does not give. Note that the hypothesis is not less general than that of Theorem 7. (The proof that "if f is *Sap* and its integral is bounded then this integral is *Uap*" is almost identical with **1**,

ch. 1, § 1, 15°.) The theorem is:

THEOREM 8. *If the integral of f exists and is Uap, and $0 < \bar{a} < 1$, then $(f_a)_{1-a}(x)$ exists for all x and is equal to that integral of f which has mean value zero; and*

$$\frac{1}{\Gamma(1-a)} \int_0^c f_a(x-v)\, v^{-a}\, dv \to (f_a)_{1-a}(x) \quad (c \to \infty)$$

uniformly for all x.

Some preliminaries are needed before proving Theorem 8. Suppose that the hypotheses of this theorem hold.

Let F be that integral of f which has mean value zero. Write

$$A(\lambda) = M\left(F(t)\, e^{-i\lambda t}\right)$$

for all real λ. Let the Fourier series of F be

$$\sum_\nu A_\nu e^{i\lambda_\nu x},$$

where $A_\nu = A(\lambda_\nu) \neq 0$, and $A(\lambda) = 0$ for all other values (including 0) of λ.

Define formally, for all $c > 0$ and all x,

$$F_c(x) = \frac{1}{\Gamma(1-a)} \int_0^c f_a(x-v)\, v^{-a}\, dv$$

$$= \frac{1}{\Gamma(a)\,\Gamma(1-a)} \int_0^c v^{-a}\, dv \int_0^\infty f(x-u-v)\, u^{a-1}\, du.$$

LEMMA 12. *For each $c > 0$, $F_c(x)$ exists for all x and is Uap; further, the set of all these functions F_c is a homogeneous set of Uap functions* [**1**].

The existence of $F_c(x)$ follows from that of

$$\int_0^c v^{-a}\, dv \int_0^{c-v} f(x-u-v)\, u^{a-1}\, du \qquad \text{(Lemma 2)}$$

and that of the right-hand member of

$$\int_0^c v^{-\bar{a}}\, dv \left| \int_{c-v}^\infty f(x-u-v)\, u^{a-1}\, du \right| \leqslant 2 D_U(F)\, \frac{|1-a|}{1-\bar{a}} \int_0^c v^{-\bar{a}} (c-v)^{\bar{a}-1}\, dv$$

(Lemma 4).

Given x and τ, and $c>0$,

$$\int_0^c v^{-\alpha}\,dv\int_0^{c-v}\Big(f(x+\tau-u-v)-f(x-u-v)\Big)\,u^{\alpha-1}\,du$$

$$=\Gamma(\alpha)\,\Gamma(1-\alpha)\int_0^c\Big(f(x+\tau-t)-f(x-t)\Big)\,dt \qquad \text{(Lemma 2)}$$

$$=\Gamma(\alpha)\,\Gamma(1-\alpha)\left(\int_{x+\tau-c}^{x+\tau}f(s)\,ds-\int_{x-c}^{x}f(s)\,ds\right)$$

$$=\Gamma(\alpha)\,\Gamma(1-\alpha)\Big\{\Big(F(x+\tau)-F(x)\Big)-\Big(F(x-c+\tau)-F(x-c)\Big)\Big\};$$

also

$$\int_0^c v^{-\bar\alpha}\,dv\left|\int_{c-v}^{\infty}\Big(f(x+\tau-u-v)-f(x-u-v)\Big)\,u^{\alpha-1}\,du\right|$$

$$\leqslant 2D_U\Big(F(t+\tau)-F(t)\Big)\,\frac{|1-\alpha|}{1-\bar\alpha}\int_0^c v^{-\bar\alpha}(c-v)^{\bar\alpha-1}\,dv \qquad \text{(Lemma 4)}$$

$$=2D_U\Big(F(t+\tau)-F(t)\Big)\,\frac{|1-\alpha|}{1-\bar\alpha}\,\Gamma(\bar\alpha)\,\Gamma(1-\bar\alpha).$$

Hence $$|F_c(x+\tau)-F_c(x)|\leqslant K_\alpha D_U\Big(F(t+\tau)-F(t)\Big),$$

where K_α depends on α only.

Therefore, for all $c>0$ and all τ,

$$D_U\Big(F_c(t+\tau)-F_c(t)\Big)\leqslant K_\alpha D_U\Big(F(t+\tau)-F(t)\Big).$$

Since F is Uap, it follows that each F_c is Uap and that the set of all functions F_c is homogeneous in the sense asserted.

LEMMA 13. *If $c>0$, λ is real but not zero, and $g(\lambda)=\lambda/|\lambda|$, then*

$$\frac{1-\alpha}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du=\frac{e^{\frac{1}{2}\pi i(1-\alpha)\,g(\lambda)}}{\Gamma(1-\alpha)}\int_{|\lambda|c}^{\infty}t^{-\alpha}e^{-ig(\lambda)t}\,dt;$$

and if $\lambda=0$ the left hand side has the value 1.

The case $\lambda=0$ is immediate. Suppose that $\lambda\neq 0$.

By partial integration of the inner integral,

$$(1-\alpha)\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du$$

$$=e^{-i\lambda c}\int_0^c v^{-\alpha}(c-v)^{\alpha-1}\,dv-i\lambda\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-1}e^{-i\lambda u}\,du,$$

the integrals on the left and the first integral on the right being absolutely

convergent.

By contour integration, considering positive and negative λ separately, we have

$$\int_0^\infty u^{\alpha-1} e^{-i\lambda u}\, du = \frac{\Gamma(\alpha)}{(i\lambda)^\alpha},$$

the power taking its principal value. Using this and Lemma 2, we have

$$\int_0^c v^{-\alpha} e^{-i\lambda v}\, dv \int_{c-v}^\infty u^{\alpha-1} e^{-i\lambda u}\, du$$

$$= \frac{\Gamma(\alpha)}{(i\lambda)^\alpha} \int_0^c v^{-\alpha} e^{-i\lambda v}\, dv - \int_0^c v^{-\alpha}\, dv \int_0^{c-v} e^{-i\lambda(u+v)} u^{\alpha-1}\, du$$

$$= \frac{\Gamma(\alpha)}{(i\lambda)^\alpha} \left(\frac{\Gamma(1-\alpha)}{(i\lambda)^{1-\alpha}} - \int_c^\infty v^{-\alpha} e^{-i\lambda v}\, dv \right) - \Gamma(\alpha)\,\Gamma(1-\alpha) \int_0^c e^{-i\lambda t}\, dt$$

$$= \frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{i\lambda} - \frac{\Gamma(\alpha)}{i\lambda} \left(ig(\lambda) \right)^{1-\alpha} \int_{|\lambda| c}^\infty t^{-\alpha} e^{-ig(\lambda)t}\, dt + \frac{\Gamma(\alpha)\,\Gamma(1-\alpha)}{i\lambda} \left(e^{-i\lambda c} - 1 \right),$$

from which the conclusion follows, since $\log\left(ig(\lambda)\right) = \frac{1}{2}\pi i g(\lambda)$.

Lemma 14. *If $c > 0$, λ is real but not zero, and $g(\lambda) = \lambda/|\lambda|$, then*

$$M\left\{ \left(F(t) - F_c(t) \right) e^{-i\lambda t} \right\} = A(\lambda)\, \frac{e^{\frac{1}{2}\pi i (1-\alpha)\, g(\lambda)}}{\Gamma(1-\alpha)} \int_{|\lambda| c}^\infty t^{-\alpha} e^{-ig(\lambda)t}\, dt;$$

and, if $\lambda = 0$, the left hand side has the value 0.

First express $F(x) - F_c(x)$ as an absolutely convergent double integral, as follows. By partial integration of the inner integral,

$$\int_0^c v^{-\alpha}\, dv \int_{c-v}^\infty f(x-u-v)\, u^{\alpha-1}\, du$$

$$= F(x-c) \int_0^c v^{-\alpha} (c-v)^{\alpha-1}\, dv - (1-\alpha) \int_0^c v^{-\alpha}\, dv \int_{c-v}^\infty F(x-u-v)\, u^{\alpha-2}\, du$$

(the left hand member was proved convergent in Lemma 12); and, by Lemma 2,

$$\int_0^c v^{-\alpha}\, dv \int_0^{c-v} f(x-u-v)\, u^{\alpha-1}\, du = \Gamma(\alpha)\,\Gamma(1-\alpha) \int_0^c f(x-t)\, dt$$

$$= \Gamma(\alpha)\,\Gamma(1-\alpha) \left(F(x) - F(x-c) \right).$$

Therefore

$$F(x)-F_c(x)=\frac{1-\alpha}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}F(x-u-v)\,u^{\alpha-2}\,du.$$

Given $T>0$ and any real λ, we have

$$\int_0^T e^{-i\lambda t}\,dt\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}F(t-u-v)\,u^{\alpha-2}\,du$$

$$=\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}u^{\alpha-2}\,du\int_0^T F(t-u-v)\,e^{-i\lambda t}\,dt$$

$$=\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du\int_0^T F(t-u-v)\,e^{-i\lambda(t-u-v)}\,dt,$$

these integrals being absolutely convergent. Therefore

$$\left|A(\lambda)\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du\right.$$

$$\left.-\frac{1}{T}\int_0^T e^{-i\lambda t}\,dt\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}F(t-u-v)\,u^{\alpha-2}\,du\right|$$

$$=\left|\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du\left(A(\lambda)-\frac{1}{T}\int_0^T F(t-u-v)\,e^{-i\lambda(t-u-v)}\,dt\right)\right|$$

$$\leqslant\int_0^c v^{-\bar{\alpha}}\,dv\int_{c-v}^{\infty}u^{\bar{\alpha}-2}\,du\left|A(\lambda)-\frac{1}{T}\int_0^T F(t-u-v)\,e^{-i\lambda(t-u-v)}\,dt\right|.$$

But this last expression between modulus signs tends to zero as T tends to infinity uniformly in u and v [**1**, ch. 1, § 3, 2°]. So, as $T\to\infty$,

$$\frac{1}{T}\int_0^T e^{-i\lambda t}\,dt\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}F(t-u-v)\,u^{\alpha-2}\,du$$

$$\to A(\lambda)\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du.$$

Consequently

$$M\left\{\left(F(t)-F_c(t)\right)e^{-i\lambda t}\right\}$$

$$=\frac{1-\alpha}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\,M\left(e^{-i\lambda t}\int_0^c v^{-\alpha}\,dv\int_{c-v}^{\infty}F(t-u-v)u^{\alpha-2}\,du\right)$$

$$=A(\lambda)\,\frac{1-\alpha}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_0^c v^{-\alpha}e^{-i\lambda v}\,dv\int_{c-v}^{\infty}u^{\alpha-2}e^{-i\lambda u}\,du,$$

from which the required conclusions follow at once (by Lemma 13, and $A(0)=0$).

We can now prove, by Parseval's theorem for *Uap* functions, that F_c converges in mean square to F as c tends to infinity, and from this we infer Theorem 8. Let

$$h(x)=\frac{2e^{\pi|\alpha-\bar{\alpha}|}}{|\Gamma(1-\alpha)|^2}\left\{\left|\int_x^\infty t^{-\alpha}\cos t\,dt\right|^2+\left|\int_x^\infty t^{-\alpha}\sin t\,dt\right|^2\right\},$$

a function which has a finite upper bound H in $x\geqslant 0$ and is $O(x^{-2\bar{\alpha}})$ in $x>0$ (Lemma 4). By Parseval's theorem [**1**] applied to the *Uap* function $F-F_c$ (Lemma 12), and by Lemma 14,

$$M\{|F(t)-F_c(t)|^2\}=\sum_\nu\left|A_\nu\frac{e^{\frac{1}{2}\pi i(\bar{\alpha}-\alpha)g(\lambda_\nu)}}{\Gamma(1-\alpha)}\right|^2\left|\int_{|\lambda_\nu|c}^\infty t^{-\alpha}e^{-ig(\lambda_\nu)t}\,dt\right|^2$$

$$\leqslant\sum_\nu h(|\lambda_\nu|c)\,|A_\nu|^2.$$

Given $\epsilon>0$, there is a $\delta>0$ such that

$$\sum_{|\lambda_\nu|<\delta}|A_\nu|^2<\frac{\epsilon}{2H},$$

since the Parseval series of F is convergent and no λ_ν is zero. So

$$M\{|F(t)-F_c(t)|^2\}\leqslant H\sum_{|\lambda_\nu|<\delta}|A_\nu|^2+O\left(\sum_{|\lambda_\nu|\geqslant\delta}|\lambda_\nu c|^{2\bar{\alpha}}|A_\nu|^2\right)$$

$$=H\sum_{|\lambda_\nu|<\delta}|A_\nu|^2+O\left(c^{-2\bar{\alpha}}\delta^{-2\bar{\alpha}}\sum_\nu|A_\nu|^2\right)$$

$$<\epsilon$$

for all c sufficiently great.

Thus F_c converges in mean square to F; and (F_c) is a homogeneous set of *Uap* functions (Lemma 12). Therefore F_c converges *uniformly* to F [**1**, ch. 1, §8, 5°]; from which the conclusions of Theorem 8 follow immediately.

9. *Conclusion.* It may be convenient to summarise the properties of Weyl's formula for *Sap* functions whose integrals are bounded, as obtained in Theorems 7, 4, and 8, exhibiting them in parallel fashion to the properties for periodic functions as stated in §1. (This is, of course, not a summary of all the results of this paper.)

Suppose that f is Sap and that its integral is bounded. If

A′. $0 < \bar{\alpha} < 1$, *then $f_\alpha(x)$ exists for almost all x, and is Sap and of mean value zero.*

B′. $\bar{\alpha} > 0$, $\bar{\beta} > 0$, $\bar{\alpha}+\bar{\beta} < 1$, *then $(f_\alpha)_\beta(x)$ exists and equals $f_{\alpha+\beta}(x)$ for all x such that the latter exists.*

C′. $0 < \bar{\alpha} < 1$, *then $(f_\alpha)_{1-\alpha}(x)$ exists for all x and is that integral of f which has mean value zero.*

References.

1. Besicovitch, A. S., *Almost periodic functions* (Cambridge, 1932).
2. Hardy, G. H., and Littlewood, J. E., "Some properties of fractional integrals: I," *Math. Zeitschrift*, 27 (1928), 565–606.
3. Love, E. R., and Young, L. C., "On fractional integration by parts," *Proc. London Math. Soc.* (2), 44 (1938), 1–35.
4. Stepanoff, W., "Über einige Verallgemeinerungen der fastperiodischen Funktionen," *Math. Annalen*, 97 (1926–7), 338–356.
5. Weyl, H., "Bemerkungen zum Begriff des Differentialquotienten gebrochener Ordnung," *Vierteljahrsschrift der naturf. Ges. in Zürich*, 62 (1917), 296–302.

Trinity College,
Cambridge.

AN INTRODUCTION TO THE THEORY OF OPERATORS: III. COMPLEX OPERATORS

By S. W. P. STEEN.

[Received 16 August, 1937.—Read 18 November, 1937.]

This paper is a continuation of two previous papers† under the same title. A knowledge of these two papers is assumed. So far we have considered operators which are capable of combining by two processes (addition and multiplication) to form new operators. We now extend the domain of operator by introducing a second kind of multiplication, so that we shall now consider operators as combining by three processes. $A+B$, $A\,.\,B$, $A:B$, to form new operators. The possibility of introducing further processes of combination, $A:.B$, $A::B$, etc., may be mentioned here. We can represent the operators considered in this paper by $n\times n$ matrices with complex elements; real operators are then represented by Hermitian matrices; and the two types of multiplication then correspond to $A\,.\,B$ for $\frac{1}{2}(A\times B+B\times A)$, and $A:B$ for $1/(2i)(A\times B-B\times A)$. Both of these are Hermitian, so that $A\,.\,B$ and $A:B$ are real operators. If we wished to extend the domain of operator still further so that square matrices with, for instance, quaternions as elements‡ satisfied the axioms, then we should have to introduce further processes of combination to correspond with the laws of combination of quaternions. Complex operators, which are defined here as ordered pairs $[A, B]$ of real operators, correspond to non-Hermitian matrices.

We define the product of two complex operators, and this corresponds to the ordinary product of two matrices. This product we denote by $Z\times Z'$. The operators considered in the two former papers form a sub-class of those considered here. This sub-class is characterised by the fact that $A:B=O$ for any two members of the sub-class. As is shown below,

† Steen, *Proc. London Math. Soc.* (2), 41 (1936), 361–392; 43 (1937), 529–543.

‡ Teichmüller, *Journal für Math.*, 174 (1935), 73–124.

when $A:B=O$, then $A.B=A\times B=B\times A=B.A$ and

$$A.(B.C)=(A.B).C,$$

so that the notation of the two former papers does not clash with that used here.

The paragraphs are numbered following on to those of the former papers. In § XIX we list the new set of axioms. This set differs from the original set by containing the axioms for $A:B$, and it reduces to the original set† if we omit axiom IV*(ix) (which postulates the existence of two operators for which $A:B\neq O$) and add as an axiom $A:B=O$ always. In § XX we introduce complex operators in the manner already described, and prove some results about the convergence of series of complex operators.

Bounded linear symmetric transformations in a Hilbert space (of an enumerable or unenumerable number of dimensions) satisfy the axioms with the exception of V*(i). Furthermore, bounded linear transformations on vector-operators satisfy the axioms except V*(i). $f, g, \ldots$ are vector-operators if they combine by the same laws as vectors, but are capable of being multiplied on the left by operators belonging to a given normal association [see Def. (vii), § XX] and if their inner product is an operator belonging to this association‡.

Unbounded self-adjoint transformations in a Hilbert space do not satisfy the axioms. This follows at once since, if A and B are self-adjoint and unbounded, then $A+B$, $A.B$, and $A:B$ need not exist. In the next paper of the series we shall then retain V*(i) and drop the axioms II (i), IV* (ia), (ib), and add axioms concerning the existence of $A+B$, $A.B$, and $A:B$. We shall thus have an algebra which is semi-ordered, with addition and two kinds of multiplication and where these operations cannot always be carried out.

† Except for a trivial alteration to V*, and that we frame VI* for sequences only. The original VI could have been for sequences only, in which case we would define $E_+ = \text{l.u.b.}\,[\text{g.l.b.}\,(nA_+, I)]$. Since the distinction between the two cases is important, it is advisable to use the axiom in its weaker form, viz. for sequences only.

‡ In the proof we follow the work of Rellich, *Math. Annalen*, 110 (1934), 342–356; F. Riesz, *Act. Litt. ac. Sci. Szeged*, 5 (1930), 23–54; Wecken, *Math. Annalen*, 110 (1934), 722–725. Also, if A_n is a sequence of bounded transformations any two of which are permutable and if $A_n \leqslant A_{n+1}$, then we define l.u.b.(A_n) as the operator which is uniquely given by

$$\tfrac{1}{4}\lim_{n\to\infty}\left\{\Big(A_n(f+g),f+g\Big)-\Big(A_n(f-g),f-g\Big)+i\left[\Big(A_n(if+g),if+g\Big)-\Big(A_n(if-g),if-g\Big)\right]\right\}.$$

We add a definition which is useful:

Let $E = \text{g.l.b.}\ (\mathfrak{E})$, where $\mathfrak{E}$ is the set of projections, E', which satisfy $E'A = A$; then $EA = A$. E we shall call the rank of A. It is easily shown that $A+I-E$ is complete.

XIX. *Skew multiplication.*

Definition (i). A set of entities satisfying the axioms §I to VI, inclusive, will be called an association of real operators†. We now extend the idea of real operator by admitting the presence of different associations‡, the individual operators of different associations being capable of combining together by addition and by two kinds of multiplication to form new operators. One of the kinds of multiplication we call direct and the other one skew.

Definition (ii). Any one of a set of entities which satisfies the axioms I* to VI* of this paragraph is called a real operator. The axioms are as follows.

I*, II*, and III* of order, addition and subtraction are the same† as before.

IV*. Multiplication;

direct	skew
(ia) $A \,.\, B$ is an operator,	(ib) $A : B$ is an operator,

both of these being uniquely determined,

(ii$a$) $A \,.\, B = B \,.\, A$,	(ii$b$) $A : B = -B : A$,
(iii$a$) $A \,.\, (B+C) = A \,.\, B + A \,.\, C$,	(iii$b$) $A : (B+C) = A : B + A : C$,
(iv$a$) if $A = B$, then $A \,.\, C = B \,.\, C$,	(iv$b$) if $A = B$, then $A : C = B : C$,

(v) if $A \neq O$, $B \neq O$, then $2A : B < A^2 + B^2$ $(A^2 = A \,.\, A)$,

(vi) if $A : B = O$, $A > O$, $B > O$, then $A \,.\, B \geqslant O$,

(vii) $A : (B : C) = -(A \,.\, B) \,.\, C + (A \,.\, C) \,.\, B$,

† Corollary (i) of § I must be taken as an axiom. It can be deduced from axioms (i) and (ii) of § I only if we assume that to each operator A there exists another operator B such that $A = B$. We must also add to § II and § IV that $A+B$, $A \times B$ are uniquely determined by A and B.

‡ Note that, if A and B belong to the same association, then $A : B = O$, as will appear later.

(viii) $(A\,.\,B):C = A\,.\,(B:C)+B\,.\,(A:C)$,

(ix) if $A\,.\,B = A:B = O$, $A \neq O$, $B \neq O$, then there is an operator C such that $C:A \neq O$, $C:B \neq O$.

The last axiom merely postulates the existence of operators for which $A:B \neq O$.

COROLLARY (i). $A:A = O$.

COROLLARY (ii). $A:O = O$.

For take $C = O$ in IV*(iii*b*).

COROLLARY (iii). $(-A):B = -(A:B)$.

For take $C = -B$ in IV*(iii*b*).

COROLLARY (iv). *If* $A:B = B:C = O$, *then* $(A\,.\,B)\,.\,C = A\,.\,(B\,.\,C)$.

From IV*(vii) we obtain $(A:B):C = -A\,.\,(B\,.\,C)+(A\,.\,C)\,.\,B$; the result now follows from Corollary (ii) and IV*(vii).

COROLLARY (v). *If* $A:B = O$, *then* $A^n:B^m = O$. $(A^{n+1} = A\,.\,A^n)$.

For, if $A:B = B:A^n = O$, we obtain from IV*(viii)

$$A^{n+1}:B = (A\,.\,A^n):B = A\,.\,(A^n:B)+A^n\,.\,(A:B) = O;$$

the desired result now follows by induction.

COROLLARY (vi). *If* $A:B = O$, *then* $A^{n_1}\,.\,B^{m_1}\ldots A^{n_k}\,.\,B^{m_k}$ *is uniquely defined for integral values of* n_i, m_i.

For by Corollaries (iv) and (v) the associative law of multiplication holds between any three terms.

COROLLARY (vii). *If* $A:B = A:C = O$, *then*

$$A:(B \pm C) = A:(B^n\,.\,C^m) = O.$$

The first is immediate from IV*(iii*b*) and Corollary (iii). The second follows from IV*(viii) and Corollary (v),

$$(B^n\,.\,C^m):A = B^n\,.\,(C^m:A)+C^m\,.\,(B^n:A) = O.$$

COROLLARY (viii). $(A\,.\,B):C+(B\,.\,C):A+(C\,.\,A):B = O$,

and $A:(B:C)+B:(C:A)+C:(A:B) = O$.

These follow from IV*(vii) and (viii).

COROLLARY (ix). *If* $A:C=O$, *then* $A:(B:C)=(A:B):C$.

This follows from Corollary (viii).

The corollaries of § IV all hold provided that each pair of operators under consideration has its skew product zero.

V*. (i) If $A:C=O$ and if A is complete (*i.e.* $A:D=A\,.\,D=O$ implies $D=O$), then there is an operator B such that $A\,.\,B=C$. (ii) There is an operator I such that $A\,.\,I=A$ for all A.

The last axiom implies the existence of a complete operator, and states that the unit operator of any given association is the same as the unit operator of any other association.

COROLLARY (i). $A:I=O$.

From IV* (viii) with $A=B=I$ we obtain $I:C=2I\,.\,(I:C)$.

COROLLARY (ii). *If* A *is complete and* $A:B=O$, *then* $A^{-1}:B=O$.

We have

$$\begin{aligned}[A^{-1}\,.\,(A:B)]\,.\,A^{-1} &= [(A^{-1}\,.\,A):B-A\,.\,(A^{-1}:B)]\,.\,A^{-1}\\ &= -[A\,.\,(A^{-1}:B)]\,.\,A^{-1}\\ &= A:[(A^{-1}:B):A^{-1}]-(A\,.\,A^{-1})\,.\,(A^{-1}:B)\\ &= A:[(A^{-1}:B):A^{-1}]+B:A^{-1}.\end{aligned}$$

From IV* Corollary (ix) we have

$$\begin{aligned}A:[(A^{-1}:B):A^{-1}] &= [A:(A^{-1}:B)]:A^{-1}=-[A:(B:A^{-1})]:A^{-1}\\ &= -[(A:B):A^{-1}]:A^{-1}.\end{aligned}$$

Thus $$[A^{-1}\,.\,(A:B)]\,.\,A^{-1}-[A^{-1}:(A:B)]:A^{-1}=B:A^{-1},$$

and the result follows.

COROLLARY (iii). *If* A *is complete, then* $A:A^{-1}=O$.

For take $B=A$ in Corollary (ii).

COROLLARY (iv). *If* A *is complete,* $A:C=O$, *then there is a unique operator* B *such that* $A\,.\,B=C$, *and* $A:B=C:B=O$.

For we may take $B=A^{-1}\,.\,C$. Then, by hypothesis and by Corollaries (ii) and (iii), the skew product of any two of the operators A, A^{-1}, C is zero.

Hence by IV*, Corollary (vii), $A:B=C:B=O$. The uniqueness of B follows as in §V, Corollary (i).

COROLLARY (v). *If $A.D=O$ implies $D=O$, and if $A:C=O$, and if $A.B=C$, then $A:B=B:C=O$ and B is unique.*

For $$A^{-1}.(A.B)=-A:(B:A^{-1})+(A.A^{-1}).B,$$

i.e. $$A^{-1}.C=-A:(B:A^{-1})+B.$$

Hence $$C=-A.[A:(B:A^{-1})]+A.B;$$

thus $A.[A:(B:A^{-1})]=O$, and so $A:(B:A^{-1})=O$. But this gives $A^{-1}.C=B$, so that $A:B=B:C=O$. The uniqueness of B follows as before.

COROLLARY (vi). *If a is a rational number, then $(aA):B=a(A:B)$ and $(aA).B=a(A.B)$.*

From IV* (iii*a*), (iii*b*) the result follows if a is an integer. If $A=mC$ then

$$A:B=m(C:B)=m\left[\left(\frac{1}{m}A\right):B\right], \text{ etc.}$$

VI*. If B_n is a sequence of operators, the skew product of any two being zero, and if there is an operator K such that $B_n \leqslant K$, $B_n:K=O$, then there is an operator U such that

(i) $B_n \leqslant U$, for all n,

(ii) if $A:B_n=O$ for all n, then $A:U=O$,

(iii) if $V:B_n=V:U=O$ for all n and if $V \not\geqslant U$, then there is a B_m such that $B_m \not\leqslant V$.

As before, U will be called l.u.b. (B_n).

We note that, if we have a set of entities which satisfy all the axioms except IV* (ix) and if the skew product of any two of the set is zero, then the set forms an association. Thus axiom IV* (ix) postulates the existence of different associations. On the other hand a sub-set of real operators (*i.e.* a set of entities satisfying I* to VI*) containing a pair A, B for which $A:B\neq O$ cannot form an association, for we cannot deduce the existence of l.u.b. (A, B) from VI* when $A:B\neq O$. The corollaries of §VI hold provided that all operators under consideration belong to the same association.

COROLLARY (i). *If a is real, $(aI):A=O$.*

If $a > o$, $aI = \text{l.u.b.}\,(p/q\,I)$, for all $o < p/q \leqslant a$; for $(p/q\,I) : A = O$, and so the result follows. If $a < o$, $(aI) : A = -(-aI) : A$.

COROLLARY (ii). *$(aA) : B = a(A : B)$, and $(aA) \,.\, B = a(A \,.\, B)$.*

For

$$(aA) : B = (aI \,.\, A) : B = aI \,.\, (A : B) + A \,.\, (aI : B) = a(A : B).$$

Also $$O = A : (B : aI) = -(A \,.\, B) \,.\, aI + (A \,.\, aI) \,.\, B.$$

COROLLARY (iii). *If*

$$A : B = A : C = \ldots = A : K = O,$$

then

$$A : R(B, C, \ldots, K) = O,$$

where $R(B, C, \ldots, K)$ is a rational function of $B, C, \ldots, K$ with real numbers as coefficients.

From IV*, Corollary (vii), VI*, Corollary (i), (ii), and V*, Corollary (ii) the result follows.

COROLLARY (iv). *If E_λ is the resolution of the identity belonging to A and if $A : B = O$, then $B : E_\lambda = O$ for all λ, and conversely.*

This follows from §§ XII, XIII, and VI* (ii).

COROLLARY (v). *If E_λ, E_μ' are the resolutions belonging to A, B respectively and if $A : B = O$, then $E_\lambda : E_\mu' = O$ for all λ, μ, and conversely.*

COROLLARY (vi). *If $A \geqslant O$, $B \geqslant O$, $A : B = O$, $B^2 \leqslant B \,.\, A$, then $B \leqslant A$.*

If B is complete the result is immediate. If not, let E be the rank of B. [Then $E : B = O$, for $E : B = (E \,.\, E) : B = 2E \,.\, (E : B)$ from IV* (viii), and $E : B = E : (B \,.\, E) = -E \,.\, (B : E)$ from IV* (viii)]. Then $B + I - E$ is complete and $B/(B + I - E) = E$. Hence, on dividing by $B + I - E$, we obtain $B \leqslant E \,.\, A \leqslant A$, since $E : A = O$ by Corollary (iv) above.

COROLLARY (vii). *If A, B, C belong to the same association, and if for all X, Y of this association† we have $A \,.\, X^2 + 2B \,.\, XY + C \,.\, Y^2 \geqslant O$, then $A \geqslant O$, $C \geqslant O$ and $B^2 \leqslant A \,.\, C$, and conversely.*

Take $Y = O$, $X = I$, and then $A \geqslant O$; similarly $C \geqslant O$. Hence

$$A(AX^2 + 2BXY + CY^2) \geqslant O,$$

† For operators belonging to the same association we shall sometimes omit the sign of multiplication.

thus $$(AX+BY)^2+(AC-B^2)\,Y^2 \geqslant O.$$

Let E be the rank of A; take $Y=A$, $X=-B$, then $(AC-B^2)\,E \geqslant O$. Further, $D=(I-E)(2BXY+CY^2) \geqslant O$ for all X, Y. If $(I-E)\,C=O$, then clearly $(I-E)\,B=O$, and so $(AC-B^2)(I-E)=O$; and the desired result follows. If $(I-E)\,C>O$, let E' be the rank of $(I-E).\,C$; then, as above, $(I-E')(I-E)\,B=O$. Now, if $(I-E)\,B=O$, the desired result follows. Let† $E''(I-E)\,B>O\,(E'' \leqslant E')$, and let $I-E'''$ be the rank of B. Take $Y=E'\,E''$, $X=-2CE'/(2B+E''')$; then

$$D=(I-E)\,E'E''(-C)<O,$$

which is a contradiction. Hence $(I-E)\,B=O$, and the result follows.

The converse follows at once; for, if $A \geqslant O$, $C \geqslant O$, $B^2 \leqslant A\,.\,C$, then‡

$$AX^2+2BXY+CY^2=\left(\sqrt{A}X+\frac{B}{\sqrt{A+E}}\,Y\right)^2+\left(C-\frac{B^2}{A+E}\right)Y^2,$$

where $A(I-E)=A$. From $B^2 \leqslant A\,.\,C$, then $B(I-E)=B$, so that $\sqrt{A}B/\sqrt{A+E}=B$. Also

$$C-\frac{B^2}{A+E}=\frac{CA-B^2+CE}{A+E} \geqslant 0,$$

and the result follows.

COROLLARY (viii). *If, for all E of a set of projections $\mathfrak{E}$, we have $EA_E=A_E$ for some operator A_E depending on E, and if $E\,.\,E'=O$ for any two (different) elements of $\mathfrak{E}$, then there is an operator A such that $AE=A_E$ for each E of $\mathfrak{E}$.*

We first prove the result for $A_E \geqslant O$. We proceed exactly as in the proof of Lemma (xi) of § XVII. The only difference is that here the set A_E may not be denumerable. We shall write $A=\Sigma A_E$.

COROLLARY (ix). (*a*) *If, for all E of $\mathfrak{E}$, $A_E \leqslant B_E$, then $A \leqslant B$, the notation being as in Corollary* (viii). (*b*) $\Sigma(A_E+B_E)=A+B$. (*c*) *If $\Sigma A_E=\Sigma AE$, then $\Sigma A_E=A$, etc.*

COROLLARY (x). *If $\sum_1^\infty A_n E$ is convergent for each E of a set $\mathfrak{E}$, where* l.u.b. $(\mathfrak{E})=I$, *then $\sum_1^\infty A_n$ is convergent.*

† The argument is similar if $E''(I-E)\,B<O$.

‡ Here and elsewhere $\sqrt{A}$, where $A \geqslant 0$, is defined in § XII, and shown to be unique.

Given $H > O$, there is an E of $\mathfrak{E}$ such that $HE > O$. Since $\sum_1^\infty A_n E$ is convergent, there is a K, $O < K \leqslant HE \leqslant H$, such that

$$\left| \sum_n^{n+p} A_n E \right| K = \left| \sum_n^{n+p} A_n \right| K \leqslant K^2.$$

Thus, from Lemma (ii) of § XV, $\sum_1^\infty A_n$ is convergent.

COROLLARY (xi). *If to each $E > O$ there corresponds an E', $O < E' \leqslant E$, then, if $\mathfrak{E}'$ is the set of such E', we have* l.u.b. $(\mathfrak{E}') = I$.

Let $E'' = \text{l.u.b.}(\mathfrak{E}')$; if $E'' < I$, then there is an E' of $\mathfrak{E}$ such that $O < E' \leqslant I - E''$. This is a contradiction.

XX. *Complex operators.*

Definition (i). An ordered pair $[A, B]$ of operators, for which $=$, $+$, $\times$ are defined as below and for which $=$ obeys the rule $[A, O] = A$, is called a complex operator.

$[A, B] = [C, D]$ if and only if $A = C$ and $B = D$,

$[A, B] + [C, D] = [A + C, B + D]$,

$[A, B] \times [C, D) = [A \,.\, C - B \,.\, D - B : C - A : D, A \,.\, D + B \,.\, C + A : C - B : D]$.

Note that, if $B = D = O$, the rule $[A, O] = A$ agrees with the definitions of $=$ and of $+$, but that

$$[A, O] \times [B, O] = [A \,.\, B, A : B].$$

This we shall write as $A \times B = [A \,.\, B, A : B]$. Thus, if $A : B = O$, then $A \times B = A \,.\, B = B \times A$. We shall often write Z for a complex operator.

COROLLARY (i). *The relation of equality between complex operators is symmetric, reflexive, and transitive.*

COROLLARY (ii). *The law of addition of complex operators is commutative, associative, and distributive with the law of multiplication.*

COROLLARY (iii). *In any formula a complex operator may be replaced by an equal one.*

COROLLARY (iv). *Subtraction of complex operators is possible and the result is unique.*

COROLLARY (v). *If A, B, C are real, then $(A \times B) \times C = A \times (B \times C)$.*

We have to show that

$$[A\,.\,B,\ A:B]\times[C,\ O]=[(A\,.\,B)\,.\,C-(A:B):C,\ (A:B)\,.\,C+(A\,.\,B):C]$$
$$=[A,\ O]\times[B\,.\,C,\ B:C]=[A\,.\,(B\,.\,C)-A:(B:C),\ A\,.\,(B:C)+A:(B\,.\,C)],$$

i.e. that

$$(A\,.\,B)\,.\,C-(A:B):C=A\,.\,(B\,.\,C)-A:(B:C),$$

and
$$(A:B)\,.\,C+(A\,.\,B):C=A\,.\,(B:C)+A:(B\,.\,C).$$

The first of these follows from IV* (vii), and the second from IV* (viii). It is now convenient to write $[A,\ B]=A+iB$. Since

$$[A,\ B]=[A,\ O]+[O,\ B]=A+[O,\ I]\times B,$$

this notation amounts to writing i or iI for $[O,\ I]$.

COROLLARY (vi). *i permutes with every operator, and $i\times i=-I$,*

$$(iA)\times B=A\times(iB)=i(A\times B),\quad i(A+iB)=-B+iA.$$

COROLLARY (vii). $Z_1\times(Z_2\times Z_3)=(Z_1\times Z_2)\times Z_3$.

If $Z_1=A+iB$, etc., we have to show that

$$(A+iB)\times[C\times E-D\times F+i(D\times E+C\times F)]$$
$$=[A\times C-B\times D+i(A\times D+B\times C)]\times(E+iF).$$

This follows from corollaries (v) and (vi) above.

Definition (ii). If $A:B=O$, we call $Z=A+iB$ normal; otherwise we call it non-normal.

Definition (iii). If $Z=A+iB$, we call $Z^*=A-iB$ the adjoint of Z.

Definition (iv). $\sqrt{(Z^*\times Z)}=\sqrt{(A^2+B^2-2A:B)}=|Z|$ is called the modulus of Z. This exists by IV* (v). Note that $|Z|=|Z^*|$ if and only if Z is normal.

Definition (v). If $Z=A+iB$, we write $A=\Re Z$, $B=\Im Z$, and call them the real and imaginary parts of Z respectively.

Definition (vi). If $Z^*\times Z=I$ and if Z is normal, then we call Z unitary; and if Z is non-normal we call Z isometric.

COROLLARY (viii). *If $Z'\times Z=Z\times Z''=I$, then $Z'=Z''$.*

For
$$Z''=(Z'\times Z)\times Z''=Z'\times(Z\times Z'')=Z'.$$

COROLLARY (ix). *If* $|Z|$ *and* $|Z^*|$ *are complete, then*

$$Z^{-1} = Z^* \times \frac{1}{|Z^*|^2} = \frac{1}{|Z^2|} \times Z^*, \quad Z^{-1} \times Z = Z \times Z^{-1} = I.$$

We call Z^{-1} the inverse of Z.

COROLLARY (x). *If* $C:D = O$, *then there is a unitary operator* U, *whose real and imaginary parts belong to the same association as* C *and* D, *such that* $(C+iD) \times U = \sqrt{(C^2+D^2)}$.

Let E be the rank of C^2+D^2, and take

$$U = (C-iD) \times \frac{1}{\sqrt{(C^2+D^2)}+I-E} + I - E.$$

COROLLARY (xi). *If for all unitary operators* U *we have*

$$|\Re\{(C+iD)\,U\}| \leqslant K,$$

and if $C:D = O$, *then* $|C+iD| \leqslant K$.

Definition (vii). All operators of the form $A+iB$, where A and B belong to the same association of real operators, is said to form a normal association.

COROLLARY (xii). *If* Z_1 *and* Z_2 *belong to the same normal association, then*

$$|Z_1 Z_2| = |Z_1| \times |Z_2|, \quad |Z_1+Z_2| \leqslant |Z_1|+|Z_2|.$$

COROLLARY (xiii). *If* $V^* \times V = I$ *and if* $(I-V)^{-1}$ *exists, then a necessary and sufficient condition for* $H = i(I+V) \times (I-V)^{-1}$ *to be real is that* V *is unitary.*

Let $V = A+iB$ and let $(I-V)^{-1} = C+iD$; then

$$H = -iI + 2i(I-V)^{-1} = i(2C-I) - 2D.$$

Hence a necessary and sufficient condition for the reality of H is $C = \frac{1}{2}I$.

If V is unitary, then $A:B = O$ and $A^2+B^2 = I$. Hence

$$(I-V)^{-1} = \frac{I-A+iB}{(I-A)^2+B^2},$$

so that

$$C = \frac{I-A}{I-2A+I} = \tfrac{1}{2}I.$$

Again, if $C = \frac{1}{2}I$, then

$$I - V = \frac{\frac{1}{2}I - iD}{\frac{1}{4}I^2 + D^2},$$

whence

$$V = \frac{D^2 - \frac{1}{4}I + iD}{\frac{1}{4}I + D^2} = A + iB,$$

with $A : B = O$, $A^2 + B^2 = I$; whence V is unitary.

COROLLARY (xiv). *If H is real, then $H + iI$ has an inverse, and*

$$\frac{H - iI}{H + iI}$$

is unitary.

We have written

$$\frac{1}{H + iI} = (H + iI)^{-1} = \frac{1}{H^2 + I} \times (H - iI).$$

Then

$$\frac{H - iI}{H + iI} = \frac{H^2 - 2iH - I}{H^2 + I} = \frac{H^2 - I}{H^2 + I} - \frac{2iH}{H^2 + I} = A + iB, \quad \text{say.}$$

Thus $A : B = O$, $A^2 + B^2 = I$, and the result follows.

Definition (viii). If $Z_n = A_n + iB_n$ is a sequence of operators belonging to a normal association and if $A_n \to A$, $B_n \to B$, then we say that $Z_n \to A + iB$. Note that $A + iB$ belongs to the same normal association.

The remaining corollaries of this paragraph deal with sequences of normal operators belonging to the same normal association.

COROLLARY (xv). *A necessary and sufficient condition that $Z_n \to Z$ is that $|Z_n - Z| \to O$.*

If $Z_n \to Z$, then from the lemmas of § XV ($Z = A + iB$)

$$(A_n - A)^2 + (B_n - B)^2 \to O, \quad i.e. \ |Z_n - Z|^2 \to O,$$

hence† $|Z_n - Z| \to O$.

If $|Z_n - Z| \to O$, then $|Z_n - Z|^2 = (A_n - A)^2 + (B_n - B)^2 \to O$, and so‡ $A_n - A \to O$, and $B_n - B \to O$, *i.e.* $Z_n \to Z$.

† If $A_n > O$ and $A_n^2 \to O$, then $A_n \to O$. For given $H^2 > O$, there is a $K = EH^2 > O$, such that $O \leqslant A_n^2 EH^2 \leqslant EH^4$; hence $O \leqslant A_n EH \leqslant EH^2$, *i.e.* $A_n \to O$.

‡ If $O \leqslant A_n \leqslant B_n$ and $B_n \to O$, then $A_n \to O$. For $O \leqslant A_n K \leqslant B_n K \leqslant K^2$.

COROLLARY (xvi). *A necessary and sufficient condition for* Z_n *to tend to a limit is that* $|Z_n - Z_{n+p}| \to O$ *as* $n \to \infty$, *uniformly in* p.

This follows from Lemma (ii), § XV and the footnotes †, ‡ on p. 409.

COROLLARY (xvii). *If* $Z_n \to Z$, $Z_n' \to Z'$, *then* $Z_n \pm Z_n' \to Z \pm Z'$ *and* $Z_n \times Z_n' \to Z \times Z'$.

These follow from Lemmas (iv), (v), and (vii) of § XV.

Definition (ix). If $\sum_1^\infty |Z_n|$ is convergent, then we say that $\sum_1^\infty Z_n$ is absolutely convergent.

COROLLARY (xviii). *If* $\sum_1^\infty |Z_n|$ *is convergent, then so is* $\sum_1^\infty Z_n$.

A necessary and sufficient condition for $\Sigma |\sum_1^\infty Z_n|$ to be convergent is that, given $H > O$, there is a K, $O < K \leqslant H$, such that

$$\sum_n^{n+p} |Z_m| K \leqslant K^2,$$

for $n > n_0(H)$ and all $p \geqslant 0$.

Now, from Corollary (xii),

$$\left| \sum_n^{n+p} Z_m \right| K \leqslant \sum_n^{n+p} |Z_m| K \leqslant K^2.$$

From Corollary (xvi) we see that $\sum_1^\infty Z_n$ is convergent.

COROLLARY (xix). *If* $\sum_1^\infty |Z_n|^2$ *and* $\sum_1^\infty |Z_n'|^2$ *are both convergent, then*

$$\left| \sum_1^\infty Z_n \times Z_n' \right| \leqslant \sqrt{\left\{ \sum_1^\infty |Z_n|^2 \cdot \sum_1^\infty |Z_n'|^2 \right\}},$$

equality holding if and only if $Z_n' = Z \times Z_n$ *for some* Z.

We have $\sum_N^M \{|Z_n| + L|Z_n'|\}^2 \geqslant O$, for all real L belonging to the same normal association as Z_n, Z_n'. Hence, from VI*, Corollary (vii), § XIX,

$$\left\{ \sum_N^M |Z_n||Z_n'| \right\}^2 \leqslant \sum_N^M |Z_n|^2 \sum_N^M |Z_n'|^2.$$

Hence, given $H > O$, there is a K, $O < K \leqslant H$, such that

$$\left| \sum_{n}^{n+p} Z_n \times Z_n' \right| K \leqslant \sum_{n}^{n+p} |Z_n||Z_n'| K \leqslant \sqrt{\left\{ \sum_{n}^{n+p} |Z_n|^2 \sum_{n}^{n+p} |Z_n'|^2 \right\}} K \leqslant K^2.$$

Thus $\sum_{1}^{\infty} Z_n \times Z_n'$ is convergent. Further

$$\left| \sum_{0}^{n} Z_n \times Z_n' \right| \leqslant \sqrt{\left\{ \sum_{0}^{n} |Z_n|^2 \sum_{0}^{n} |Z_n'|^2 \right\}} \leqslant \sqrt{\left\{ \sum_{0}^{\infty} |Z_n|^2 \sum_{0}^{\infty} |Z_n'|^2 \right\}},$$

and the result follows from the definition of a limit.

Christ's College,
Cambridge.

ON THE PRODUCT OF THREE HOMOGENEOUS LINEAR FORMS (II)

By H. DAVENPORT.

[Received and read 20 January, 1938.]

1. *Introduction.*

Let L_1, L_2, L_3 be three real homogeneous linear forms in u, v, w with determinant 1. Let†

$$M = \min |L_1 L_2 L_3|$$

when u, v, w run through all sets of integral values except 0, 0, 0. It was proved by Minkowski‡ that

$$M \leqslant \tfrac{4}{19}. \tag{1}$$

In the previous paper§, I showed that this could be improved to

$$M \leqslant 8\{(3+\sqrt{2})\sqrt{(2\sqrt{2}-1)}+1\}^{-2} \quad \left(= \frac{1}{6{\cdot}07\ldots}\right).$$

In this paper it will be proved that

$$M \leqslant \tfrac{1}{7}. \tag{2}$$

This result is best possible. For if $\theta = 2\cos 2\pi/7$, $\phi = 2\cos 4\pi/7$, $\psi = 2\cos 6\pi/7$ are the roots of $t^3+t^2-2t-1=0$, then

$$L_1 = \lambda_1(\theta u+\phi v+\psi w),$$

$$L_2 = \lambda_2(\phi u+\psi v+\theta w),$$

$$L_3 = \lambda_3(\psi u+\theta v+\phi w),$$

---

† "min" means lower bound, whether attained or not.

‡ *Göttinger Nachrichten* (1904), 311–355, or *Ges. Abh.* II, 3–42.

§ *Journal London Math. Soc.*, 13 (1938), 139–145.

where $\lambda_1\lambda_2\lambda_3 = \frac{1}{7}$, are linear forms with determinant 1. For all integral values of u, v, w except 0, 0, 0 we have

$$|L_1 L_2 L_3| \geqslant \tfrac{1}{7},$$

since $7L_1 L_2 L_3$ is the norm of $\theta u+\phi v+\psi w$, which is an integer ($\neq 0$) of the field $\mathbf{P}(\theta)$.

Actually, rather more will be proved in this paper than the mere inequality (2). First, it will be proved that there exists an absolute constant $\mu < \frac{1}{7}$ such that either $M \leqslant \mu$ or $M = \frac{1}{7}$. Secondly, it will be proved that, if $M = \frac{1}{7}$, then L_1, L_2, L_3 are equivalent to the special forms given above.

I wish to express my thanks to Prof. Mordell for suggesting the problem to me, and to Dr. Heilbronn for some valuable suggestions in connection with some of the lemmas of § 3.

2. *Notation.*

θ, ϕ, ψ are the numbers already defined; their approximate numerical values are $\theta = 1{\cdot}24698\ldots$, $\phi = 0-{\cdot}44504\ldots$, $\psi = -1{\cdot}80194\ldots$. They are easily seen to satisfy

$$\theta\phi\psi = 1,$$

$$(1+\theta)(1+\phi)(1+\psi) = -1,$$

$$(-1+\theta)(-1+\phi)(-1+\psi) = 1,$$

$$(2+\theta)(2+\phi)(2+\psi) = 1.$$

ϵ is an arbitrarily small positive number, and ϵ_1, ϵ_2, ... are positive numbers *depending only on* ϵ which tend to zero with ϵ.

3. *Inequalities.*

Throughout this section x, y, z are real numbers such that

$$x \geqslant y \geqslant z \tag{3}$$

and

$$|(n+x)(n+y)(n+z)| > 1-\epsilon \tag{4}$$

for every integer n (positive, negative, or zero). These conditions on x, y, z are unaltered if x, y, z are replaced by $x+m$, $y+m$, $z+m$, or by $-z+m$, $-y+m$, $-x+m$, where m is any integer. Two sets related in

either of these ways will be called equivalent. Obviously $x-z$ is the same for two equivalent sets.

Any interval $(l, l+1)$, where l is an integer, which contains one of the numbers x, y, z will be called an I-interval.

LEMMA 1. *All three I-intervals cannot be consecutive.*

Proof. If all three I-intervals are consecutive, we can take them to be $(-2, -1)$, $(-1, 0)$, $(0, 1)$. By (4),

$$P_2 = (2+x)(2+y)(2+z) > 1-\epsilon,$$

$$P_1 = (1+x)(1+y)(1+z) < -1+\epsilon,$$

$$P_0 = xyz > 1-\epsilon,$$

$$P_{-1} = (-1+x)(-1+y)(-1+z) < -1+\epsilon.$$

But it is easily verified that

$$-P_2+3P_1-3P_0+P_{-1} = -6.$$

Hence $-6 < -8+8\epsilon$, which is a contradiction.

LEMMA 2. *If $x-z < 3+\frac{1}{5}$, the I-intervals are all different.*

Proof. If the I-intervals are not all different, there exists a set, equivalent to the given set, for which

$$0 < z \leqslant y < 1, \quad y \leqslant x.$$

By (4),

$$xyz > 1-\epsilon, \quad |x-1|(1-y)(1-z) > 1-\epsilon.$$

Let $t = \frac{1}{2}(y+z)$; then $0 < t < 1$ and

$$xt^2 > 1-\epsilon, \quad |x-1|(1-t)^2 > 1-\epsilon.$$

If $t < \frac{1}{2}$, then $x > 4-4\epsilon$ and $z < \frac{1}{2}$, whence $x-z > \frac{7}{2}-4\epsilon$. If $t \geqslant \frac{1}{2}$, then $x > 5-4\epsilon$ and $z < 1$, whence $x-z > 4-4\epsilon$.

LEMMA 3. *If $x-z < 3+\frac{1}{5}$, then $[x]-[z] = 3$.*

Proof. Plainly $[x]-[z] \leqslant 4$. By Lemmas 1, 2, $[x]-[z] \geqslant 3$. Suppose now that $[x]-[z] = 4$. Then there exists a set, equivalent to the given set, for which

$$0 < z < 1, \quad 1 < y \leqslant \tfrac{5}{2}, \quad 4 < x < 5.$$

By (4),

$$1-\epsilon < |(x-1)(y-1)(z-1)| < 4 \cdot \tfrac{3}{2} \cdot (1-z),$$

$$1-\epsilon < |(x-4)(y-4)(z-4)| < (x-4) \cdot 3 \cdot 4.$$

Hence $$x-z > 3+\tfrac{1}{6}(1-\epsilon)+\tfrac{1}{12}(1-\epsilon) > 3+\tfrac{1}{5},$$

contrary to hypothesis.

It follows from Lemmas 1, 2 and 3 that, if $x-z < 3+\frac{1}{5}$, then there exists an equivalent set for which the I-intervals are

$$(-2, -1), \quad (-1, 0), \quad (1, 2).$$

These are the same as the I-intervals of the set θ, ϕ, ψ. In the remainder of this section we shall be concerned only with x, y, z which satisfy (4) and

(3′) $$-2 < z < -1, \quad -1 < y < 0, \quad 1 < x < 2.$$

Such a set x, y, z will be called a *normal* set.

For a normal set, the inequalities (4) for $n = -1, 0, 1, 2$ can be written

(5) $$x(-y)(-z) > 1-\epsilon = (1-\epsilon)\theta(-\phi)(-\psi),$$

(6) $$(x-1)(1-y)(1-z) > 1-\epsilon = (1-\epsilon)(\theta-1)(1-\phi)(1-\psi),$$

(7) $$(x+1)(y+1)(-1-z) > 1-\epsilon = (1-\epsilon)(\theta+1)(\phi+1)(-1-\psi),$$

(8) $$(x+2)(y+2)(z+2) > 1-\epsilon = (1-\epsilon)(\theta+2)(\phi+2)(\psi+2),$$

the content of each bracket being positive.

LEMMA 4. *For any normal set we have* $x+y+z > -1-\epsilon$. *If*

$$x+y+z < -1+2\epsilon,$$

then $$\max(|x-\theta|, \quad |y-\phi|, \quad |z-\psi|) < \epsilon_1.$$

Proof. Adding (6) and (7), we obtain

(9) $$-2-2(xy+yz+zx) > 2-2\epsilon.$$

Adding and (7) (8),

(10) $$7+3(x+y+z)+(xy+yz+zx) > 2-2\epsilon.$$

These give $x+y+z > -1-\epsilon$.

If $x+y+z < -1+2\epsilon$, then (9) and (10) give

$$-2-8\epsilon < xy+yz+zx < -2+\epsilon.$$

From (5) and (7) we obtain

$$1-\epsilon < xyz < 1+10\epsilon.$$

Hence the cubic equation whose roots are x, y, z has the form

$$t^3+pt^2+qt+r = 0,$$

where $$\max(|p-1|, \quad |q+2|, \quad |r+1|) < 10\epsilon.$$

It follows that

$$\max(|x-\theta|, \quad |y-\phi|, \quad |z-\psi|) < \epsilon_1,$$

and this establishes the Lemma.

We now consider the possible relations of magnitude between x, y, z and θ, ϕ, ψ:

(A) $$x \leqslant \theta, \quad y \leqslant \phi, \quad z \leqslant \psi,$$

(B) $$x \leqslant \theta, \quad y \leqslant \phi, \quad z \geqslant \psi,$$

(C) $$x \leqslant \theta, \quad y \geqslant \phi, \quad z \leqslant \psi,$$

(D) $$x \leqslant \theta, \quad y \geqslant \phi, \quad z \geqslant \psi,$$

(E) $$x \geqslant \theta, \quad y \leqslant \phi, \quad z \leqslant \psi,$$

(F) $$x \geqslant \theta, \quad y \leqslant \phi, \quad z \geqslant \psi,$$

(G) $$x \geqslant \theta, \quad y \geqslant \phi, \quad z \leqslant \psi,$$

(H) $$x \geqslant \theta, \quad y \geqslant \phi, \quad z \geqslant \psi.$$

LEMMA 5. *In cases* (A), (B), (C), (D) *we have*

$$\max(|x-\theta|, \quad |y-\phi|, \quad |z-\psi|) < \epsilon_2.$$

Proof. In case (A), the result follows easily from (8) (or alternatively from Lemma 4). In case (B), the result follows easily from (7). In case (D), the result follows easily from (5) [or alternatively from (6)].

In case (C), we argue as follows. Let

$$x = \theta-\alpha, \quad y = \phi+\beta, \quad z = \psi-\gamma.$$

By Lemma 4,

(11) $$\beta > \alpha+\gamma-\epsilon.$$

By (5),

$$(\theta-\alpha)(-\phi-\beta)(-\psi+\gamma) > (1-\epsilon)\theta(-\phi)(-\psi),$$

whence $$(-\phi-\beta)(-\psi+\gamma) > (-\phi)(-\psi)(1-\epsilon),$$

and therefore

$$(-\phi)\gamma > (-\psi)\beta-(-\phi)(-\psi)\epsilon.$$

In view of the numerical values of ϕ and ψ, this implies that

$$\gamma > 3\beta-2\epsilon. \tag{12}$$

(11) and (12) prove that α, β, γ are all less than ϵ_2.

LEMMA 6. *In cases* (E), (G) *we have* (*trivially*) $x-z \geqslant \theta-\psi$. *If* $x-z < \theta-\psi+\frac{1}{30}$, *then*

$$\max(|x-\theta|,\ |y-\phi|,\ |z-\psi|) < \tfrac{1}{30}+\epsilon.$$

Proof. In case (E), write $x=\theta+\alpha$, $y=\phi-\beta$, $z=\psi-\gamma$; then

$$\alpha+\gamma < \tfrac{1}{30},$$

and, by Lemma 4, $\beta+\gamma < \alpha+\epsilon$. Hence α, β, γ are all less than $\frac{1}{30}+\epsilon$.

In case (G), write $x=\theta+\alpha$, $y=\phi+\beta$, $z=\psi-\gamma$. Then $\alpha+\gamma < \frac{1}{30}$. By (5),

$$(\theta+\alpha)(-\phi-\beta)(-\psi+\gamma) > (1-\epsilon)\theta(-\phi)(-\psi),$$

whence

$$1-\frac{\beta}{(-\phi)} > \frac{1-\epsilon}{\left(1+\frac{\alpha}{\theta}\right)\left(1+\frac{\gamma}{(-\psi)}\right)}$$

$$\geqslant (1-\epsilon)\left(1-\frac{\alpha}{\theta}-\frac{\gamma}{(-\psi)}\right)$$

$$\geqslant 1-\frac{\alpha}{\theta}-\frac{\gamma}{(-\psi)}-\epsilon.$$

Thus

$$\beta < \frac{(-\phi)}{\theta}\alpha+\frac{(-\phi)}{(-\psi)}\gamma+(-\phi)\epsilon$$

$$< \alpha+\gamma+\epsilon$$

$$< \tfrac{1}{30}+\epsilon.$$

This establishes the result.

LEMMA 7. *In case* (F), *we have* $x-z>\theta-\psi-3\epsilon$. *If*

$$x-z<\theta-\psi+\tfrac{1}{30},$$

then
$$\max(|x-\theta|,\ |y-\phi|,\ |z-\psi|)<\tfrac{1}{10}+\epsilon_3.$$

Proof. Write $x=\theta+\alpha,\ y=\phi-\beta,\ z=\psi+\gamma$. By Lemma 4

$$\beta<\alpha+\gamma+\epsilon.$$

By (7),

$$(\theta+1+\alpha)(-\psi-1-\gamma)>(1-\epsilon)(\theta+1)(-\psi-1),$$

whence
$$(-\psi-1)\alpha>(\theta+1)\gamma-(\theta+1)(-\psi-1)\epsilon.$$

Using the numerical values of θ and ψ, this implies

$$\alpha>2\gamma-3\epsilon.$$

Hence $x-z=\theta-\psi+\alpha-\gamma>\theta-\psi-3\epsilon$.

If $x-z<\theta-\psi+\frac{1}{30}$, then $\alpha-\gamma<\frac{1}{30}$, whence

$$\gamma<\alpha-\gamma+3\epsilon<\tfrac{1}{30}+3\epsilon,$$

$$\alpha<\gamma+\tfrac{1}{30}<\tfrac{2}{30}+3\epsilon,$$

$$\beta<\alpha+\gamma+\epsilon<\tfrac{3}{30}+7\epsilon.$$

LEMMA 8. *In case* (H), *we have* $x-z>\theta-\psi-\epsilon_4$. *If*

$$x-z<\theta-\psi+\tfrac{1}{30},$$

then
$$\max(|x-\theta|,\ |y-\phi|,\ |z-\psi|)<\tfrac{1}{10}+\epsilon_5.$$

Proof. Write $x=\theta+\alpha,\ y=\phi+\beta,\ z=\psi+\gamma$.

By (5),

$$(\theta+\alpha)(-\phi-\beta)(-\psi-\gamma)>(1-\epsilon)\theta(-\phi)(-\psi),$$

whence
$$1+\frac{\alpha}{\theta}>\frac{1-\epsilon}{\left(1-\dfrac{\beta}{(-\phi)}\right)\left(1-\dfrac{\gamma}{(-\psi)}\right)}$$

$$\geqslant(1-\epsilon)\left(1+\frac{\beta}{(-\phi)}+\frac{\gamma}{(-\psi)}\right)$$

$$>1+\frac{\beta}{(-\phi)}+\frac{\gamma}{(-\psi)}-\epsilon_6.$$

In view of the numerical values of θ, ϕ, ψ, this implies that

(13)
$$\cdot36\alpha>\beta+\cdot24\gamma-\epsilon_7.$$

By (7),

$$(\theta+1+\alpha)(\phi+1+\beta)(-\psi-1-\gamma) > (1-\epsilon)(\theta+1)(\phi+1)(-\psi-1),$$

whence

$$1-\frac{\gamma}{(-\psi-1)} > \frac{1-\epsilon}{\left(1+\frac{\alpha}{\theta+1}\right)\left(1+\frac{\beta}{\phi+1}\right)}$$

$$\geqslant (1-\epsilon)\left(1-\frac{\alpha}{\theta+1}-\frac{\beta}{\phi+1}\right)$$

$$\geqslant 1-\frac{\alpha}{\theta+1}-\frac{\beta}{\phi+1}-\epsilon.$$

In view of the numerical values of θ, ϕ, ψ, this gives

(14) $$\beta > \cdot 68\gamma - \cdot 25\alpha - \epsilon_8.$$

By (13), (14),

$$\cdot 61\alpha > \cdot 92\gamma - \epsilon_7 - \epsilon_8,$$

whence $$\alpha > \tfrac{3}{2}\gamma - \epsilon_9.$$

Hence $$x-z = \theta-\psi+\alpha-\gamma > \theta-\psi-\epsilon_9.$$

If $x-z < \theta-\psi+\frac{1}{30}$, then $\alpha-\gamma < \frac{1}{30}$, and so

$$\tfrac{1}{2}\gamma < \alpha-\gamma+\epsilon_9 < \tfrac{1}{30}+\epsilon_9,$$

$$\gamma < \tfrac{2}{30}+2\epsilon_9,$$

$$\alpha = \alpha-\gamma+\gamma < \tfrac{3}{30}+2\epsilon_9,$$

$$\beta < \cdot 36\alpha+\epsilon_7 \leqslant \alpha+\epsilon_7 < \tfrac{3}{30}+2\epsilon_9+\epsilon_7.$$

This completes the proof of Lemma 8.

Let $\epsilon_{10} = \max(2\epsilon_2, 3\epsilon, \epsilon_4)$, and $\epsilon_{11} = \max(\epsilon_3, \epsilon_5)$.

LEMMA 9. *For any normal set,*

$$x-z > \theta-\psi-\epsilon_{10};$$

and if

$$x-z < \theta-\psi+\tfrac{1}{30},$$

then

$$\max(|x-\theta|, |y-\phi|, |z-\psi|) < \tfrac{1}{10}+\epsilon_{11}.$$

Proof. Lemmas 5, 6, 7, 8.

4. *The main lemma.*

LEMMA 10. *If ξ, η, ζ are real numbers satisfying*

$$|(\xi+n)(\eta+n)(\zeta+n)| > 1-\epsilon$$

for every integer n, and also satisfying

$$|\xi-\eta| < \theta-\psi-\epsilon_{10}, \quad |\eta-\zeta| < \theta-\psi-\epsilon_{10}, \tag{15}$$

$$|\xi-\zeta| < \theta-\psi+\tfrac{1}{30}, \tag{16}$$

then there exist numbers ξ_1, η_1, ζ_1, which are either of the form

$$\xi_1 = \xi+m, \quad \eta_1 = \eta+m, \quad \zeta_1 = \zeta+m,$$

or of the form

$$\xi_1 = -\xi+m, \quad \eta_1 = -\eta+m, \quad \zeta_1 = -\zeta+m,$$

where m is an integer such that either

$$\max(|\xi_1-\theta|, |\eta_1-\phi|, |\zeta_1-\psi|) < \tfrac{1}{10}+\epsilon_{11},$$

or

$$\max(|\xi_1-\psi|, |\eta_1-\phi|, |\zeta_1-\theta|) < \tfrac{1}{10}+\epsilon_{11}.$$

Also ξ_1, η_1, ζ_1 satisfy

$$\xi_1+\eta_1+\zeta_1 > -1-\epsilon;$$

and, if

$$\xi_1+\eta_1+\zeta_1 < -1+2\epsilon$$

then either

$$\max(|\xi_1-\theta|, |\eta_1-\phi|, |\zeta_1-\psi|) < \epsilon_1,$$

or

$$\max(|\xi_1-\psi|, |\eta_1-\phi|, |\zeta_1-\theta|) < \epsilon_1.$$

Proof. ξ, η, ζ satisfy the conditions imposed on x, y, z at the beginning of §3, except for the condition $x \geqslant y \geqslant z$. Let ξ', η', ζ' be a permutation of ξ, η, ζ for which $\xi' \geqslant \eta' \geqslant \zeta'$. Since

$$\xi'-\zeta' = \max(|\xi-\eta|, |\eta-\zeta|, |\xi-\zeta|) < \theta-\psi+\tfrac{1}{30} < 3+\tfrac{1}{5},$$

the remark which follows Lemma 3 shows that there is a *normal* set ξ'', η'', ζ'' equivalent to ξ', η', ζ'. By Lemma 9,

$$\xi'-\zeta' = \xi''-\zeta'' > \theta-\psi-\epsilon_{10}.$$

Comparing this with (15), it is plain that either

$$\xi' = \xi, \quad \eta' = \eta, \quad \zeta' = \zeta, \quad \text{or} \quad \xi' = \zeta, \quad \eta' = \eta, \quad \zeta' = \xi.$$

Now ξ'', η'', ζ'' are either of the form

$$\xi'+m, \quad \eta'+m, \quad \zeta'+m,$$

or of the form

$$-\zeta'+m, \quad -\eta'+m, \quad -\xi'+m.$$

Hence they are of one of the forms

$$\begin{gathered}\xi+m, \quad \eta+m, \quad \zeta+m,\\ \zeta+m, \quad \eta+m, \quad \xi+m,\\ -\xi+m, \quad -\eta+m, \quad -\zeta+m,\\ -\zeta+m, \quad -\eta+m, \quad -\xi+m.\end{gathered}$$

In the first two cases we define

$$\xi_1=\xi+m, \quad \eta_1=\eta+m, \quad \zeta_1=\zeta+m,$$

and in the last two cases we define

$$\xi_1=-\xi+m, \quad \eta_1=-\eta+m, \quad \zeta_1=-\zeta+m.$$

Then either $\xi_1=\xi'', \quad \eta_1=\eta'', \quad \zeta_1=\zeta'',$

or $\xi_1=\zeta'', \quad \eta_1=\eta'', \quad \zeta_1=\xi''.$

Since ξ'', η'', ζ'' is a normal set, and since, by (16), $\xi''-\zeta''<\theta-\psi+\frac{1}{30}$, the assertions of Lemma 10 follow from Lemmas 9 and 4.

5. *The volume of a certain domain.*

LEMMA 11. *If* $\lambda>0$, $0<\delta<\lambda$, $\rho>0$, *then the inequalities*

$$\begin{gathered}|\xi+\eta+\zeta|<\rho,\\ |\xi-\eta|<\lambda, \quad |\eta-\zeta|<\lambda,\\ |\xi-\zeta|<\lambda+\delta\end{gathered}$$

define a convex domain in (ξ, η, ζ)*-space, of volume*

$$\rho(2\lambda^2+\tfrac{4}{3}\lambda\delta-\tfrac{2}{3}\delta^2).$$

Proof. The convexity is obvious. To calculate the volume, make the change of variable

$$X=\xi+\eta+\zeta, \quad Y=\xi-\eta, \quad Z=\eta-\zeta,$$

for which $\dfrac{\partial(X, Y, Z)}{\partial(\xi, \eta, \zeta)}=3, \quad \dfrac{\partial(\xi, \eta, \zeta)}{\partial(X, Y, Z)}=\frac{1}{3}.$

The required volume is $\frac{2}{3}\rho$ times the area

$$|Y|<\lambda, \quad |Z|<\lambda, \quad |Y+Z|<\lambda+\delta$$

in the (Y, Z)-plane. The half of this area for which $Z>0$ is

$$\begin{aligned}\int_{-\lambda}^{\lambda} \min(\lambda, \lambda+\delta-Y)\,dY &= \int_{-\lambda}^{\delta} \lambda\,dY+\int_{\delta}^{\lambda}(\lambda+\delta-Y)\,dY \\ &= \lambda(\lambda+\delta)+(\lambda+\delta)(\lambda-\delta)-\tfrac{1}{2}\lambda^2+\tfrac{1}{2}\delta^2 \\ &= \tfrac{3}{2}\lambda^2+\lambda\delta-\tfrac{1}{2}\delta^2.\end{aligned}$$

This establishes the result.

LEMMA 12. *For sufficiently small ϵ, the volume of the convex domain D_1 defined by*

$$|\xi+\eta+\zeta|<3,$$

$$|\xi-\eta|<\theta-\psi-\epsilon_{10}, \quad |\eta-\zeta|<\theta-\psi-\epsilon_{10},$$

$$|\xi-\zeta|<\theta-\psi+\tfrac{1}{30}$$

is greater than 56·1.

Proof. By Lemma 11, the volume is

$$3\{2(\theta-\psi-\epsilon_{10})^2+\tfrac{4}{3}(\theta-\psi-\epsilon_{10})(\tfrac{1}{30}+\epsilon_{10})-\tfrac{2}{3}(\tfrac{1}{30}+\epsilon_{10})^2\},$$

which tends to

$$6(\theta-\psi)^2+\tfrac{4}{30}(\theta-\psi)-\tfrac{2}{900}$$

as $\epsilon\to 0$. Since

$$6(\theta-\psi)^2>55\cdot71, \quad \tfrac{4}{30}(\theta-\psi)>\cdot4, \quad \tfrac{2}{900}<\cdot01,$$

we have the result.

6. *Proof of Theorem* 1.

THEOREM 1. *Let L_1, L_2, L_3 be three real homogeneous linear forms in u, v, w with determinant* 1. *Let*†

$$M=\min|L_1 L_2 L_3|$$

when u, v, w run through all sets of integral values except 0, 0, 0. *Then*

$$M\geqslant\frac{1}{7\cdot01} \quad \textit{implies} \quad M=\frac{1}{7}.$$

† See footnote † on p. 412.

Proof. For any $\epsilon > 0$ there exist integral values of u, v, w for which the corresponding values of L_1, L_2, L_3, say L_1^*, L_2^*, L_3^*, satisfy

$$M \leqslant |L_1^* L_2^* L_3^*| < \frac{M}{1-\epsilon}.$$

For all integral u, v, w except 0, 0, 0 we have

$$|L_1 L_2 L_3| \geqslant M,$$

and therefore

$$\left| \frac{L_1}{L_1^*} \cdot \frac{L_2}{L_2^*} \cdot \frac{L_3}{L_3^*} \right| > 1-\epsilon.$$

Write $\xi = L_1/L_1^*$, $\eta = L_2/L_2^*$, $\zeta = L_3/L_3^*$. The points in (ξ, η, ζ)-space which correspond to integral values of u, v, w form a lattice with the following properties:

(a) (0, 0, 0) and (1, 1, 1) are lattice points.

(b) For every lattice point except (0, 0, 0), $|\xi\eta\zeta| > 1-\epsilon$. It follows that for every lattice point not of the form (l, l, l), where l is an integer,

$$|(n+\xi)(n+\eta)(n+\zeta)| > 1-\epsilon$$

for every integer n.

(c) The determinant of the lattice has absolute value

$$\frac{1}{|L_1^* L_2^* L_3^*|} \leqslant \frac{1}{M} \leqslant 7{\cdot}01.$$

Since the domain D_1 of Lemma 12 has volume greater than 56·1, and

$$56{\cdot}1 > 8 \times 7{\cdot}01,$$

it follows from the classical theorem of Minkowski that D_1 contains a lattice point other than (0, 0, 0). This lattice point, say (ξ, η, ζ), cannot be of the form (l, l, l), where l is an integer. Hence, by (b), ξ, η, ζ satisfy the conditions of Lemma 10. Therefore there exists a lattice point (ξ_1, η_1, ζ_1) such that either

(17α) $$\max(|\xi_1-\theta|, |\eta_1-\phi|, |\zeta_1-\psi|) < \tfrac{1}{10}+\epsilon_{11},$$

or

(17β) $$\max(|\xi_1-\psi|, |\eta_1-\phi|, |\zeta_1-\theta|) < \tfrac{1}{10}+\epsilon_{11}.$$

These two possibilities are mutually exclusive, since

$$\theta-\psi > 2(\tfrac{1}{10}+\epsilon_{11}).$$

By the same argument, applied to the domain D_2 defined by

$$|\xi+\eta+\zeta| < 3,$$

$$|\eta-\xi| < \theta-\psi-\epsilon_{10}, \quad |\zeta-\xi| < \theta-\psi-\epsilon_{10},$$

$$|\eta-\zeta| < \theta-\psi+\tfrac{1}{30},$$

there exists a lattice point (ξ_2, η_2, ζ_2) such that either

$$\max(|\xi_2-\phi|, |\eta_2-\psi|, |\zeta_2-\theta|) < \tfrac{1}{10}+\epsilon_{11}, \tag{18α}$$

or

$$\max(|\xi_2-\phi|, |\eta_2-\theta|, |\zeta_2-\psi|) < \tfrac{1}{10}+\epsilon_{11}. \tag{18β}$$

Similarly, there exists a lattice point (ξ_3, η_3, ζ_3) such that either

$$\max(|\xi_3-\psi|, |\eta_3-\theta|, |\zeta_3-\phi|) < \tfrac{1}{10}+\epsilon_{11}, \tag{19α}$$

or

$$\max(|\xi_3-\theta|, |\eta_3-\psi|, |\zeta_3-\phi|) < \tfrac{1}{10}+\epsilon_{11}. \tag{19β}$$

It is plain that the three lattice points are all different, since

$$\min(|\theta-\phi|, |\phi-\psi|, |\psi-\theta|) > 2(\tfrac{1}{10}+\epsilon_{11}).$$

We now prove that, if (17α) holds, then neither (18β) nor (19β) holds. If (17α) and (18β) held, we should have

$$|\xi_1-\xi_2| < |\theta-\phi|+2(\tfrac{1}{10}+\epsilon_{11}),$$

$$|\eta_1-\eta_2| < |\theta-\phi|+2(\tfrac{1}{10}+\epsilon_{11}),$$

$$|\zeta_1-\zeta_2| < 2(\tfrac{1}{10}+\epsilon_{11}),$$

whence

$$\begin{aligned}|(\xi_1-\xi_2)(\eta_1-\eta_2)(\zeta_1-\zeta_2)| &< \{|\theta-\phi|+\tfrac{2}{10}+2\epsilon_{11}\}^2(\tfrac{2}{10}+2\epsilon_{11}) \\ &< (1{\cdot}25+{\cdot}45+{\cdot}2)^2({\cdot}2)+\epsilon_{12} \\ &< 1-\epsilon,\end{aligned}$$

and this contradicts (b), since $(\xi_1-\xi_2, \eta_1-\eta_2, \zeta_1-\zeta_2)$ is a lattice point other than (0, 0, 0). In the same way, if (17α) and (19β) held, we should have

$$\begin{aligned}|(\xi_1-\xi_3)(\eta_1-\eta_3)(\zeta_1-\zeta_3)| &< (|\phi-\psi|+\tfrac{2}{10}+2\epsilon_{11})^2(\tfrac{2}{10}+2\epsilon_{11}) \\ &< (-\cdot44+1\cdot81+\cdot2)^2(\cdot2)+\epsilon_{13} \\ &< 1-\epsilon,\end{aligned}$$

which is again contrary to (b).

Similarly, if (17β) holds, then neither (18α) nor (19α) holds. The proof is exactly the same, except that θ and ψ are interchanged.

Hence the three lattice points satisfy either (17α), (18α), (19α) or (17β), (18β), (19β).

Let (ξ_4, η_4, ζ_4) be the lattice point defined by

$$\xi_4=\xi_1+\xi_2+\xi_3, \quad \eta_4=\eta_1+\eta_2+\eta_3, \quad \zeta_4=\zeta_1+\zeta_2+\zeta_3.$$

In each of the two cases (α) and (β), we have (since $\theta+\phi+\psi=-1$)

$$\max(|\xi_4+1|, |\eta_4+1|, |\zeta_4+1|) < \tfrac{3}{10}+3\epsilon_{11}, \tag{20}$$

whence

$$|(\xi_4+1)(\eta_4+1)(\zeta_4+1)| < (\tfrac{3}{10}+3\epsilon_{11})^3 < 1-\epsilon.$$

Hence $\xi_4=\eta_4=\zeta_4=-1$, for otherwise this would contradict (a) and (b).

Let $s_i=\xi_i+\eta_i+\zeta_i$ $(i=1, 2, 3, 4)$. By Lemma 10,

$$s_i > -1-\epsilon \quad \text{for} \quad i=1, 2, 3.$$

Also $$s_1+s_2+s_3=s_4=-3.$$

Hence $$s_i < -1+2\epsilon \quad \text{for} \quad i=1, 2, 3.$$

Therefore, again by Lemma 10, the three lattice points satisfy either

$$\max\begin{Bmatrix} |\xi_1-\theta|, & |\eta_1-\phi|, & |\zeta_1-\psi|, \\ |\xi_2-\phi|, & |\eta_2-\psi|, & |\zeta_2-\theta|, \\ |\xi_3-\psi|, & |\eta_3-\theta|, & |\zeta_3-\phi| \end{Bmatrix} < \epsilon_1, \tag{21α}$$

or the corresponding inequality (21β) obtained by interchanging θ and ψ.

Let $$\Delta=\begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix}, \qquad D=\begin{vmatrix} \theta & \phi & \psi \\ \phi & \psi & \theta \\ \psi & \theta & \phi \end{vmatrix}.$$

It is easily seen that D is unaltered when θ and ψ are interchanged, and that $D = 7$. By (21α) or (21β), we have

$$|\Delta - D| < \epsilon_{14},$$

i.e.

$$|\Delta - 7| < \epsilon_{14}.$$

But, since (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), (ξ_3, η_3, ζ_3) are three lattice points, Δ is a multiple of the determinant of the lattice, *i.e.* of $1/|L_1^* L_2^* L_3^*|$. By (1) (or alternatively by Minkowski's more elementary result $M \leqslant \frac{2}{9}$), we have

$$|L_1^* L_2^* L_3^*| < \frac{\frac{2}{9}}{1-\epsilon},$$

whence

$$\frac{2}{|L_1^* L_2^* L_3^*|} > 9(1-\epsilon) > 7 + \epsilon_{14}.$$

Hence

$$\Delta = \frac{1}{|L_1^* L_2^* L_3^*|},$$

and

$$\left| \frac{1}{|L_1^* L_2^* L_3^*|} - 7 \right| < \epsilon_{14}.$$

Since

$$\frac{1-\epsilon}{M} < \frac{1}{|L_1^* L_2^* L_3^*|} \leqslant \frac{1}{M},$$

and ϵ is arbitrarily small†, this proves that $M = \frac{1}{7}$.

7. *Some further lemmas.*

In what follows, $c_1, c_2, \ldots$ denote suitable positive absolute constants.

LEMMA 13. *If $0 < \epsilon < c_1$, and if a lattice in (ξ, η, ζ)-space satisfies the conditions* (*a*), (*b*), (*c*) *in* § 6, *then there exist three generating lattice points*‡ (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), (ξ_3, η_3, ζ_3) *which satisfy*

$$(22) \qquad \begin{cases} \xi_1+\xi_2+\xi_3 = \eta_1+\eta_2+\eta_3 = \zeta_1+\zeta_2+\zeta_3 = -1, \\ -1-\epsilon < \xi_i+\eta_i+\zeta_i < -1+2\epsilon \quad (i = 1, 2, 3), \end{cases}$$

and which also satisfy either (21α) *or* (21β).

† As emphasized in § 2, $\epsilon_1, \epsilon_2, \ldots, \epsilon_{14}$ depend only on ϵ, and tend to zero with ϵ.

‡ *I.e.* three lattice points such that every lattice point is expressible in the form

$$(f\xi_1+g\xi_2+h\xi_3, \quad \eta_1+g\eta_2+h\eta_3, \quad f\zeta_1+g\zeta_2+h\zeta_3)$$

with integral f, g, h.

This is simply a restatement of what was proved in the course of the proof of Theorem 1.

LEMMA 14. *If* $0 < \epsilon < c_2$, *then Lemma* 4 *is valid with* $\epsilon_1 = c_3 \epsilon$.

Proof. In the proof of Lemma 4, it was seen that x, y, z were the roots of $t^3+pt^2+qt+r = 0$, where

$$\max(|p-1|, |q+2|, |r+1|) < 10\epsilon.$$

Let t and t_0 denote either x and θ, or y and ϕ, or z and ψ. Then

$$t^3+pt^2+qt+r = {t_0}^3+{t_0}^2-2t_0-1 = 0,$$

whence

$$(t-t_0)\left(t^2+tt_0+{t_0}^2+p(t+t_0)+q\right) = -(p-1){t_0}^2-(q+2)t_0-(r+1).$$

The right-hand side has its absolute value less than

$$10\epsilon({t_0}^2+|t_0|+1) < 70\epsilon,$$

since $|t_0| < 2$.

There exists a number c_2 such that, if $0 < \epsilon < c_2$,

$$|t^2+tt_0+{t_0}^2+p(t+t_0)+q| > \tfrac{1}{2}|3{t_0}^2+2t_0-2|.$$

Also $$\min(|3\theta^2+2\theta-2|, |3\phi^2+2\phi-2|, |3\psi^2+2\psi-2|) > 2.$$

Hence, if $0 < \epsilon < c_2$,

$$\max(|x-\theta|, |y-\phi|, |z-\psi|) < 70\epsilon = c_3\epsilon.$$

LEMMA 15. *If* $0 < \epsilon < c_2$, *and if* x_1, y_1, z_1 *and* x_2, y_2, z_2 *are two normal sets*† *satisfying*

$$x_1+y_1+z_1 < -1+2\epsilon, \quad x_2+y_2+z_2 < -1+2\epsilon,$$

then

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ y_2 & z_2 & x_2 \\ 1 & 1 & 1 \end{vmatrix} < -7+6\epsilon+c_4\epsilon^2.$$

Proof. Write

$$x_i = \theta+\alpha_i, \quad y_i = \phi+\beta_i, \quad z_i = \psi+\gamma_i \quad (i = 1, 2).$$

† *I.e.* two sets of real numbers satisfying (4) and (3′).

By hypothesis

$$\alpha_i+\beta_i+\gamma_i<2\epsilon; \tag{23}$$

and by Lemmas 4 and 14,

$$\max(|\alpha_1|, \ldots, |\gamma_2|)<c_3\epsilon. \tag{24}$$

Since x_1, y_1, z_1 satisfy (6), we have

$$(\theta-1+\alpha_1)(\phi-1+\beta_1)(\psi-1+\gamma_1)>1-\epsilon.$$

Hence, using (24),

$$(\phi-1)(\psi-1)\alpha_1+(\psi-1)(\theta-1)\beta_1+(\theta-1)(\phi-1)\gamma_1>-\epsilon-c_5\epsilon^2.$$

Now

$$\begin{aligned}(\phi-1)(\psi-1)&=\phi\psi-\phi-\psi+1=(\phi+\theta)-\phi-\psi+1\\&=\theta-\psi+1,\end{aligned}$$

and similarly for the other products. Hence

$$(\theta-\psi+1)\alpha_1+(\phi-\theta+1)\beta_1+(\psi-\phi+1)\gamma_1>-\epsilon-c_5\epsilon^2.$$

By (23), this gives

$$(\psi-\theta)\alpha_1+(\theta-\phi)\beta_1+(\phi-\psi)\gamma_1<3\epsilon+c_5\epsilon^2. \tag{25}$$

Since x_2, y_2, z_2 satisfy (5), we have

$$(\theta+\alpha_2)(\phi+\beta_2)(\psi+\gamma_2)>1-\epsilon,$$

whence

$$\phi\psi\alpha_2+\psi\theta\beta_2+\theta\phi\gamma_2>-\epsilon-c_6\epsilon^2,$$

i.e.

$$(\phi+\theta)\alpha_2+(\psi+\phi)\beta_2+(\theta+\psi)\gamma_2>-\epsilon-c_6\epsilon^2. \tag{26}$$

Also, since x_2, y_2, z_2 satisfy (7), we have

$$(\theta+1+\alpha_2)(\phi+1+\beta_2)(\psi+1+\gamma_2)<-1+\epsilon,$$

whence

$$(\phi+1)(\psi+1)\alpha_2+(\psi+1)(\theta+1)\beta_2+(\theta+1)(\phi+1)\gamma_2<\epsilon+c_7\epsilon^2.$$

Now

$$(\phi+1)(\psi+1)=(\phi+\theta)+\phi+\psi+1=\phi,$$

and similarly for the other products. Hence

$$\phi\alpha_2+\psi\beta_2+\theta\gamma_2<\epsilon+c_7\epsilon^2. \tag{27}$$

It follows from (26) and (27) that

$$(28) \qquad (\phi-\theta)\alpha_2+(\psi-\phi)\beta_2+(\theta-\psi)\gamma_2<3\epsilon+(c_6+2c_7)\epsilon^2.$$

The determinant with which we are concerned is

$$\begin{vmatrix} \theta+\alpha_1 & \phi+\beta_1 & \psi+\gamma_1 \\ \phi+\beta_2 & \psi+\gamma_2 & \theta+\alpha_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \theta & \phi & \psi \\ \phi & \psi & \theta \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \phi & \psi & \theta \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \theta & \phi & \psi \\ \beta_2 & \gamma_2 & \alpha_2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ 1 & 1 & 1 \end{vmatrix}.$$

By (24), the absolute value of the last determinant is less than $c_8\epsilon^2$. The first determinant has the value

$$\phi\psi+\psi\theta+\theta\phi-\theta^2-\phi^2-\psi^2=-7.$$

The second and third determinants are

$$(\psi-\theta)\alpha_1+(\theta-\phi)\beta_1+(\phi-\psi)\gamma_1<3\epsilon+c_5\epsilon^2,$$

and

$$(\phi-\theta)\alpha_2+(\psi-\phi)\beta_2+(\theta-\psi)\gamma_2<3\epsilon+c_9\epsilon^2,$$

by (25) and (28). This establishes Lemma 15.

8. *The case* $M=\frac{1}{7}$.

THEOREM 2. *If* $M=\frac{1}{7}$, *then* (*apart from order*) L_1, L_2, L_3 *have the form*

$$L_1=\lambda_1(\theta U+\phi V+\psi W),$$

$$L_2=\lambda_2(\phi U+\psi V+\theta W),$$

$$L_3=\lambda_3(\psi U+\theta V+\phi W),$$

where $\lambda_1\lambda_2\lambda_3=\frac{1}{7}$, *and* U, V, W *are obtained from* u, v, w *by an integral unimodular linear transformation.*

Proof. Choose $\delta>0$ such that

$$(29) \qquad \delta<c_1, \quad \delta<c_2, \quad c_4\delta<1.$$

There exist integral values of u, v, w such that the corresponding values of L_1, L_2, L_3, say L_1^*, L_2^*, L_3^*, satisfy

$$|L_1^* L_2^* L_3^*| = \frac{1}{7(1-\epsilon')},$$

where $$0 \leqslant \epsilon' < \delta.$$

Write $\xi = L_1/L_1^*$, $\eta = L_2/L_2^*$, $\zeta = L_3/L_3^*$. For all integral values of u, v, w, except 0, 0, 0, we have

$$|\xi\eta\zeta| \geqslant 1-\epsilon'.$$

The points in (ξ, η, ζ)-space which correspond to integral values of u, v, w form a lattice which satisfies the conditions (a), (b), (c) of § 6, provided that we take $\epsilon > \epsilon'$ in (b).

Hence, by Lemma 13 and (29), with any ϵ such that $\epsilon' < \epsilon < \delta$ there exist three generating lattice points (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), (ξ_3, η_3, ζ_3) which satisfy (22) and which also satisfy either (21α) or (21β). On the right of (21α) or (21β) we can take $\epsilon_1 = c_3 \epsilon$, by Lemma 14.

The inequality (21β) is obtained from (21α) by interchanging θ and ψ. This is equivalent to replacing L_1, L_2, L_3 by L_3, L_2, L_1 and also interchanging (ξ_2, η_2, ζ_2) and (ξ_3, η_3, ζ_3). Hence we can restrict ourselves without loss of generality to the case in which (21α) holds.

Since the three lattice points generate the lattice, we have

$$\Delta = \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ \xi_3 & \eta_3 & \zeta_3 \end{vmatrix} = 7(1-\epsilon'). \tag{30}$$

By (22),

$$-\Delta = \begin{vmatrix} \xi_1 & \eta_1 & \zeta_1 \\ \xi_2 & \eta_2 & \zeta_2 \\ 1 & 1 & 1 \end{vmatrix} = \Delta_1, \tag{31}$$

say.

By (b) and (21α), ξ_1, η_1, ζ_1 is a normal set, and ζ_2, ξ_2, η_2 is a normal set; and each satisfies (22). Hence, by Lemma 15,

$$\Delta_1 < -7 + 6\epsilon + c_4 \epsilon^2. \tag{32}$$

By (30), (31), (32),

$$7\epsilon' < 6\epsilon + c_4 \epsilon^2.$$

Since this is valid for all ϵ satisfying $\epsilon' < \epsilon < \delta$, we have

$$7\epsilon' \leqslant 6\epsilon' + c_4 \epsilon'^2 \leqslant 6\epsilon' + c_4 \delta \epsilon'.$$

Since $c_4 \delta < 1$, it follows that $\epsilon' = 0$.

Hence $$|L_1^* L_2^* L_3^*| = \tfrac{1}{7},$$

and the three generating lattice points satisfy (21α) for any $\epsilon > 0$. Thus the three generating lattice points are (θ, ϕ, ψ), (ϕ, ψ, θ), (ψ, θ, ϕ), and so the lattice is given by

$$\xi = \theta U + \phi V + \psi W,$$
$$\eta = \phi U + \psi V + \theta W,$$
$$\zeta = \psi U + \theta V + \phi W,$$

where U, V, W take all integral values. Hence U, V, W are connected with u, v, w by an integral unimodular linear transformation, and L_1, L_2, L_3 have the form stated in the enunciation.

The University,
Manchester.

A METHOD OF VARIATION WITHIN THE FAMILY OF SIMPLE FUNCTIONS

By MENAHEM SCHIFFER.

[Received 12 October, 1937.—Read 18 November, 1937.]

1. *Introduction.* In the theory of conformal representation various problems of extremum have been stated and solved. There are two possible ways of investigating such problems. In the first method, all conformal representations of the type considered are discussed, and inequalities for them are established. Then it is shown that, for a certain representation, the bound of the inequality is actually attained, and this representation solves the corresponding extremum problem. An example of this method is the proof of the "Verzerrungssatz" of Koebe. In the second method, on the other hand, the extremal representation is discussed with particular reference to the extremal character of its adjoint function. This method is similar to that used in the classical calculus of variations. It has been used by Grötzsch†, Marty‡, and others.

In the present paper a new and general method of the second type is described. It is a kind of calculus of variation applied to the family of all normalized functions which are simple (univalent) in a given domain. This method proves to be of great applicability to a large number of problems involving simple functions. With its aid, differential equations can be found, either for the functions of the family with certain extremal properties, or, at least, for the boundary curves of the domain conformally mapped with their aid from the given domain. In this way numerous problems in simple functions can be solved; in this paper, the method will be applied to the problem of the coefficients of simple functions, and will be used to develop a generalization for multiply connected domains of some theorems of growth of simple functions.

† H. Grötzsch, "Über ein Variationsproblem der konformen Abbildung", *Leipziger Berichte*, 82 (1930), 251–263.

‡ F. Marty, "Sur le module des coefficients de MacLaurin d'une fonction univalente", *Comptes rendus*, 198 (1934), 1569–1571.

2. *The fundamental lemma.* Our method consists of a comparison of the extremal function with infinitesimally neighbouring functions of the same family. For this purpose the following lemma is very useful:

LEMMA. *There exists a set of positive constants*

$$K^*,\ A^* \leqslant 4^2,\ B^* \leqslant 4^3,\ \ldots,$$

such that the following statements hold:

(*a*) *Let C be a bounded continuum in the z-plane, containing more than one point and possessing as complement the domain D. Then, for each $z_0 \subset C$ and for each value of the parameter $\rho > 0$, there exist functions*

$$(1) \qquad F_\rho(z, z_0) = z - z_0 + K\rho + \frac{A\rho^2}{z-z_0} + \frac{B\rho^3}{(z-z_0)^2} + \ldots,$$

which are simple and regular in D, the pole at infinity excepted, and for which the moduli of the coefficients K, A, B, ... are bounded by K^*, A^*, B^*, ...

(*b*) *Let the function $s(z)$ be analytic and not identically vanishing on C. Suppose that, at each fixed point $z_0 \subset C$, for each sequence of functions $F_{\rho_\nu}(z, z_0)$ of the family* (1) *for which $\lim_{\rho_\nu \to 0} A = a$ exists and is different from* 0, *we have $R\{as(z_0)\} \geqslant 0$. Then C is an analytic curve $z(t)$, satisfying the differential equation*

$$\left(\frac{dz}{dt}\right)^2 s(z) + 1 = 0.$$

Proof. I. We first prove (*a*). Without loss of generality we can suppose that $z_0 = 0$. (This will be assumed throughout I–IV.) Let p and q be two different points of C. Then

$$(2) \qquad F(z) = \frac{p-q}{\log \dfrac{1-p/z}{1-q/z}} = z - \frac{p+q}{2} - \frac{(p-q)^2}{12z} - \frac{\Pi_3(p, q)}{z^2} - \frac{\Pi_4(p, q)}{z^3} \cdots$$

is simple in D and regular, the pole at ∞ excepted. $\Pi_\nu(p, q)$ denotes a homogeneous polynomial of degree ν in p and q. Putting $\rho = \max(|p|, |q|)$, we can then write (2) in the form

$$(2') \qquad F_\rho(z) = z - \frac{p+q}{2\rho}\rho - \left(\frac{p}{\rho} - \frac{q}{\rho}\right)^2 \frac{\rho^2}{12z} - \Pi_3\left(\frac{p}{\rho}, \frac{q}{\rho}\right)\frac{\rho^3}{z^2} - \ldots,$$

and thus show that (2) is a function of the type (1) for the case considered $z_0 = 0$. Further, ρ can be chosen arbitrarily small, because in each

neighbourhood of O distinct points p, q can be found. This proves (a), if K^*, A^*, B^*, ... are taken as the bounds for the coefficients

$$\Pi_\nu\left(\frac{p}{\rho}, \frac{q}{\rho}\right)$$

of (2′).

But to prove (b) we shall have to consider also another class of functions (1) which are obtained in the following way. We choose a sub-continuum I_ρ of C containing the point O, the complement of which is a domain D_ρ, with the mapping radius $\rho > 0$. We map D_ρ conformally on $|\eta| > \rho$ by means of the simple function

$$\eta = \zeta(z) = z + K(\rho)\rho + \frac{A(\rho)\rho^2}{z} + \frac{B(\rho)\rho^3}{z^2} + \frac{C(\rho)\rho^4}{z^3} + \cdots. \tag{3}$$

The inverse function of (3),

$$z = \zeta^{-1}(\eta) = \eta - K(\rho)\rho - \frac{A(\rho)\rho^2}{\eta} - \frac{\hat{B}(\rho)\rho^3}{\eta^2} - \frac{\hat{C}(\rho)\rho^4}{\eta^3} - \cdots, \tag{4}$$

is simple and regular in $|\eta| > \rho$, the pole at infinity excepted. By the principle of areas (Flächensatz) of Bieberbach-Faber†, we have therefore

$$|A(\rho)|^2 + 2|\hat{B}(\rho)|^2 + 3|\hat{C}(\rho)|^2 + \cdots \leqslant 1. \tag{5}$$

Hence all the coefficients $A(\rho)$, $\hat{B}(\rho)$, $\hat{C}(\rho)$, ... of (4) are bounded, independently of ρ and the choice of I_ρ; applying the same principle to the function $z = \sqrt{\{\rho\zeta^{-1}(\eta^2/\rho)\}}$, which is also regular and simple in $|\eta| > \rho$, we prove also that $|K(\rho)| \leqslant 2$; from this, the boundedness of the coefficients $B(\rho)$, $C(\rho)$, ... of (3) can be proved. In particular we find that $|A(\rho)| \leqslant 1$. On the other hand, the functions

$$h_\gamma(\eta) = \eta + \frac{\gamma\rho^2}{\eta} \tag{6}$$

are simple in $|\eta| > \rho$ and regular for each $|\gamma| \leqslant 1$, the pole at ∞ excepted. Therefore the function

$$F_\rho(z) = h_\gamma\big(\zeta(z)\big) = z + K(\rho)\rho + \frac{\big(A(\rho)+\gamma\big)\rho^2}{z} + \frac{\tilde{B}(\rho,\gamma)\rho^3}{z^2} + \cdots \tag{7}$$

---

† See, for example, Bieberbach, *Lehrbuch der Funktionentheorie*, 2 (2. Aufl., 1931), 72–74, or Titchmarsh, *The theory of functions*, (1932), 209.

is simple in D_ρ and regular for each $|\gamma| \leqslant 1$, the point ∞ excepted. Its coefficients $\tilde{B}(\rho, \gamma)$, $\tilde{C}(\rho, \gamma)$, ... are bounded for all $\rho < 0$, $|\gamma| \leqslant 1$, independently of I_ρ. The function (7) therefore also belongs to the class (1), the constants K^*, A^*, B^*, ... being suitably chosen.

We now fix K^*, A^*, B^*, ... so that they give simultaneously upper bounds for the moduli of the coefficients of (2′) and (7). It is obvious that we may take $A^* = 4^2$, $B^* = 4^3$, ... For the functions (2′) and (7) are simple outside of the circle $|z| \leqslant 4\rho$, and by the principle of areas (5) we get the above upper bounds for the coefficients. We assert that part (b) of the lemma holds also for these constants.

II. We suppose henceforth that $s(0) \neq 0$, and we put

$$\operatorname{sgn} s(0) = \frac{s(0)}{|s(0)|} = e^{i\tau}.$$

We consider the function (2′) for which $p \to 0$ and $q = 0$. Then

$$\rho = \max(|p|, |q|) = |p|,$$

and (2′) can be written in the form

$$(2'') \qquad F_\rho(z) = z - \tfrac{1}{2}(\operatorname{sgn} p)\rho - \frac{(\operatorname{sgn} p)^2 \rho^2}{12z} - \frac{\Pi_3(\operatorname{sgn} p, 0)\rho^3}{z^2} - \ldots.$$

We call the direction of $e^{i\kappa}$ a limit direction at O with respect to points of C, if a sequence of points $p_\nu \subset C$ with $p_\nu \to 0$ can be chosen for which

$$\lim_{\nu \to \infty} \operatorname{sgn} p_\nu = e^{i\kappa}.$$

Let us consider the functions (2″) for which $p = p_\nu$, $\rho = \rho_\nu = |p_\nu|$, and $\lim_{\nu \to \infty} \operatorname{sgn} p_\nu = e^{i\kappa}$. In consequence of the hypothesis of (b) we must have $\frac{1}{2}\pi \leqslant 2\kappa + \tau \leqslant \frac{3}{2}\pi$; for

$$\lim_{\nu \to \infty} \left[-\frac{\operatorname{sgn} p^2}{12} \right] = \frac{e^{i(2\kappa + \pi)}}{12} = a$$

is not zero. Hence κ must lie either in the closed interval

$$[\tfrac{1}{4}\pi - \tfrac{1}{2}\tau, \ \tfrac{3}{4}\pi - \tfrac{1}{2}\tau], \quad \text{or in} \quad [\tfrac{5}{4}\pi - \tfrac{1}{2}\tau, \ \tfrac{7}{4}\pi - \tfrac{1}{2}\pi].$$

All limit directions in O lie therefore in the two vertically opposite right angles which are bisected by the straight line l through O with the direction $e^{\frac{1}{2}i(\pi - \tau)}$. Hence, if $|p|$ is sufficiently small, all points $p \subset C$

have their arguments in the interior of the intervals

$$[\tfrac{1}{4}\pi-\tfrac{1}{2}\tau-\epsilon,\ \tfrac{3}{4}\pi-\tfrac{1}{2}\tau+\epsilon],\quad [\tfrac{5}{4}\pi-\tfrac{1}{2}\tau-\epsilon,\ \tfrac{7}{4}\pi-\tfrac{1}{2}\tau+\epsilon],$$

for each given $\epsilon > 0$.

We show, in addition, that 0 is a boundary point of C.

III. We consider the function (7) which is simple in the domain D_ρ. Since $|A(\rho)| \leqslant 1$ and since γ may take any value such that $|\gamma| \leqslant 1$, it follows from the hypothesis of (b) that $\lim_{\rho\to 0} |A(\rho)| = 1$ for each sequence of continua I_ρ. For, in the contrary case, a sub-sequence I_{ρ_ν} could be chosen, satisfying $\lim_{\nu\to\infty} |A(\rho_\nu)| < 1$; and, consequently, by a proper choice of γ_ν, the argument of $\alpha = \lim_{\nu\to\infty} \{A(\rho_\nu)+\gamma_\nu\}$ could be fixed arbitrarily, contradicting the hypothesis of (b).

We now choose a sub-sequence I_{ρ_ν} yielding a convergent sub-sequence $A(\rho_\nu)$ with $\lim_{\nu\to\infty} A(\rho_\nu) = e^{i\vartheta}$. Also, by choosing γ_ν properly, we can make the argument of α take an arbitrarily given value in the open interval $(\vartheta-\frac{1}{2}\pi,\ \vartheta+\frac{1}{2}\pi)$. By the hypothesis of (b), all the values of

$$(\tau+\vartheta-\tfrac{1}{2}\pi,\ \tau+\vartheta+\tfrac{1}{2}\pi)$$

must lie in the interval $[-\frac{1}{2}\pi,\ +\frac{1}{2}\pi]$. This is only possible when these two intervals of length π coincide, *i.e.* $\tau+\vartheta = 0$. This condition determines ϑ uniquely; and therefore we have

$$\lim_{\rho\to 0} A(\rho) = e^{-i\tau}. \tag{8}$$

Thus, the functions (4) mapping $|\eta| > \rho$ conformally on D_ρ necessarily have coefficients $A(\rho)$ tending to $e^{-i\tau}$ as $\rho \to 0$. The functions

$$z' = \eta' - K(\rho) - \frac{A(\rho)}{\eta'} - \frac{\hat{B}(\rho)}{\eta'^2} - \dots \tag{4'}$$

map the circle $|\eta'| > 1$ on domains D_ρ' which is obtained from D_ρ by the magnification $z' = z/\rho$. In virtue of (8) and (5) $\hat{B}(\rho)$ and $\hat{C}(\rho)$, all tend to 0 as $\rho \to 0$; and so (4′) can be written in the form

$$z' = \eta' - K(\rho) - \frac{e^{-i\tau}}{\eta'} + \delta(\rho)\, H_\rho\left(\frac{1}{\eta'}\right). \tag{4''}$$

Here $|H_\rho(1/\eta')| \leqslant M_\epsilon$ in each domain $|\eta'| \geqslant 1+\epsilon > 1$, M_ϵ being independent of I_ρ, while $\delta(\rho) \to 0$ as $\rho \to 0$. Choosing ϵ sufficiently small and making a

corresponding choice of ρ, we can make the image of the circle $|\eta'| = 1+\epsilon$ lie in an arbitrary vicinity of a segment of a straight line

$$[-2e^{\frac{1}{2}i(\pi-\tau)}-K(\rho),\ +2e^{\frac{1}{2}i(\pi-\tau)}-K(\rho)],$$

which is obtained from $|\eta'| = 1$ by the conformal representation

$$z' = \eta' - K(\rho) - \frac{e^{-i\tau}}{\eta'}.$$

Thus, *a fortiori*, the boundary B_ρ' of D_ρ' also lies, in an arbitrary neighbourhood of this segment; for it lies in the interior of the image of $|\eta'| = 1+\epsilon$, obtained by (4''). In other words, the boundary in question lies in an arbitrary neighbourhood of a straight line l_ρ with the direction $e^{\frac{1}{2}i(\pi-\tau)}$. On the other hand, B_ρ' contains the point O, this being a boundary point of C, and therefore also of D_ρ; hence, B_ρ' lies in an arbitrary neighbourhood of the straight line l with the direction $e^{\frac{1}{2}i(\pi-\tau)}$ *through the origin*. Therefore the distance of each point of B_ρ' from l is less than $d(\rho)$; and this converges with ρ to zero. The boundary points of D_ρ itself which also form the boundary of I_ρ have distances from l less than $\rho d(\rho)$, and the same holds for all interior points of I_ρ, if any such exist.

If we now describe round $z = 0$ a circle Γ_ρ with radius ρ, then, for ρ sufficiently small, at least one point $P(\rho)$ of C lies on Γ_ρ, and it can be connected with O by a continuum I_σ entirely interior to Γ_ρ whose mapping radius is σ. Since the mapping radius is monotonic, we have $\sigma \leqslant \rho$. $P(\rho)$ has, therefore, as shown above, a distance less than $\sigma d(\sigma)$ from the straight line l; thus, the line $\overline{OP(\rho)}$ forms with l an angle $\omega(\rho)$ which satisfies the inequality $|\sin\omega(\rho)| < d(\sigma)$. Hence, $\lim_{\rho\to 0}\omega(\rho)$ is 0 or π, and the secants $\overline{OP(\rho)}$ converge to l as $\rho \to 0$. This shows that at least one of the directions $\pm e^{\frac{1}{2}i(\pi-\tau)}$ is a limit direction for C in $z = 0$.

IV. We now take the two vertically opposite angular domains containing all the points with arguments in the intervals $[\frac{1}{4}\pi-\frac{1}{2}\tau-\epsilon,\ \frac{3}{4}\pi-\frac{1}{2}\tau+\epsilon]$ and $[\frac{5}{4}\pi-\frac{1}{2}\tau-\epsilon,\ \frac{1}{4}\pi-\frac{1}{2}\tau+\epsilon]$ for a fixed value of $\epsilon > 0$, and cut two sectors out of these by means of a circle Γ_{ρ_0} with radius ρ_0 and centre 0. If ρ_0 is chosen sufficiently small, all points of C in the interior of Γ_{ρ_0} lie in these two sectors, by II. If, further, ρ_0 is chosen so small that not all points of C are inside Γ_{ρ_0}, there lies in at least one of the two sectors, say in S_1, and on Γ_{ρ_0} a point of C which can be connected with O by a continuum entirely in Γ_{ρ_0} and therefore also in S_1. On each circle Γ_ρ with $\rho < \rho_0$ there lies also at least one point $P(\rho)$ which can be connected with O by a continuum which lies

in Γ_ρ and S_1. The mapping radius of the connecting continuum I_σ is $\sigma \leqslant \rho$. By the expansion in III the points $P(\rho)$ have distances less than $\sigma d(\sigma)$ from the straight line l; the secants $\overline{OP(\rho)}$ therefore form angles $\omega(\rho)$ with that ray l_1 of l which lies in S_1, $\omega(\rho)$ being such that $\lim_{\rho\to 0} \omega(\rho) = 0$. We choose for each ρ exactly one $P(\rho)$; and so we get a sequence of points which converge monotonically to O with ρ and for which the secants $\overline{OP(\rho)}$ converge to the ray l_1. Now it will be shown that each sequence of points Q_ν in the same sector S_1 which converge to O has the same property, viz. that the secants $\overline{OQ_\nu}$ converge to the ray l_1. It will thus be proved that the direction l_1 is the only limit direction in the sector S_1.

To prove this assertion, we consider the sequences of points

$$Q_1, \quad Q_2, \quad \ldots, \quad Q_n, \quad \ldots \to O$$

and $P_1 = P(|Q_1|)$, $P_2 = P(|Q_2|)$, ..., $P_n = P(|Q_n|) \to 0$; we have $|P_\nu| = |Q_\nu|$ and $\lim_{\nu\to\infty} \overline{OP_\nu} = l_1$.

We construct the functions (2′) with $p = P_\nu$, $q = Q_\nu$, and $\rho = |P_\nu|$. Then

$$(9) \quad F_\rho(z) = z - \frac{P_\nu + Q_\nu}{2\rho}\rho - \frac{(\operatorname{sgn} P_\nu - \operatorname{sgn} Q_\nu)^2\rho^2}{12z} - \frac{\Pi_3(\operatorname{sgn} P_\nu, \operatorname{sgn} Q_\nu)\rho^3}{z^2} - \ldots$$

is a function of the form (1). For ν sufficiently large, the argument of P_ν is arbitrarily near to $\frac{1}{2}(\pi - \tau)$ (if S_1 lies in this direction from 0) and $\arg Q_\nu$ is in the interval $[\frac{1}{4}\pi - \frac{1}{2}\tau - \epsilon, \frac{3}{4}\pi - \frac{1}{2}\tau + \epsilon]$; the argument of $(\operatorname{sgn} P_\nu - \operatorname{sgn} Q_\nu)$ then lies, as is most easily seen from geometric considerations, in one of the intervals $[\pi - \frac{1}{2}\tau - \delta_\nu, \frac{9}{8}\pi - \frac{1}{2}\tau + \delta_\nu]$ or $[-\frac{1}{8}\pi - \frac{1}{2}\tau - \delta_\nu, -\frac{1}{2}\tau + \delta_\nu]$ with $\delta_\nu > 0$ and $\delta_\nu \underset{\nu}{\to} 0$. Therefore, the argument of $\{-(\operatorname{sgn} P_\nu - \operatorname{sgn} Q_\nu)^2 e^{i\tau}\}$ lies in the interval $[\frac{3}{4}\pi, \frac{5}{4}\pi]$. This and the hypothesis of (*b*) yield the result that

$$(10) \qquad \lim_{\nu\to\infty} \{\operatorname{sgn} P_\nu - \operatorname{sgn} Q_\nu\} = 0;$$

for otherwise a sub-sequence Q_ν could be chosen for which

$$\lim_{\nu\to\infty} \{-(\operatorname{sgn} P_\nu - \operatorname{sgn} Q_\nu)^2\} = a \neq 0$$

without satisfying the condition $R\{as(z_0)\} \geqslant 0$. But (10) shows that the angle $P_\nu O Q_\nu$ converges to zero; therefore also $\lim_{\nu\to\infty} \overline{OQ_\nu} = l_1$, which was the assertion. An analogous property holds, if a sequence $P(\rho)$ in the sector S_2 is at our disposal.

V. Abandoning now the restriction $z_0 = 0$, we can summarize all the results proved hitherto in the following statement:

At each point z_0 with $s(z_0) \neq 0$, C possesses limit directions which can be classified into two groups. First a limit direction h which is one of the two directions $\pm e^{\frac{1}{2}i(\pi-\tau)} = \operatorname{sgn}\{\pm 1/\sqrt{-s(z_0)}\}$; second all the other directions, if any such exist. These lie in that right angle which is bisected by h produced backwards. The nearest limit direction different from h, therefore, forms with h an angle greater than or equal to $\frac{3}{4}\pi$. These geometric facts are invariant with regard to conformal representation; for such a representation transforms limit directions into limit directions and leaves the angles between different limit directions unchanged. Thus, at each point z_0, the system of limit directions is subjected only to a rigid rotation. We use this fact to obtain a simpler rule for the system of limit directions at different points of C.

Let $z_0 \subset C$, and let z_0 be not a zero of $s(z)$; then the function

$$u = y(z) = \int_{z_0}^{z} \sqrt{-s(\zeta)}\, d\zeta \tag{11}$$

is regular and simple for $|z-z_0| \leqslant k$ and k sufficiently small. A subcontinuum T of C, which lies entirely in this circular domain, contains z_0 and does not reduce to this point, is mapped by (11) on a continuum T' in the u-plane which contains $u = 0$. At each point of T' the limit directions of the image points of C are separated into two groups. First into a limit direction parallel to the real axis, and secondly into limit directions which are separated from the first by an empty domain of aperture greater than or equal to $\frac{3}{4}\pi$.

For let z^* be an arbitrary point of T, and suppose that, for a sequence $z_\nu \to z^*$, $z_\nu \subset C$,

$$\lim_{\nu\to\infty} \operatorname{sgn}(z_\nu - z^*) = e^{i\psi}; \tag{12}$$

then, for $u_\nu = y(z_\nu)$ and $u^* = y(z^*)$, we have

$$\lim_{\nu\to\infty} \operatorname{sgn}(u_\nu - u^*) = e^{i\psi} \operatorname{sgn} \sqrt{-s(z^*)}. \tag{12'}$$

The limit direction $e^{i\psi} = \pm\operatorname{sgn}\{1/\sqrt{-s(z^*)}\}$ is therefore transformed into a real direction.

By a theorem of Haslam-Jones†, the projection of T' on the imaginary axis in the u-plane has measure zero. T' is a continuum and contains the

† Tangential properties of a plane set of points", *Quart. J. of Math.*, 7 (1936), 116–123.

origin; therefore it lies entirely on the real axis. The corresponding part T of C in the z-plane therefore lies on the curve

$$\int_{z_0}^{z} \sqrt{-s(z)}\, dt = t,$$

with t a real number, which is got from the real u-axis by an analytical representation.

This holds for $|z-z_0| \leqslant k$; but this integral, along an arbitrary part of C which does not contain zeros of $s(z)$, can now be calculated by regarding this part as composed of sufficiently small pieces for which the above consideration holds. Finally, we can even integrate over the whole curve by regarding it as composed of parts which do not contain zeros of $s(z)$ in their interior. Along the whole curve C we therefore have

$$\int_{z_0}^{z} \sqrt{-s(z)}\, dz = t, \tag{13}$$

a real number if $z_0 \subset C$. For an analytic branch of C, t can be chosen as the parameter of the curve. Differentiating (13), we get the differential equation

$$\left(\frac{dz}{dt}\right)^2 s(z) + 1 = 0 \tag{14}$$

for all points z which are not zeros of $s(z)$.

This is the assertion of the lemma.

From the proof it is obvious that C can have branch points or cusps only at the zeros of $s(z)$.

3. *The problem of the coefficients.* Let us now apply this lemma to the problem of the coefficients of a simple function. We imagine a given domain Δ of finite or infinite connectivity in the ζ-plane; it is supposed only that $\zeta = 0$ belongs to Δ. Now consider the family of all functions

$$z = f(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \ldots + a_n\zeta^n + \ldots \tag{15}$$

which are regular and simple in Δ. This family is normal and therefore, for each n, there exists at least one function $f_n(\zeta)$ of the family for which $|a_n|$ attains its greatest value within the family. An important property of these functions may now be pointed out.

The function $f(\zeta)$ maps the domain Δ on a domain D in the z-plane the complements of which are certain continua R_ν and points P_ν. We choose a fixed continuum R among the R_ν (*i.e.* R contains more than one point)

and select on it a point $z^* \neq \infty$. Round z^* we describe a circle Γ_r with radius $r < |z^*|$ and a concentric circle $\Gamma_{\frac{1}{3}r}$ with radius $\frac{1}{3}r$. Hence the origin lies outside Γ_r. Consider now the sub-continuum C of R which contains z^* and lies entirely inside $\Gamma_{\frac{1}{3}r}$. We apply the lemma to C: for each $\rho > 0$ and each $z_0 \subset C$ there exist functions (1) which are simple and regular in the whole z-plane, C and ∞ excepted. The expansion (1) converges for all points z outside Γ_r, ∞ excepted. Therefore the functions

$$(16)\quad F_\rho\{f(\zeta);\, z_0\} = f(\zeta) - z_0 + K\rho + \frac{A\rho^2}{f(\zeta)-z_0} + \frac{B\rho^3}{\{f(\zeta)-z_0\}^2} + \frac{C\rho^4}{\{f(\zeta)-z_0\}^3} + \cdots$$

are regular and simple in Δ and their expansion (16) converges uniformly in the vicinity of $\zeta = 0$. For, to $\zeta = 0$ there corresponds $z = 0$, and this point lies outside Γ_r. Further, the coefficients B, C, ... are bounded for all values of ρ in consequence of § 2, I; and so, in the neighbourhood of $\zeta = 0$, we may write

$$(16')\qquad F_\rho\{f(\zeta);\, z_0\} = f(\zeta) - z_0 + K\rho + \frac{A\rho^2}{f(\zeta)-z_0} + O(\rho^3),$$

where $O(\rho^3)$ denotes terms of at least the third order in ρ.
Normalizing this function according to (15), we get

$$(16'')\qquad f^*(\zeta) = f(\zeta) - \frac{A\rho^2}{z_0}\left\{\frac{f(\zeta)^2}{z_0^2} + \frac{f(\zeta)^3}{z_0^3} + \cdots\right\} + O(\rho^3).$$

This is a new normalized and simple function in Δ. We now put

$$(17)\qquad [f(\zeta)]^r = \sum_{\nu=r}^{\infty} a_\nu^{(r)} \zeta^\nu = \zeta^r + r a_2 \zeta^{r+1} + \cdots.$$

Then we can write (16'') in the form

$$(18)\qquad f^*(\zeta) = \sum_{n=1}^{\infty}\left\{a_n - \frac{A\rho^2}{z_0}\sum_{k=2}^{\infty}\frac{a_n^{(k)}}{z_0^k} + O(\rho^3)\right\}\zeta^n.$$

This rearrangement of (13'') is legitimate in view of the uniform convergence of (16'') in the vicinity of the origin. (18) is a function of the family (15) which is regular and simple in Δ.

If now, in particular, $f(\zeta)$ is an extremal function $f_n(\zeta)$ with the coefficient a_n of the largest absolute value in the family, then for each z_0 we have the inequality

$$(19)\qquad |a_n^*| = \left|a_n - \frac{A\rho^2}{z_0}\sum_{k=2}^{n}\frac{a_n^{(k)}}{z_0^k} + O(\rho^3)\right| \leqslant |a_n|;$$

hence, obviously, $a_n \neq 0$ and

$$R\left\{A\frac{1}{a_n}\sum_{k=2}^{n}\frac{a_n^{(k)}}{z_0^{k+1}}+O(\rho)\right\}\geqslant 0. \tag{19'}$$

Take any sequence $F_{\rho_\nu}(z;\, z_0)$ of functions (1), with $\rho_\nu \to 0$, for which

$$a = \lim_{\nu\to\infty} A_\nu$$

exists and is different from zero. For each A_ν the inequality (19′) holds, and so we get, in the limit when $\rho \to 0$,

$$R\left\{a\frac{1}{a_n}\sum_{k=2}^{n}\frac{a_n^{(k)}}{z_0^{k+1}}\right\}\geqslant 0. \tag{19''}$$

Thus, the supposition of part (*b*) of the lemma holds with

$$s(z)=\frac{1}{a_n}\sum_{k=2}^{n}\frac{a_n^{(k)}}{z_0^{k+1}}.$$

$s(z)$ does not vanish identically since $a_n^{(n)}=1$. Therefore C is an analytic curve $z(t)$ satisfying the differential equation

$$\left(\frac{dz}{dt}\right)^2\frac{1}{a_n}\sum_{k=2}^{n}\frac{a_n^{(k)}}{z(t)^{k+1}}+1=0. \tag{20}$$

This holds for that part of R which lies inside $\Gamma_{\frac{1}{3}r}$. By regarding R as composed of parts for which (20) holds, we can show that R itself is also an analytic curve satisfying (20). The only critical points that the curve R can have are the zeros of $s(z)$ which lie on it. At them the curve can fork into different branches and it can possess cusps. We also see that D has no exterior points, *i.e.* it is a slit domain. Thus we have proved the following theorem:

THEOREM I. *Let Δ be a domain containing $\zeta=0$. Among all functions* (15) *which are simple and regular in Δ, there exists at least one, $f_n(\zeta)$, with the n-th coefficient a_n largest in absolute value. This function maps Δ on a slit domain D, the boundary of which consists of curves satisfying the differential equation* (20) *and possibly single points.*

This theorem includes the theorem due to Marty†, that each extremal function $f_n(\zeta)$ maps Δ on a slit domain; another special case of Theorem I was recently published by the writer‡.

† F. Marty, *loc. cit.*

‡ "Un calcul de variation dans une famille de fonctions univalentes", *Comptes rendus*, 205 (1937), 709–711.

4. *The case* $n = 2$. In the case $n = 2$ it follows from (20) that, for each boundary continuum of D which contains more than one point, the following differential equation holds:

$$\left(\frac{dz}{dt}\right)^2 \frac{1}{a_2 z(t)^3} + 1 = 0. \tag{21}$$

Let z_1 be a fixed point in one of these continua; then, for every other point $z(t)$ of this same continuum, the integration of (21) yields

$$z(t) = [z_1^{-\frac{1}{2}} \pm \tfrac{1}{2} it \sqrt{a_2}]^{-2}. \tag{22}$$

Thus, by means of a real parameter t which is restricted to a certain interval, we can represent each boundary continuum containing more than one point in the form

$$z(t) = [z_\nu^{-\frac{1}{2}} + it \sqrt{a_2}]^{-2}, \tag{22'}$$

where z_ν is a point on the continuum considered. Now the extremal function $f_2(\zeta)$ maps Δ on a slit domain. On the other hand this function has no pole in Δ. Therefore, the point ∞ is always a boundary point and lies on one of the continua considered. This continuum is either a single point or a curve (22'). In the latter case this continuum must have the parametric representation $z = -(1/a_2)t$; for this is the only unbounded curve (22'). Thus we obtain the following

THEOREM II. *The extremal function whose coefficient* a_2 *has the largest absolute value in the family maps the domain* Δ *on a slit region which is bounded by curves* (22'); *if the point* ∞ *is not a single boundary point, it lies on the rectilinear slit* $z = -(1/a_2)t$.

It may be added that this result can also be derived from a theorem of de Possel† which we shall prove in § 7. If the initial domain Δ is the interior of the unit circle, an obvious conclusion from (22') is the theorem of Bieberbach‡: $|a_2| \leqslant 2$.

5. *The case* $n = 3$. In this case, the differential equation

$$\left(\frac{dz}{dt}\right)^2 \frac{1}{a_3} \left\{\frac{1}{z(t)^4} + \frac{2a_2}{z(t)^3}\right\} + 1 = 0 \tag{23}$$

† R. de Possel, "Zum Parallelschlitztheorem unendlich-vielfach zusammenhängender Gebiete", *Göttinger Nach.* (1931), 199–202.

‡ L. Bieberbach, "Über die Koeffizienten derj. Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln", *Berliner Berichte* (1916), 940–955 (946).

must be considered. It shows that if $a_2 \neq 0$ the continua of the boundary of D are analytic curves which can have singularities only at the points $z = -1/(2a_2)$ and $z = \infty$. By the transformation $z = 1/u$ it is easily shown that, if ∞ is not a single point, the curve containing it converges toward it in the asymptotic direction $\mu = \arg(-a_2/a_3)$, it does not, however, pass beyond but retraces its track. The point ∞ is, therefore, either a single point, or the endpoint of a boundary slit. If, on the other hand, in the neighbourhood of $z_0 = -1/(2a_2)$ we put $u = z - z_0$, we have near $u = 0$ the differential equation

$$\left(\frac{du}{dt}\right)^2 \frac{2a_2}{a_3} \frac{u}{(z_0+u)^4} + 1 = 0. \tag{23'}$$

From this we infer that the curve under consideration can fork at the point z_0 into at most three analytic branches which meet at z_0 at angles of $2\pi/3$. At all points except $-1/(2a_2)$ and ∞, the continua are regular analytic curves. The parameters a_2, a_3 of the equation (23) are, of course, different for each given domain Δ. We shall now continue the analysis of the special case in which the domain Δ is the interior of the unit circle. We shall show by this example how in many cases differential equations for the extremal function can be derived from the differential equations for the boundary.

Let $z = f_3(\zeta)$ be the extremal function considered. From the above-mentioned properties of the boundary of D we infer that $f_3(\zeta)$ is regular everywhere on the periphery of the unit circle, except at the points which correspond to $z = \infty$ and to $z = -1/(2a_2)$. To the point $z = \infty$ corresponds a double pole on the unit circle; to the point $z = -1/(2a_2)$ there correspond algebraic singularities of $f_3(\zeta)$, as is evident from the behaviour, in view of (23'), of the boundary curves in the neighbourhood of the critical point $-1/(2a_2)$. It can be seen from (23') that the part of the boundary which lies in a sufficiently small neighbourhood of $-1/(2a_2)$ is obtained by a conformal representation from the real axis in the t-plane with the help of a mapping function of the local parameter $t^{\frac{3}{2}}$; thus the conformal representation of the periphery of the unit circle on this same piece of the boundary, which is furnished by $f_3(\zeta)$, must be expansible in the neighbourhood of that point ζ_0 which corresponds to $-1/(2a_2)$ in the form

$$f_3(\zeta) = -\frac{1}{2a_2} + C_1(\zeta - \zeta_0)^{\frac{3}{2}} + \cdots. \tag{24}$$

Since $f_3(\zeta)$ is analytic on the unit circle, except at the points mentioned above, we can there write $\zeta = e^{is}$ and introduce this s as the real parameter of the boundary curve by means of the relation $z(s) = f(e^{is})$. s is obtained

from the old parameter t of the curve by a real parameter transformation, and therefore, in place of (23), we can write

$$g(\zeta)=\frac{1}{a_3}\,\frac{\zeta^2 f_3'(\zeta)^2}{f_3(\zeta)^4}\{2a_2 f_3(\zeta)+1\}\geqq 0 \quad (\zeta=e^{is}). \tag{25}$$

The function $g(\zeta)$ is regular in the interior of the unit circle except for a double pole at $\zeta=0$. By (25) it can be continued without ambiguity over the whole plane by Schwarz's principle of reflection. Its only poles are $\zeta=0$ and $\zeta=\infty$; for, in the unit circle, the simple function $f_3(\zeta)$ has no zeros and no poles except the above mentioned double pole. This does not, however, give rise to singularities of $g(\zeta)$. The branch points of $f_3(\zeta)$, also, do not give rise to singular points of $g(\zeta)$; for, on the Riemann surface of $f_3(\zeta)$, $g(\zeta)$ has poles only in the vicinity of the branch-points ζ_0; but from the expansion (24) results that $g(\zeta)$ is regular even at these points; finally, $g(\zeta)$ is uniform in the ζ-plane, so that ζ_0 is also not a branch-point for it; therefore $g(\zeta)$ is regular at ζ_0. Hence, $g(\zeta)$ is a rational function which has conjugate values at points symmetric with respect to the unit circle and possesses double poles at $\zeta=0$ and $\zeta=\infty$. Therefore

$$g(\zeta)=\frac{A}{\zeta^2}+\frac{B}{\zeta}+C+\bar{B}\zeta+\bar{A}\zeta^2, \tag{26}$$

with C real.

By comparing the coefficients, supposing that $a_3>0$, and using (15), we find that

$$\frac{\zeta^2 f_3'(\zeta)^2}{f_3(\zeta)^4}\{2a_2 f_3(\zeta)+1\}=\frac{1}{\zeta^2}+\frac{2a_2}{\zeta}+2a_3+2\bar{a}_2\zeta+\zeta^2. \tag{27}$$

This is a differential equation for $f_3(\zeta)$ which still contains the two unknown parameters a_2 and a_3. All the other coefficients of $f_3(\zeta)$ are to be calculated successively from (27). But an algebraic relation connecting a_2 and a_3 is at once evident. To the finite endpoint of the boundary slit of D must correspond a zero of $f_3'(\zeta)$ on the unit circle. From (27) it follows that the polynomial $P(\zeta)=\zeta^4+2\bar{a}_2\zeta^3+2a_3\zeta^2+2a_2\zeta+1$ possesses at least a double zero, and therefore its discriminant vanishes. We thus find that a_3 satisfies the equation

$$a_3{}^4-R(a_2{}^2)a_3{}^3-\left(5|a_2|^2-\frac{|a_2|^4}{4}+2\right)a_3{}^2 \tag{28}$$

$$+9R(a_2{}^2)\left(1+\frac{|a_2|^2}{2}\right)a_3-(|a_2|^2-1)^3-\frac{27}{4}R(a_2{}^2)^2=0.$$

The extremal function for the case $n = 3$ is already known; Löwner† has proved that the function

$$p(\zeta) = \frac{\zeta}{(1-\zeta)^2} = \sum_{n=1}^{\infty} n\zeta^n \tag{29}$$

possesses the third coefficient largest in absolute value. In this paper, we have only shown that the function $f_3(\zeta)$ possesses some characteristic properties of (29). But on the other hand, our method can immediately be generalized from $n = 3$ to arbitrary n.

6. *On the conjecture* $|a_n| \leqslant n$: It has been conjectured that the function (29) is, for each n, the extremal function $f_n(\zeta)$ in the unit circle. We shall now prove that (29) satisfies the differential equation (20) for each n. Since the function (29) maps the periphery of the unit circle on the section of the negative axis from $-\frac{1}{4}$ to $-\infty$, it is sufficient to show that, for $z < -\frac{1}{4}$, we always have

$$s_n(z) = \sum_{k=2}^{n} \frac{a_n^{(k)}}{z^{k+1}} \leqslant 0 \quad (n = 2, 3, \ldots), \tag{30}$$

the $a_n^{(k)}$ being the coefficients belonging to $p(\zeta)$.

By (16″), all $s_n(z)$ are obtained by means of the generating function

$$\frac{p(\zeta)^2}{\{p(\zeta)-z\}z^2} = -\sum_{n=2}^{\infty} s_n(z)\,\zeta^n. \tag{31}$$

It must be shown therefore that the function (31) has positive coefficients in the expansion in powers of ζ for all $z < -\frac{1}{4}$. Disregarding a positive factor, the left-hand side of (31) is

$$B(\zeta) = \frac{\zeta^2}{(1-\zeta)^2\{1-(2-t)\zeta+\zeta^2\}}, \tag{32}$$

where $0 \leqslant t \leqslant 4$. Putting $2-t = 2\cos\phi$, we find that

$$B(\zeta) = \frac{\zeta^2}{(1-\zeta)^2} \sum_{\nu=0}^{\infty} \frac{\sin(\nu+1)\phi}{\sin\phi}\,\zeta^\nu = \sum_{n=2}^{\infty} t_n \zeta^n, \tag{33}$$

where

$$t_n = \sum_{\nu=1}^{n} \frac{\sin\nu\phi}{\sin\phi}(n-\nu) = \frac{1}{4\sin^2\frac{1}{2}\phi}\left\{n - \frac{\sin n\phi}{\sin\phi}\right\}. \tag{34}$$

† K. Löwner, "Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I", *Math. Annalen*, 89 (1923), 103–121 (121).

Now, disregarding a positive factor, t_n is equal to $-s_n(z)$. From (34), it follows that t_n is positive for all values of $z < -\frac{1}{4}$, and therefore $s_n(z)$ is negative. We have thus proved

THEOREM III. *The function* (29) *satisfies the differential equations* (20) *for each extremal function* $f_n(\zeta)$.

7. *Generalizations of the coefficients problem.* Hitherto we have used the lemma of § 2 to estimate the coefficients of simple functions with the normalization (15). It is evident that more difficult problems on extrema of simple functions can also be treated in the same way. For example: Let $P(a_2, a_3, \ldots, a_m)$ be a polynomial of arbitrary order in the variables $a_2, a_3, \ldots, a_m$. We seek that function $f(\zeta)$ which is simple in the domain Δ and for which $|P(a_2, a_3, \ldots, a_m)|$ becomes a maximum. By the above considerations, we obtain

THEOREM IV. *The function* (15) *which is regular and simple in* Δ *and for which* $|P(a_2, a_3, \ldots, a_m)|$ *has its maximum within the family, maps* Δ *on a domain* D *whose boundary continua satisfy a differential equation*

$$\left(\frac{dz}{dt}\right)^2 \sum_{\nu=3}^{m+1} \frac{c_\nu}{\{z(t)\}^\nu} + 1 = 0, \tag{35}$$

where the c_ν *are certain constants depending on the domain* Δ *and on the polynomial* P.

The problem of the coefficients can also be set up for other normalizations than (15). For example: Let Δ be a domain in the ζ-plane containing the point at infinity, and consider the family of all functions which are simple in Δ and normalized in the form

$$f(\zeta) = \zeta + \frac{a}{\zeta} + \frac{b}{\zeta^2} + \cdots. \tag{36}$$

By the method of § 3, we construct from $f(\zeta)$ a new function

$$F_\rho\{f(\zeta)\} = f(\zeta) + \frac{A\rho^2}{f(\zeta) - z_0} + O(\rho^3) \tag{37}$$

which is simple in Δ. This development is convergent in the vicinity of ∞, and therefore $F_\rho\{f(\zeta)\}$ admits the expansion

$$F_\rho\{f(\zeta)\} = \zeta + \frac{a + A\rho^2 + O(\rho^3)}{\zeta} + \cdots. \tag{38}$$

This is a correctly normalized function (36) and the consideration of § 3 can be applied. For each extremal function a differential equation of the boundary of the mapped domain D can be found.

E.g., we seek that function (36) which is simple in Δ and for which $|a|$ assumes the largest possible value. For it the inequality

$$R\left\{\frac{A}{-a}\right\} \geqslant 0 \tag{39}$$

must hold at each point z_0 of the complement of D. By the lemma, D is therefore a slit domain whose slits satisfy the differential equation

$$-\frac{1}{a}\left(\frac{dz}{dt}\right)^2 + 1 = 0. \tag{40}$$

We thus prove that the function (36), simple in Δ and with the coefficient a largest in absolute value among the family, maps Δ on a domain bounded by parallel slits with the common direction $\operatorname{sgn}\sqrt{a}$.

This theorem is due to de Possel† who used it to establish the possibility of mapping an arbitrary domain on a domain bounded by parallel slits.

8. *On the growth of simple functions*: We content ourselves with the following remarks. We show that the lemma can also be used to prove certain theorems concerning the growth of simple functions. It is known that, among all functions (15) which are simple and regular in the interior of the unit circle, the function (29) has the quickest growth in the direction of the positive axis and assumes, at each point of it, values of larger modulus than those of every other function of the family. We investigate now the generalized problem which has the following form. Let Δ once more be an arbitrary domain in the ζ-plane not containing the point at infinity. Let us consider the family (15) of functions which are regular and simple in Δ. We fix a point ζ_0 in Δ, and seek that function of the family for which $|f(\zeta_0)|$ assumes the largest possible value within the family. Using the formulae (16′) and (16″) and concluding as in § 3, we obtain

$$\left(\frac{dz}{dt}\right)^2 \frac{f(\zeta_0)}{z^2\{z-f(\zeta_0)\}} + 1 = 0 \tag{41}$$

as the differential equation for the extremal function, which must hold for each of the mapped boundary slits.

---

† R. de Possel, *loc. cit.*

If z_0 a fixed point on a given boundary slit in the image-plane, we get

$$\int_{z_0}^{z} \frac{dx}{x\sqrt{(1-x/\gamma)}} = t, \tag{42}$$

a real number with $\gamma = f(\zeta_0)$. Carrying out the integration, we find that

$$\log\left\{\frac{1-\sqrt{(1-z/\gamma)}}{1+\sqrt{(1-z/\gamma)}}\right\} - \log\left\{\frac{1-\sqrt{(1-z_0/\gamma)}}{1+\sqrt{(1-z_0/\gamma)}}\right\} = t, \tag{43}$$

or writing

$$C = \frac{1+\sqrt{(1-z_0/\gamma)}}{1-\sqrt{(1-z_0/\gamma)}},$$

and taking s as a real parameter instead of t

$$z = f(\zeta_0)\,\frac{4sC}{(1+sC)^2}. \tag{44}$$

Here s varies in a certain real interval and (44) furnishes the corresponding points of the boundary continuum considered.

As in §§ 4 and 5, it is evident that one of the boundary continua must pass through ∞; but the only unbounded curve (44) is the straight line $z = f(\zeta_0)\,T$, T being another real parameter. Hence

THEOREM V. *If ζ_0 is a point in a given domain Δ, the function* (15) *which is regular and simple in Δ and assumes at ζ_0 the largest modulus within the family maps Δ on a domain bounded by curves with the parametric representation* (44). *In particular the straight line $z = f(\zeta_0)\,T$, with T varying in a certain real interval, is included among them.*

A completely analogous statement can be made concerning that function (15) which is simple in Δ and assumes in ζ_0 the least modulus within the family; the same statement is also valid for the simple function (15) which possesses the boundary point B whose distance from the origin is the least possible.

The Hebrew University,
Jerusalem.

ON THE COEFFICIENTS OF SIMPLE FUNCTIONS

By MENAHEM SCHIFFER.

[Received 24 February, 1938.—Read 17 March, 1938.]

1. A classical problem concerning simple functions may be formulated in the following way: Let Δ be a domain in the ζ-plane containing $\zeta = 0$, and let

$$z = f(\zeta) = \zeta + a_2\zeta^2 + a_3\zeta^3 + \dots a_n\zeta^n + \dots \tag{1}$$

be regular and simple in Δ. The set of all normalized functions (1) forms a normal family, and there exist therefore functions $f_n(\zeta)$ of the type (1) for which $|a_n|$ possesses the largest value among the family. What is then this maximal value of $|a_n|$ and what can be stated of the extremal functions $f_n(\zeta)$ possessing the maximal $|a_n|$?

Each function (1) can be characterized by the domain D in the z-plane on which Δ is mapped with the aid of (1); and the problem is solved when the domains D_n belonging to the extremal functions $f_n(\zeta)$ are determined. By a method due to Marty* it is easily shown that the domains D_n are, for each n, slit domains. In a recent paper† it was shown that the boundary slits of D_n are analytic curves $z(t)$ satisfying the differential equation

$$\left(\frac{1}{z}\frac{dz}{dt}\right)^2 S_n(z) + 1 = 0. \tag{2}$$

Here, $S_n(z)$ is a polynomial of the degree $n-1$ in $1/z$,the coefficients of which are formed in a certain manner from the coefficients of $f_n(\zeta)$. This

---

* F. Marty, "Sur le module des coefficients de MacLaurin d'une fonction univalente", *Comptes rendus*, 198 (1934), 1569.

† M. Schiffer, "A method of variation within the family of simple functions", *Proc. London Math. Soc.* (2), 44 (1938), 433–449.

polynomial may be obtained in the following way. We form the function

$$F_z(\zeta) = \frac{f_n(\zeta)}{1-f_n(\zeta)/z} = \sum_{\nu=1}^{\infty} a_\nu \{1+S_\nu(z)\}\zeta^\nu \quad (a_1 = 1), \tag{3}$$

and in this development $S_n(z)$ is the polynomial* just mentioned.

The only possible critical points of the boundary curves of D_n are the zeros of $S_n(z)$ lying on these. So far as is known at present they may be branch points of these curves, and this paper is devoted to the investigation of these critical points.

2. If z is a boundary point of D_n the function (3) is simple and regular in Δ, and belongs therefore to the family (1). In view of the extremal property of $f_n(\zeta)$, we have

$$|1+S_n(z)| \leqslant 1. \tag{4}$$

If, at a boundary point $z_0 \neq \infty$, we have $S_n(z_0) = 0$, then equality holds in (4), and $F_{z_0}(\zeta)$ is also an extremal function. Therefore, all theorems on $f_n(\zeta)$ hold also for $F_{z_0}(\zeta)$ and in particular there exists an equation analogous to (2).

The image of Δ given by $F_{z_0}(\zeta)$ is obtained from D_n by the linear transformation

$$y = \frac{z}{1-z/z_0}. \tag{5}$$

Its boundary curves $y(\tau)$ satisfy the differential equation

$$\left(\frac{1}{y}\frac{dy}{d\tau}\right)^2 T_n(y)+1 = 0, \tag{2'}$$

where $T_n(y)$ is obtained from the generating function

$$G_y(\zeta) = \frac{F_{z_0}(\zeta)}{1-F_{z_0}(\zeta)/y} = \sum_{\nu=1}^{\infty} a_\nu \Big(1+S_\nu(z_0)\Big)\Big(1+T_\nu(y)\Big)\zeta^\nu. \tag{3'}$$

If z is connected with y by (5), we have also, by (3),

$$G_y(\zeta) = \frac{f_n(\zeta)}{1-f_n(\zeta)/z} = \sum_{\nu=1}^{\infty} a_\nu \Big(1+S_\nu(z)\Big)\zeta^\nu; \tag{3''}$$

hence, comparing (3') with (3'') and assuming that $S_n(z_0) = 0$, we have

$$T_n(y) = S_n(z). \tag{6}$$

* The author is much indebted to Miss M. L. Cartwright for calling his attention to this relation.

Substituting for ζ in terms of z in (2′) with the aid of (5) and (6), we get a new differential equation (involving a new parameter τ)

$$\left(\frac{1}{z}\frac{dz}{d\tau}\right)^2 \frac{1}{(1-z/z_0)^2} S_n(z)+1=0 \tag{7}$$

for the boundary curves of D_n; and, since t and τ are both real, (2) and (7) together yield

$$\left(1-\frac{z}{z_0}\right)^2 \geqslant 0 \tag{8}$$

for all z on the boundary curves. This shows that

$$z = Tz_0, \tag{9}$$

with real T. So we find that if $S_n(z_0)=0$ holds for a finite boundary point z_0, all the boundary slits which contain more than one point lie on a ray through the origin.

Therefore the boundary curves of the domain D_n are analytic curves which have no finite critical points.

The Hebrew University,
Jerusalem.

INVOLUTORISCHE TRANSFORMATIONEN UND SELBSTREZIPROKE FUNKTIONEN

Von H. KOBER.

[Received 13 December, 1937.—Read 16 December, 1937.]

Die Funktionen $f(x)$, $g(x)$, $h(x)$ sollen zum Raume $\mathfrak{L}_2$, der Menge aller quadratisch von Null bis Unendlich Lebesgue-integrierbaren Funktionen, gehören. Sei $T[\chi]$†, $g = Tf$ die Transformation‡

$$\int_0^x g(\xi)\,d\xi = \int_0^\infty \frac{\chi(tx)}{t} f(t)\,dt, \quad \frac{\chi(x)}{x} \subset \mathfrak{L}_2. \tag{1.1}$$

Die Frage nach den Eigenwerten und Eigenfunktionen, d.h. nach den endlichen Zahlenwerten von λ, für welche die Gleichung $f - \lambda Tf = 0$ mindestens eine von Null verschiedene Lösung $f(x)$ hat, und nach diesen Lösungsfunktionen $f(x)$, ferner die Frage nach der Darstellung einer willkürlichen Funktion $h(x)$ durch Eigenfunktionen, findet hier die Beantwortung:

† Herr E. Frola hat die etwas speziellere Transformation

$$g(x) = \int_0^\infty A(tx) f(t)\,dt$$

behandelt und u.a. die hier als Satz I und Satz II'*a* ausgesprochenen Tatsachen bewiesen, I für ein engeres, II'*a* für ein anderes Feld von Funktionen als hier, *Annali di Matematica* IV, 16 (1937), 127–155, s. Theor. X, VI und Corollar II.

‡ Für $\chi(x) = (2/\pi)^{\frac{1}{2}} \sin x$ bezw. $(2/\pi)^{\frac{1}{2}}(1 - \cos x)$ ist $T[\chi]$ die Fourier-cos- bezw. sin-Transformation, für reellwertiges χ bei Erfüllung von (1 . 2) sind allgemeiner die $T[\chi]$ genau G. N. Watson's "General Transforms", *Proc. London Math. Soc.* (2), 35 (1933), 156–199. Dann und nur dann ist $T[\chi]$ stetig (beschränkt also), wenn $\omega(t)$ (s. Abschn. 1) beschränkt ist in $(-\infty, \infty)$, Verf. I, bezw. II, *Quart. J. of Math.*, 8 (1937), 172–185, bezw. 9 (1938), 41–52. Man messe die folgenden an den wohlbekannten Ergebnissen für Integralgleichungen mit einem symetrischen Kern vom Hilbert-Schmidt-Typ., E. Schmidt, *Math. Annalen*, 63 (1907), 433–76, *cf.* M. H. Stone, "Linear transformations in Hilbert space", *Amer. Math. Soc. Colloquium Publications*, 15 (1932), 196, und an M. Plancherels Ergebnissen, *Rend. di Palermo*, 30 (1910), 289–335.

SATZ I. *Damit für die Integralgleichung* (1.1) *mindestens eine Eigenfunktion existiert, deren Mellinsche Transformierte fast überall von Null verschieden ist, ist notwendig und hinreichend, dass für alle positiven u und v*

$$(1.21) \qquad \int_0^\infty \frac{\chi(ux)\chi(vx)}{x^2}\,dx = c\,\min(u, v) \quad (c \neq 0)$$

ist. Dann gibt es genau zwei Eigenwerte, $\lambda = \pm c^{-\frac{1}{2}}$.

Wählt man, ohne Einschränkung der Allgemeinheit, $c = 1$, so nennt man die zu $\lambda_1 = 1$ bezw. $\lambda_2 = -1$ gehörigen Eigenfunktionen selbst-† bezw. schiefreziprok, die Bedingung (1.21) wird

$$(1.2) \qquad \int_0^\infty \frac{\chi(ux)\chi(vx)}{x^2}\,dx = \min(u, v).$$

Die schiefreziproken Funktionen von $T[\chi]$ sind selbstreziprok für $T[-\chi]$.

SATZ II. *Sei* $T[\chi]$ *eine durch* (1.1) *und* (1.2) *definierte Transformation. Wenn* $T[\chi]$ *beschränkt ist, dann und nur dann ist jede Funktion* $h(x)$ *aus* $\mathfrak{L}_2$ *darstellbar*

(*a*) *als Summe aus je einer selbst- und eine schiefreziproken Funktion von* $T[\chi]$,

(*b*) *in der Form* $\Sigma a_n \phi_n$ *mit* $\Sigma |a_n|^2 < \infty$ *in dem Sinne*

$$\left\| h(x) - \sum_{n=0}^{m} a_n \phi_n(x) \right\| = \left(\int_0^\infty \left| h(x) - \sum_{n=0}^{m} a_n \phi_n(x) \right|^2 dx \right)^{\frac{1}{2}} \to 0 \quad (m \to \infty),$$

wobei die Folge $\{\phi_n\}$ *eine feste Folge von Eigenfunktionen ist.*

Die Darstellungen IIa und, nach Festlegung der Folge $\{\phi_n\}$, *IIb sind eindeutig.*

An Stelle von Satz II wird ein allgemeinerer Satz II′ über involutorische (d.h. $T^2 f = f$) lineare Transformationen bewiesen werden. Mit dessen Hilfe beherrscht man alle Transformationen dieser Art und kann ein Verfahren angeben zur Konstruktion sowohl sämtlicher Transformationen dieser Klasse wie zum Aufbau solcher mit gewissen speziellen Eigenschaften (Abschnitt 6).

SATZ III. *Sei T eine lineare abgeschlossene Transformation, beschränkt oder nicht, mit einem in* $\mathfrak{L}^2$ *überall dichten Bereiche* $\mathfrak{D}_T$, *und* $T^2 f = f$ $(f \subset \mathfrak{D}_T)$.

† G. H. Hardy and E. C. Titchmarsh, *Quart. J. of Math.*, 1 (1930), 196–231.

Jede Funktion aus $\mathfrak{L}_2$ *ist eindeutig darstellbar*

(*a*) *als Summe je einer selbstreziproken Funktion von* T *und einer schiefreziproken von* T^*, *der Adjungierten von* T (*s.* 5.1, *Abschn.* 5),

(*b*) *in der Form* $a_0\phi_0+a_1\phi_1+\ldots$, *mit* $\Sigma|a_n|^2<\infty$ *im Sinne von Satz IIb, wobei* ϕ_0, ϕ_2, $\phi_4\ldots$ *ein vollständiges orthonormales fest gewähltes System selbstreziproker Funktionen von* T, ϕ_1, ϕ_3, ϕ_5, ... *ein solches von* $-T^*$ *ist.*

Ersichtlich können nicht mehr Eigenwerte als $\lambda=+1$ und $\lambda=-1$ auftreten, da für $g=\lambda Tg$ doch $g=\lambda^2 g$, $\lambda=\pm 1$ folgt.

Die Transformationen $T[\chi]$ der Art (1.1) und (1.2) erfüllen die Voraussetzungen des Satzes III (Verf. II, A, B und hier 1.3).

SATZ IV. *Sei* $T[\chi]$ *eine Transformation, definiert durch* (1.1) *und* (1.2). *Die in Bezug auf* $T[\chi]$ *selbstreziproken Funktionen bilden einen Hilbertschen Raum*†.

Auf Grund von Satz IV gibt es also eine unendliche Folge $\phi_0(x)$, $\phi_1(x)$, ... derart, dass

$$(1)\qquad \int_0^\infty \phi_m(x)\,\bar{\phi}_n(x)\,dx = 0 \quad \text{für} \quad m\neq n,$$
$$= 1 \quad \text{für} \quad m=n$$
$$(m=0,\ 1,\ 2,\ \ldots;\ \ n=0,\ 1,\ 2,\ \ldots).$$

(2) Eine Funktion $h(x)$ dann und nur dann selbstreziprok ist, wenn sie mit $\Sigma|a_n|^2<\infty$ darstellbar ist in der Form $h(x)=\Sigma a_n\phi_n(x)$ im Sinne von II*b*. Die Folge $\{\phi_n\}$ heisst† ein vollständiges orthonormales System des Raumes der selbstreziproken Funktionen. Im Falle der Fourier-cos-Transformation‡, $\chi(x)=(2/\pi)^{\frac{1}{2}}\sin x$, bilden die Funktionen

$$\phi_n(x)=c_n e^{\frac{1}{2}x^2}\left(\frac{d}{dx}\right)^n e^{-x^2},\qquad c_n=(-1)^n\,(2^{n-1}n!\,\sqrt{\pi})^{-\frac{1}{2}},$$

für $n=0,\ 4,\ 8,\ \ldots$, ein vollständiges orthonormales System des Raumes der selbstreziproken Funktionen, für $n=2,\ 6,\ 10,\ \ldots$, des der schiefreziproken

† Dessen Definition und Eigenschaften s. J. v. Neumann, *Math. Annalen*, 102 (1930), 49–131, hier als "Neumann I" zitiert, und Stone, *loc. cit.* So ist z.B. $\mathfrak{L}_2$ ein solcher Raum, wenn der "Abstand" zweier Elemente f und g als $\|f-g\|$ (s. Schluss d. Einleitung) und so die Metrik des Raumes festgelegt wird. "Neumann II" ist *Annals of Math.* (2), 33 (1932), 294–310.

‡ Das ist wohl schon bekannt, jedenfalls leicht herzuleiten aus z.B. Theor. 55, 57 u. §9.12 in E. C. Titchmarsh, *Introduction to the theory of Fourier integrals* (Oxford, 1937). *Cf.* E. Feldheim, *Bull. Sci. Math.* 2, 61 (1937), 10–18.

Funktionen. Entsprechendes gilt für $n = 1, 5, 9, \ldots$ bezw. $n = 3, 7, 11, \ldots$ im Falle der Fourier-sin-Transformation.

Im fünften Abschnitt dieser Arbeit werden einige allgemeine Sätze über Transformationen bewiesen; sie werden z.T. in dieser und einer anschliessenden Arbeit gebraucht, sie sind z.T. für das volle Verständnis der Eigenschaften hier auftretender Transformationen erforderlich; in der Literatur sind sie noch nicht zu finden.

Die Mellinsche Transformierte Uf von $f(x)$ und die Bezeichnungen (f, g), $\|f\|$ mögen bedeuten

$$Uf = F(t) = \underset{n\to\infty}{\text{l.i.m.}} \int_{1/n}^{n} f(x)\, x^{-\frac{1}{2}+it}\, dx \quad \left(\begin{matrix}\text{in mean square}\\ \text{Quadratmittel}\end{matrix}\right),$$

$$(0.1) \qquad (f, g) = \int_0^\infty f(x)\,\bar{g}(x)\, dx, \quad \|f\| = (f, f)^{\frac{1}{2}}.$$

Die Funktion $\mathfrak{u}(x)$ sei gleich 1 für $0 \leqslant x < 1$, sonst gleich Null. Äquivalente Funktionen f und g $(f \equiv g)$ sollen als eine Funktion gelten, $f_n \to f$ soll $\|f - f_n\| \to 0$ für $n \to \infty$ bedeuten.

1. *Beweis von Satz I.* Sei

$$F(t) = Uf, \quad G(t) = Ug, \quad \omega(t)/(\tfrac{1}{2} - it) = U\left(\chi(x)/x\right).$$

Die Transformation (1.1) ist äquivalent† mit der Gleichung

$$(1.11) \qquad G(t) = \omega(t)\, F(-t); \quad G,\ F,\ \omega(t)/(\tfrac{1}{2} - it) \subset L^2(-\infty, \infty).$$

Der Bereich $\mathfrak{D}$ von $T[\chi]$, d.h. die Menge $\mathfrak{D}$ der Funktionen $f(x)$, für die Tf Sinn hat‡, ist durch die Bedingungen $F(t)$, $F(t)\,\omega(-t) \subset L^2(-\infty, \infty)$ bestimmt, der Vorrat von T ist die Menge der zugeordneten $g = Tf$. Dann und nur dann ist $\mathfrak{D}$ der ganze Raum $\mathfrak{L}_2$, wenn T beschränkt (stetig) ist, also dann und nur dann, wenn $\omega(t)$ beschränkt ist§.

Aus der Gleichung $f - \lambda Tf = 0$ folgt wegen (1.11) sofort

$$(1.22) \qquad F(t) = \lambda\omega(t)\, F(-t), \quad F(t) = \lambda^2\, \omega(t)\, \omega(-t)\, F(t).$$

Wenn also wenigstens eine Eigenfunktion $f(x)$ der Art existiert, dass $F(t) = Uf$ auf keiner Menge der t von positivem Masze verschwindet, muss

$$(1.23) \qquad \omega(t)\,\omega(-t) = \lambda^{-2} = c \quad (0 < |c| < \infty)$$

sein, und nach bekannten Sätzen ist (1.23) äquivalent mit (1.21)‖.

† Verf. I, Hilfssätze 2 (ii) und (iii).

‡ $Tf = g$ muss also eine Funktion g aus $\mathfrak{L}_2$ darstellen; *cf.* Verf. II, Abschn. 3.

§ Verf. I, Satz. 2; II, Satz D.

‖ I. W. Busbridge, *Journal London Math. Soc.*, 9 (1934), 179–87 (2.4.1).

Die Bedingung ist auch hinreichend. Sei $F(t)=(\frac{1}{2}+it)^{-1}$ für $t>0$, $\eta=\pm 1$, $F(t)=\eta c^{-\frac{1}{2}}\omega(t)(\frac{1}{2}-it)^{-1}$ für $t<0$, so ist $F(t)$ aus $L^2(-\infty,\infty)$, und sie ist überall von Null verschieden und erfüllt (1.22), wie man leicht sieht wegen (1.11) ist $f(x)=U^{-1}F$ eine Eigenfunktion. Satz I ist bewiesen.

Normiert man $\chi(x)$ durch Multiplikation mit $c^{-\frac{1}{2}}$, so werden (1.21), (1.23):

$$\int_0^\infty \frac{\chi(ux)\chi(vx)}{x^2}\,dx=\min(u,v); \tag{1.2}$$

$$\omega(t)\,\omega(-t)=1. \tag{1.2'}$$

Aus (1.11) folgt dann sofort $F(t)=G(-t)\,\omega(t)$, also für $g=Tf$

$$f=Tg,\quad T^2f=TTf=f\quad (f\subset\mathfrak{D}). \tag{1.3}$$

2. *Beweis von Satz II.* Da eine Transformation $T[\chi]$ der Art (1.1) dann und nur dann beschränkt ist, wenn ihr Bereich $\mathfrak{L}_2$ ist (Verf. II, Satz D), folgt Satz II aus dem allgemeinen

SATZ II'. *Sei T linear und $T^2f=f$†. Alle Funktionen aus dem Bereiche $\mathfrak{D}_T$ von T und nur solche sind darstellbar*

(a) als Summe aus je einer selbst- und einer schiefreziproken Funktion von T,

(b) wenn T abgeschlossen ist, in der Form $\Sigma a_n\phi_n$ mit $\Sigma|a_n|^2<\infty$, wobei $\{\phi_n\}$ eine feste Folge von Eigenfunktionen ist.

Die Darstellungen IIa und, nach Festlegung von $\{\phi_n\}$, IIb sind eindeutig.

Wenn T abgeschlossen ist, sind die Mengen $\mathfrak{R}_T$ und $\mathfrak{R}_{-T}$ der selbst- bezw. schiefreziproken Funktionen abgeschlossene lineare Mannigfaltigkeiten.

Beweis. Wegen der Linearität von T und wegen $T^2f=f$ ist für

$$\phi=\tfrac{1}{2}(f+Tf),\quad \psi=\tfrac{1}{2}(f-Tf)\quad (f\subset\mathfrak{D}_T) \tag{2.1}$$

$T\phi=\phi$ (Hardy-Titchmarsh, *loc. cit.*), $T\psi=-\psi$,

$$f=\phi+\psi\quad (\phi\subset R_T,\ \psi\subset\mathfrak{R}_{-T}). \tag{2.2}$$

Ist umgekehrt eine Funktion f in der Form (2.2) darstellbar, so muss, da $T\phi$ und $T\psi$ Sinn haben, auch Tf Sinn haben, also f zu $\mathfrak{D}_T$ gehören. Aus (2.2) folgt $Tf=\phi-\psi$, hieraus und aus (2.2) folgt (2.1), die Darstellung ist eindeutig.

† Für alle f aus $\mathfrak{D}_T$ soll also Tf zu $\mathfrak{D}_T$ gehören, also $T^2f=T(Tf)$ Sinn haben.

Sei nun T auch abgeschlossen. Die Linearität der Menge $\mathfrak{R}_T$ z.B. folgt aus der von T; ebenso ihre Abgeschlossenheit aus der von T: für $\phi_n \subset \mathfrak{R}_T$, $\phi_n \to \phi$ folgt $T\phi = \phi$ aus $T\phi_n = \phi_n$; also $\phi \subset \mathfrak{R}_T$. Also ist $\mathfrak{R}_T$ eine abgeschlossene lineare Mannigfaltigkeit in $\mathfrak{L}_2$ und somit entweder ein m-dimensionaler ($m \geqslant 0$) unitärer Raum oder ein Hilbertscher Raum†, ebenso $\mathfrak{R}_{-T}$.—Übrigens muss mindestens einer dieser Räume ein Hilbertscher sein, falls die lineare Mannigfaltigkeit $\mathfrak{D}_T$ in irgendeinem Hilbertschen Raume überall dicht ist (*cf.* Satz IV), da sonst wegen Satz II′ a nur endlich viele lineare unabhängige Elemente in $\mathfrak{D}_T$ vorhanden wären, ein offenbarer Widerspruch. Im Falle $Tf = -f$ z.B. ist $\mathfrak{R}_T$ leer, $\mathfrak{R}_{-T} = \mathfrak{L}_2$.—Seien $\phi_0, \phi_2, \phi_4, \ldots$ bezw. $\phi_1, \phi_3, \phi_5, \ldots$ vollständige orthonormale Systeme der Räume $\mathfrak{R}_T$ bezw. $\mathfrak{R}_{-T}$, so sind alle selbst- bezw. schiefreziproken Funktionen und nur solche von der Form $a_0\phi_0 + a_2\phi_2 + \ldots$ bezw. $a_2\phi_1 + a_3\phi_3 + \ldots$ mit $\Sigma|a_n|^2 < \infty$, aus II′a folgt jetzt leicht II′b.

3. *Beweis von Satz III.* Wenn T linear abgeschlossen ist und der Bereich in $\mathfrak{L}_2$ überall dicht ist, hat T^* dieselben Eigenschaften (Neumann II, Satz 3). Seien $\mathfrak{D}_T$ und $\mathfrak{D}_T{}^*$ die Bereiche von T und T^*. Wenn ausserdem $T^2 f = f$ ($f \subset \mathfrak{D}_T$) ist, so ist für alle g aus $\mathfrak{D}_T{}^*$

$$(f, g) = \big(T(Tf), g\big) = (Tf, T^*g),$$

also auf Grund der Definition der Adjungierten $g = T^*T^*g = (T^*)^2 g$.

HILFSSATZ. *Die Menge $\mathfrak{R}^*_T$ aller schiefreziproken Funktionen von T^* ist mit der Menge der zu allen selbstreziproken Funktionen von T orthogonalen Funktionen aus $\mathfrak{L}_2$ identisch*, $\mathfrak{R}^*_T = \mathfrak{L}_2 - \mathfrak{R}_T$.

Beweis. Für g aus $\mathfrak{R}^*_T$ ist $T^*g = -g$, also $(f, g) = 0$ für alle f aus $\mathfrak{R}_T$ wegen

$$(f, g) = (Tf, g) = (f, T^*g) = (f, -g) = -(f, g).$$

Ist umgekehrt $(f, g) = 0$ für alle f aus $\mathfrak{R}_T$, so ist für jedes k aus $\mathfrak{D}_T$ sicher $k + Tk$ selbstreziprok, also $k + Tk$ ein solches f

$$0 = (k + Tk, g) = (k, g) + (Tk, g).$$

Aus $(k, g) = -(Tk, g)$ folgt durch die Definition der Adjungierten $T^*g = -g$, $g \subset \mathfrak{R}^*_T$.

† Stone *loc. cit.*, Theor. 1.19.

Genau wie im Beweise von Satz II folgt jetzt, dass $\mathfrak{R}_T$ und $\mathfrak{R}^*_{\pm T}$ abgeschlossene lineare Mannigfaltigkeiten sind, nach einem allgemeinen Satze (Neumann I, Satz 13) folgt die eindeutige Darstellbarkeit jeder Funktion aus $\mathfrak{L}_2$ in der Form $h = f+g$, $f \subset \mathfrak{R}_T$, $g \subset \mathfrak{R}^*_{\pm T}$. Satz III$a$ ist bewiesen.

Es folgt leicht (*cf.* Abschn. 2), dass mindestens einer der beiden Räume $\mathfrak{R}_T$ und $\mathfrak{R}^*_{\pm T}$ ein Hilbertscher Raum ist; z.B. für $Tf = f$ ist $Tg^* = g$, $\mathfrak{R}^*_{-T}$ leer, $\mathfrak{R}_T = \mathfrak{L}_2$.

Satz IIIb folgt leicht aus IIIa. Wenn nun $T[\chi]$ eine Transformation der Art (1.1) und (1.2) ist, ist $T^* = T[\bar{\chi}]$ (Verf. II, Satz C), wenn also eine Funktion g zu $\mathfrak{R}^*_{\pm T}$ gehört, gehört $\bar{g}$ zu $\mathfrak{R}_{-T}$, jede Funktion $h(x)$ ist also mit $\Sigma|a_n|^2 < \infty$ in der Form darstellbar

$$h = a_0\phi_0 + a_1\bar{\phi}_1 + a_2\phi_2 + a_3\bar{\phi}_3 + \ldots \quad (\phi_{2n} \subset \mathfrak{R}_T,\ \phi_{2n+1} \subset \mathfrak{R}_{-T}).$$

4. *Beweis von Satz IV.* Auf Grund des letzten Teiles von Satz II′ muss nur gezeigt werden, dass die Anzahl linear unabhängiger selbstreziproker Funktionen einer Transformation $T[\chi]$ der Art (1.1) und (1.2) nicht endlich ist. Sei $f(x)$ aus $\mathfrak{R}[\chi] = \mathfrak{R}_T$, also wegen (1.11)

$$Uf = F(t) = \omega(t)\,F(-t). \tag{4.1}$$

Sei $h(x)$ irgend eine Funktion aus $\mathfrak{L}_2$ mit $h(x)\,\omega(-x) \subset \mathfrak{L}_2$, dann erfüllt die Funktion $F(t)$, die gleich $h(t)$ bezw. $h(-t)\,\omega(t)$ für $t > 0$ bezw. $t < 0$ ist, (4.1), und es ist (*cf.* Verf. I, 178)

$$\|h(x)\| = \|F(x)\| \leqslant \left(\int_{-\infty}^{\infty} |F(t)|^2\,dt\right)^{\frac{1}{2}} = C\,\|f(x)\| \quad [C = \sqrt{(2\pi)}], \tag{4.2}$$

$$f(x) = U^{-1}F = \frac{1}{2\pi}\,\underset{n\to\infty}{\text{l.i.m.}} \int_0^n h(t)\,x^{-\frac{1}{2}}\{x^{-it} + \omega(-t)\,x^{it}\}\,dt. \tag{4.3}$$

Da nun $h(t)$ auch durch $F(t)$ bestimmt ist, besteht eine ein-eindeutige Zuordnung S zwischen den Funktionen h und den f aus $\mathfrak{R}[\chi]$,

$$f = Sh, \quad h = S^{-1}f, \quad \|h\| \leqslant C\,\|f\|. \tag{4.4}$$

Gäbe es nur endlich viele, etwa n, linear unabhängige Elemente $f \subset \mathfrak{R}[\chi]$, so müsste das Gleiche für die Funktionen h gelten; aber bereits die $n+1$ Funktionen $h_r(x) = \mathfrak{u}(x/r)$, $r = 1, 2, \ldots, n+1$, sind linear unabhängig, und $h(x)$ gehört zu $\mathfrak{L}_2$, ebenso $h_r(x)\,\omega(-x)$ wegen $\omega(x)/(\frac{1}{2} - ix) \subset L^2(-\infty, \infty)$. Satz IV ist bewiesen.

5. *Allgemeine Transformationen.*

Seien $\mathfrak{H}$ und $\mathfrak{K}$ zwei Hilbertsche Funktionenräume, in jedem sei die Bedeutung des "inneren Produktes" (*cf.* 0.1), und so die Metrik festgelegt, seien $f(x)$, $g(x)$, $h(x)$ jetzt stets Funktionen aus $\mathfrak{H}$, $\phi(x)$, $\gamma(x)$, $\eta(x)$ aus $\mathfrak{K}$, $\phi = Af$ eine Transformation aus $\mathfrak{H}$ nach $\mathfrak{K}$. Mit folgendem Verfahren lassen sich die bekannten Sätze, die für eine Transformation $g = Bf$ aus $\mathfrak{H}$ nach $\mathfrak{H}$ gelten, ganz leicht auf den allgemeinen Fall $\phi = Af$ übertragen†.

Seien $\{h_n\}$ bezw. $\{\eta_n\}$ vollständige orthonormale Systeme von $\mathfrak{H}$ bezw. $\mathfrak{K}$. Sei $g(x)$ gegeben, also $g = \Sigma a_n h_n$ mit $\Sigma|a_n|^2 < \infty$, so ordne man $g(x)$ in $\mathfrak{K}$ die Funktion $\gamma(x) = \Sigma a_n \eta_n(x)$ zu. Diese Transformation $\gamma = Ug$ aus $\mathfrak{H}$ nach $\mathfrak{K}$ ist unitär, denn

$$\|\gamma(x)\|^2 = (\gamma, \gamma) = \Sigma|a_n|^2 = (g, g) = \|g(x)\|^2, \quad \|Ug\| = \|g\|,$$

ihr Bereich ist ganz $\mathfrak{H}$, ihr Vorrat ganz $\mathfrak{K}$, sie ist natürlich stetig und hat eine stetige Umkehrung. Als Beispiel soll ein für den Fall $\mathfrak{K} = \mathfrak{H}$ wohlbekannter Satz verallgemeinert werden (Neumann II, Satz 12):

SATZ α. *Eine lineare abgeschlossene Transformation aus $\mathfrak{H}$ nach $\mathfrak{K}$, deren Bereich ganz $\mathfrak{H}$ ist, ist beschränkt (stetig).*

Sei $\phi = Af$, A linear und abgeschlossen, der Bereich sei $\mathfrak{H}$, der Vorrat ein Teil von $\mathfrak{K}$. Sei $B = U^{-1}A$, also aus $\mathfrak{H}$ nach $\mathfrak{H}$ mit ganz $\mathfrak{H}$ als Bereich. B ist linear, da U und A es sind, und abgeschlossen, d.h. aus

$$r_n(x) = Bf_n(X), \quad f_n \to f \subset \mathfrak{H}, \quad r_n \to r \subset \mathfrak{H}$$

folgt $r = Bf$. Denn aus $r_n \to r$ folgt $Ur_n \to Ur$, also

$$Af_n = UBf_n = Ur_n \to Ur;$$

da A abgeschlossen ist, folgt aus $f_n \to f$, $Af_n \to Ur$ nun $Af = Ur$, daher $Bf = U^{-1}Af = r$. Nach dem bekannten Satze ist somit B beschränkt, folglich auch A:

$$\|Af\| = \|UBf\| = \|Bf\| \leqslant c\|f\|.$$

Die Adjungierte A^* einer Transformation A, deren Bereich $\mathfrak{D}_A$ in $\mathfrak{H}$ überall dicht ist, wird definiert durch $A^*\eta = h^*$, wobei η, h^* ein von der Wahl von f in $\mathfrak{D}_A$ unabhängiges Lösungspaar der Gleichung†

$$(Af, \eta) = (f, h^*) \tag{5.1}$$

† Alle diese Sätze sind durch ein anderes Verfahren übertragen worden von F. J. Murray, *Trans. American Math. Soc.*, 37 (1935), 301–338.

ist; A^* ist eine Transformation aus $\mathfrak{R}$ nach $\mathfrak{H}$. Es gelten z.B.

SATZ β. *Sei A l.a.d. aus $\mathfrak{H}^*$ nach $\mathfrak{R}$. Dann existiert die Adjungierte A^* und ist l.a.d. aus $\mathfrak{R}$ nach $\mathfrak{H}$, die Adjungierte von A^* ist A.*

SATZ γ. *Sei A l.a.d. aus $\mathfrak{H}$ nach $\mathfrak{R}$. Dann und nur dann ist A umkehrbar, also die Zuordnung $\phi = Af$ ein-eindeutig, wenn der Vorrat der Adjungierten in $\mathfrak{H}$ überall dicht ist.* (Verf. II, s. Beweis von H).

Der Satz IV steht in unmittelbarer Beziehung zu dem allgemeinen

SATZ V. *Sei A l.a.d. aus $\mathfrak{H}$ nach $\mathfrak{R}$ und $\|Af\| \geqslant c\|f\|$, $c > 0$, für alle f aus dem Bereiche von A. Dann existiert die Umkehrung A^{-1}, ist linear und stetig, und der Vorrat $\mathfrak{A}$ von A ist ein Hilbertscher Raum. Die Adjungierte A_0^* aus $\mathfrak{A}$ nach $\mathfrak{H}$ hat den Vorrat $\mathfrak{H}$ und ebenfalls eine stetige Umkehrung.*

Wegen $\|Af\| \geqslant c\|f\|$ kann Af nur für $f = 0$ verschwinden, aus der Linearität folgt dann die Umkehrbarkeit. Sei $Af = \phi$, also $f = A^{-1}\phi$, so ist $\|A^{-1}\phi\| \leqslant c^{-1}\|\phi\|$, daher A^{-1} beschränkt (stetig).

Der Raum $\mathfrak{A}$ ist ersichtlich linear. Als Teilraum von $\mathfrak{R}$ ist $\mathfrak{A}$ separabel. $\mathfrak{A}$ ist auch abgeschlossen: Sei $\{\phi_n\}$ aus $\mathfrak{A}$, $\|\phi_m - \phi_n\| \leqslant \epsilon$ für $m \geqslant N(\epsilon)$, $n \geqslant N(\epsilon)$, so konvergiert $\{\phi_n\}$ in $\mathfrak{R}$ gegen eine Grenzfunktion ϕ, die Folge $\{f_n\} = \{A^{-1}\phi_n\}$ wegen $\|f_m - f_n\| \leqslant c^{-1}\|\phi_m - \phi_n\| \leqslant \epsilon/c$ in $\mathfrak{H}$ gegen eine Grenzfunktion f; da A abgeschlossen ist, muss $\phi = Af$ sein, also ϕ zu $\mathfrak{A}$ gehören. Die Dimensionszahl von $\mathfrak{A}$ ist unendlich; sonst würde aus der Umkehrbarkeit von A (*cf.* Abschn. 4) eine endliche Dimensionszahl des Bereiches $\mathfrak{D}_A$ von A, und, da $\mathfrak{D}_A$ im Hilbertschen Raume $\mathfrak{H}$ überall dicht ist, unschwer ein Widerspruch folgen. $\mathfrak{A}$ erfüllt die Postulate eines Hilbertschen Raumes.

Sei A^* aus $\mathfrak{R}$ nach $\mathfrak{H}$ die Adjungierte von A; man verengere ihren Bereich, sodass die Adjungierte A_0^* von A aus $\mathfrak{A}$ nach $\mathfrak{H}$ mit dem Bereiche $\mathfrak{D}_0^* \subseteq \mathfrak{A}$ entsteht, A_0^* ist l.a.d. aus $\mathfrak{A}$ nach $\mathfrak{H}$ (Satz β). Für jedes $\eta(x)$ aus $\mathfrak{D}_0^*$ ist $\|A_0^*\eta\| \geqslant c\|\eta\|$: für $f = A^{-1}\eta$, $Af = \eta$ ist $\|f\| \leqslant c^{-1}\|\eta\|$,

$$\|\eta\|^2 = (Af, \eta) = (f, A_0^*\eta) \leqslant \|f\| \cdot \|A_0^*\eta\| \leqslant c^{-1}\|\eta\| \cdot \|A_0^*\eta\|.$$

Nach dem ersten Teile dieses Satzes existiert $(A_0^*)^{-1}$ und ist stetig, der Vorrat von A_0^* ist ein Hilbertscher Raum; dieser ist in $\mathfrak{H}$ überall dicht, weil A umkehrbar ist (Satz γ), ist also $\mathfrak{H}$. Der Vorrat von A^* muss erst recht $\mathfrak{H}$ sein; falls $\mathfrak{A}$ echter Teil von $\mathfrak{R}$, also sicher nicht überall dicht in $\mathfrak{R}$ ist, ist A^* (Satz γ) nicht umkehrbar.

---

* Abgekürzt für: Sei A linear abgeschlossen mit in $\mathfrak{H}$ überall dichtem Bereiche.

Jetzt lässt sich auch die Transformation (4.4) $f = Sh$ vollständig überblicken. S ist aus $\mathfrak{L}_2$ nach $\mathfrak{L}_2$, linear, abgeschlossen, da sogar für $f_n = Sh_n$ aus $f_n \to f$ die Selbstreziprozität von f (Abschn. 2) und somit die Existenz einer Funktion $h = S^{-1}f$ und $h_n \to h$ wegen (4.4) folgt. Der Bereich $\mathfrak{D}_S$ von S ist in $\mathfrak{L}_2$ überall dicht, da h jede Treppenfunktion sein darf. So ist der Vorrat $\mathfrak{R}[\chi]$ von S wegen (4.4) nach Satz V ein Hilbertscher Raum. Mit Rücksicht auf (4.1) ist, wenn ϕ die Mellinsche Transformierte von $\psi \subset \mathfrak{L}_2$ ist,

$$\int_0^\infty f\bar{\psi}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)\,\bar{\phi}(t)\,dt = \frac{1}{2\pi}\int_0^\infty h(t)\{\bar{\phi}(t)+\omega(-t)\,\bar{\phi}(-t)\}\,dt,$$

also wegen (5.1) die Adjungierte S^* aus $\mathfrak{L}_2$ nach $\mathfrak{L}_2$

$$S^*\psi = (2\pi)^{-1}\{\phi(t)+\bar{\omega}(-t)\,\phi(-t)\} \quad (0<t<\infty), \tag{5.2}$$

ihr Bereich durch $\phi(t) \subset \mathfrak{L}_2$ und $\phi(-t)\,\bar{\omega}(-t) \subset \mathfrak{L}_2$ bestimmt. In der Tat ist $S^*\phi$ nicht umkehrbar, denn $S^*\phi$ ist für jede bezüglich $T[-\bar{\chi}]$ selbstreziproke Funktion $\phi(x)$ und nur für solche $\phi \subset \mathfrak{R}[-\bar{\chi}]$ Null, weil die Gleichung $\phi(t) = -\bar{\omega}(-t)\,\phi(-t)$ mit $\psi(x) \subset \mathfrak{R}[-\bar{\chi}]$ äquivalent ist. Bildet man aber die Adjungierte S_0^* aus $\mathfrak{R}[\chi]$ nach $\mathfrak{L}_2$, so haben $\mathfrak{R}[\chi]$ und $\mathfrak{R}[-\bar{\chi}]$ nur das Element Null gemeinsam (Satz IIIa), also ist $S_0^* k = 0$ nur für $k = 0$, S_0^* umkehrbar. Setzt man $T[\chi]$ und damit $\omega(t)$ beschränkt voraus, so bildet S den ganzen Raum $\mathfrak{L}_2$ ein-eindeutig und stetig auf ganz $\mathfrak{R}[\chi]$ ab, S_0^* aber ganz $\mathfrak{R}[\chi]$ auf ganz $\mathfrak{L}_2$; nur wenn $T[\chi]$ unitär, also $|\omega(t)| = 1$ ist, ist auch $\pi^{\frac{1}{2}} S$ unitär, man findet unschwer für jedes $h \subset \mathfrak{L}_2$ mit $h(x)\,\omega(-x) \subset \mathfrak{L}_2$:

$$S_0^* Sh = S^* Sh = (2\pi)^{-1} h(x)\left(1+|\omega(-x)|^2\right).$$

Mit Heranziehung von Satz α beweist man die Umkehrung von V:

SATZ V_B. *Sei A linear abgeschlossen aus $\mathfrak{H}$ nach $\mathfrak{K}$, umkehrbar und der Vorrat $\mathfrak{A}$ ein Hilbertscher Raum. Dann ist $\|Af\| \geqq c\|f\|$ für jedes f aus dem Bereiche $\mathfrak{D}$ von A, $c > 0$, und es gibt einen Hilbertschen Raum $\mathfrak{H}_1$, der ein Teil von $\mathfrak{H}$ und in dem $\mathfrak{D}$ überall dicht ist.*

Ferner beweist man unter Heranziehung der Sätze α und V:

SATZ V_C†. *Seien $\mathfrak{H}$ und $\mathfrak{K}$ gegebene Hilbertsche Räume, A l.a.d. aus $\mathfrak{H}$ nach $\mathfrak{K}$ und daher die Adjungierte A^* l.a.d. aus $\mathfrak{K}$ nach $\mathfrak{H}$, $\mathfrak{D}$ der Bereich von A,*

† Satz V_C (ii) und (iii) ist für den Fall $\mathfrak{H} = \mathfrak{K}$, A linear und beschränkt mit dem Bereiche ganz $\mathfrak{H}$, bekannt: E. Hellinger und O. Toeplitz, *Encyklopädie d. Math. Wiss.* II_3, *Heft* 9 (1923–27), Abschn. 18. Herrn Hellinger spreche ich für seine Hilfe meinen besten Dank aus. *Cf.* für allgemeinere Raüme J. Radon, *Akademieberichte Wien, Math. Klasse*, 122, IIa (1913), 92–101 mit Satz V, V_B, 101–104 mit V_C.

$\mathfrak{D}^*$ *von* A^*. *Damit* A *umkehrbar und der Vorrat ganz* $\mathfrak{K}$ *sei, ist jedes der folgenden Bedingungspaare notwendig und hinreichend*:

(i) *Der Vorrat ist überall dicht in* $\mathfrak{K}$. $\|Af\| \geqslant c\|f\|$ *für alle* $f \subset \mathfrak{D}$, $c > 0$.

(ii) $\|Af\| \geqslant c\|f\|$ *für alle* $f \subset \mathfrak{D}$; $c > 0$. $\|A^*\eta\| \geqslant c\|\eta\|$ *für alle* $\eta \subset \mathfrak{D}^*$.

(iii) $\|Af\| \geqslant c\|f\|$ *für alle* $f \subset \mathfrak{D}$; $c > 0$. $A^*\eta = 0$ *nur für* $\eta = 0$, $\eta \subset \mathfrak{K}$.

Stets ist dann A^{-1} *beschränkt und hat den Bereich* $\mathfrak{K}$†.

Anwendung: Die Transformation $T[\chi]$ der Art (1.1) ist l.a.d. aus $\mathfrak{L}_2$ nach $\mathfrak{L}_2$, ihre Adjungierte ist $T[\bar{\chi}]$ (Verf. II, *C*). Eine notwendige und hinreichende Bedingung, dass T^{-1} existiert und den Bereich $\mathfrak{L}_2$ hat, lautet:

(ii)′ $\qquad \|T[\chi]f\| \geqslant c\|f\|$ für alle $f \subset \mathfrak{D}_T$; $c > 0$.

Denn aus dieser Bedingung folgt leicht für alle η aus $\mathfrak{D}^*$, dass auch T^* positiv nach unten beschränkt ist, also ist Satz V_C (ii) anwendbar. Die Bedingung (ii)′ ist äquivalent mit $|\omega(t)| \geqslant c > 0$ (Verf. II, *K*).

6. Hier sollen (Satz VI) alle linearen involutorischen Transformationen in einem beliebig gegebenen Hilbertschen Raume $\mathfrak{H}$ konstruiert werden, durch die Sätze VI′ *a*, *b*, *c* und *d* alle abgeschlossenen bezw. stetigen bezw. "isometrischen" (Stone, Def. 2.19) bezw. unitären Transformationen dieser Art. Besonders einfache Beispiele für die letzten sind ausser den Fourier-sin- und cos- die unitären Transformationen

$$g(x) = Tf = -f(2\pi - x) \quad [\mathfrak{H} = L^2(0, 2\pi)],$$

für welche die Folgen $\{\pi^{-\frac{1}{2}} \sin nx\}$, $n = 1, 2, \ldots$, ein vollständiges orthonormales System der selbstreziproken, $(2\pi)^{-\frac{1}{2}}$, $\pi^{-\frac{1}{2}} \cos x$, $\pi^{-\frac{1}{2}} \cos 2x$, ... der schiefreziproken Funktionen bilden, und

$$g(x) = Tf = f(\pi - x) \quad [\mathfrak{H} = L^2(0, \pi)],$$

für welche entsprechende Folgen

$\{(2/\pi)^{\frac{1}{2}} \sin nx\}$ $(n = 1, 3, 5, \ldots)$ bezw. $\{(2/\pi)^{\frac{1}{2}} \sin nx\}$ $(n = 2, 4, 6, \ldots)$,

oder auch

$\pi^{-\frac{1}{2}}$, $(2/\pi)^{\frac{1}{2}} \cos 2x$, $(2/\pi)^{\frac{1}{2}} \cos 4x, \ldots$ bezw. $\{(2/\pi)^{\frac{1}{2}} \cos nx)\}$ $(n = 1, 3, 5, \ldots)$

sind.

† Entsprechend den Metriken der Raüme $\mathfrak{H}$ und $\mathfrak{K}$ können $\|f\|$ und $\|Af\|$ verschiedenen Sinn haben, ebenso $\|A^*\eta\|$ und $\|\eta\|$, (f, g) and (ϕ, γ).

SATZ VI. *Seien $\mathfrak{M}_1$, $\mathfrak{M}_2$ lineare Mannigfaltigkeiten in $\mathfrak{H}$, die bis auf $f(x)\equiv 0$ kein Element gemeinsam haben. Sei $\mathfrak{D}$ die durch ihre Vereinigungsmenge bestimmte lineare Mannigfaltigkeit, $\mathfrak{D}$ als Bereich von T gewählt, einem f aus $\mathfrak{D}$ folgendes $g=Tf$ zugeordnet:*

$$f=u+v \quad (u\subset\mathfrak{M}_1,\ v\subset\mathfrak{M}_2), \quad g=Tf=u-v.$$

Wählt man $\mathfrak{M}_1$ und $\mathfrak{M}_2$ auf alle möglichen Arten, so erhält man alle linearen involutorischen Transformationen in $\mathfrak{H}$.

Beweis. Die Darstellung $f=u+v$ ist auf Grund der Voraussetzung eindeutig, T linear und

$$T^2f=T(u-v)=T\big(u+(-v)\big)=u+v=f.$$

Umgekehrt erhält man nach Satz II′*a* so jede lineare involutorische Transformation indem, man $\mathfrak{M}_1=\mathfrak{R}_T$, $\mathfrak{M}_2=\mathfrak{R}_{-T}$ wählt.

SATZ VI′. *Wählt man auf alle möglichen Arten die linearen Mannigfaltigkeiten $\mathfrak{M}_1$ und $\mathfrak{M}_2$ des Satzes VI als (a) abgeschlossen, (b) völlig getrennt†, (c) zueinander orthogonal, so erhält man alle linearen involutorischen Transformationen in $\mathfrak{H}$, die (a) abgeschlossen bezw. (b) stetig bezw. (c) "isometrisch" sind.*

Beweis von VI′*a*. Sei T die in VI konstruierte Transformation. Für $\{f_n\}\subset\mathfrak{D}$, $f_n\to f$ und $g_n\to g$ ist $Tf=g$; denn aus $f_n=u_n+v_n\to f$, $g_n=u_n-v_n\to g$ folgt $u_n\to\frac{1}{2}(f+g)=u$, $v_n\to\frac{1}{2}(f-g)=v$, und da $\mathfrak{M}_1$, $\mathfrak{M}_2$ abgeschlossen sind, $u\subset\mathfrak{M}_1$, $v\subset\mathfrak{M}_2$, also $f=u+v\subset\mathfrak{D}$, nach Definition daher $Tf=u-v$, also $Tf=g$. Umgekehrt erhält man auf Grund von Satz II′ so alle Transformationen der gewünschten Art.

VI′*b*. Sei $\epsilon>0$ gegeben, so ist für ein $\delta>0$ für alle u aus $\mathfrak{M}_1$ und v aus $\mathfrak{M}_2$ mit $\|u-v\|<\delta$ dann $\|u\|<\frac{1}{2}\epsilon$, $\|v\|<\frac{1}{2}\epsilon$, also für alle $f_i\subset\mathfrak{D}$, $f_i=u_i+v_i$, $i=1, 2$ mit $\|f_1-f_2\|<\delta$ wegen

$$f_1-f_2=(u_1-u_2)+(v_1-v_2), \quad g_1-g_2=(u_1-u_2)-(v_1-v_2)$$

$$\|u_1-u_2\|<\tfrac{1}{2}\epsilon, \quad \|v_1-v_2\|<\tfrac{1}{2}\epsilon, \quad \|g_1-g_2\|<\epsilon, \tag{6.1}$$

somit $T(f_1-f_2)<\epsilon$. Also ist T stetig.

Ist umgekehrt T stetig und daher beschränkt, also $\|Tf\|\leqslant A\|f\|$ für alle f aus $\mathfrak{D}$, so sind $\mathfrak{M}_1=\mathfrak{R}_T$ und $\mathfrak{M}_2=\mathfrak{R}_{-T}$ völlig getrennt. Denn aus

† D.h. zu jedem $\epsilon>0$ gibt es ein $\eta>0$ der Art, dass für alle u aus $\mathfrak{M}_1$, alle v aus $\mathfrak{M}_2$ mit $\|u-v\|<\eta$ dann $\|u\|<\epsilon$, $\|v\|<\epsilon$ ist, dass also der Abstand der beiden Mannigfaltigkeiten nur in der Nähe des Nullpunktes beliebig klein wird.

$u \subset \mathfrak{M}_1$, $v \subset \mathfrak{M}_2$, $\|u-v\| < \epsilon/\max(A, 1)$ folgt wegen $u+v = T(u-v)$ doch $\|u+v\| < \epsilon$, also

$$\|u\| = \|\tfrac{1}{2}(u+v)+\tfrac{1}{2}(u-v)\| < \epsilon, \quad \|v\| = \|\tfrac{1}{2}(u+v)-\tfrac{1}{2}(u-v)\| < \epsilon.$$

VI′*c*. Seien $\mathfrak{M}_1$ und $\mathfrak{M}_2$ zueinander orthogonal, also $(u, v) = 0$ für alle $u \subset \mathfrak{M}_1$, $v \subset \mathfrak{M}_2$, $f_i = u_i+v_i \subset \mathfrak{D}$, $i = 1, 2$. Dann ist

$$(Tf_1, Tf_2) = (u_1-v_1, u_2-v_2) = (u_1+v_1, u_2+v_2) = (f_1, f_2),$$

also Tf isometrisch. Umgekehrt folgt aus $(Tf_1, Tf_2) = (f_1, f_2)$ dann $(Tu, Tv) = (u, v)$ für alle u aus $\mathfrak{M}_1 = \mathfrak{R}_T$, v aus $\mathfrak{M}_2 = \mathfrak{R}_{-T}$; wegen $Tu = u$, $Tv = -v$ ist $(Tu, Tv) = -(u, v)$; so folgt $(u, v) = 0$. Die Räume $\mathfrak{M}_1$ und $\mathfrak{M}_2$ sind wegen $\|u-v\|^2 = \|u\|^2+\|v\|^2$ natürlich völlig getrennt. Aus VI′*a*, *c* folgt leicht der Satz:

VI′*d*. *Wenn* $\mathfrak{M}_1$ *und* $\mathfrak{M}_2$ *linear, abgeschlossen und zueinander orthogonal sind, ist* $\mathfrak{D}$ *eine abgeschlossene lineare Mannigfaltigkeit in* $\mathfrak{H}$, T *eine unitäre involutorische Transformation in* $\mathfrak{D}$.

Breslau,
Frobenstr. 9.

NOTES ON A NET OF QUADRIC SURFACES: III. THE SCROLL OF TRISECANTS OF THE JACOBIAN CURVE

By W. L. EDGE.

[Received 14 January, 1938.—Read 20 January, 1938.]

This note is concerned with the scroll R^8 generated by the trisecants of the Jacobian curve of a net of quadric surfaces. It is shown in § 3 that, when the lines of [3] are represented by the points of a quadric Ω in [5], the curve which is the locus of those points of Ω which represent the generators of R^8 is the intersection of Ω and a Veronese surface. This configuration, of a quadric and a Veronese surface in [5], is then amplified by means of a reciprocation with respect to the quadric; there thus arises a self-dual configuration which has an interest of its own, as also has the terse description of it which the matrix notation affords. Our chief concern, however, is with the interpretation of this configuration in terms of line-geometry in [3]; and, when the figure in [3] is considered in the light of the amplified configuration in [5], there are immediately disclosed a second scroll ρ^8, also of order eight, and two cubic complexes. This scroll and the two complexes are related covariantly to the net of quadric surfaces, and they are here encountered for the first time.

The work falls into three sections, the first section consisting only of § 1 and § 2. In § 1 two properties of canonical sets of generators of R^8 are obtained. These are only two of several properties which are disclosed almost instantaneously once the geometrical technique proper to the present note has been developed. None the less § 1 is an appropriate opening to the note; both because, in virtue of its allusions to some of the geometrical features of a net of quadrics which are already recognised, and in virtue of the mode of reasoning employed, it serves as a reminder of how much is at present known, and therefore opens the proceedings, so to speak, at the proper level; and also because the more elementary reasoning of § 1 serves to set the more powerful reasoning which is used later in its right perspective.

In § 2 the fundamental result concerning the line-coordinates of a generator of R^8 is established; this is the pivot on which the whole of the subsequent work turns.

The second section, consisting of §§ 3–9, is concerned with the construction of the figure in [5] and with the exposition of its chief properties; this figure consists essentially of a quadric and two Veronese surfaces which, through the agency of the quadric, are symmetrically related to each other. The curves in which the Veronese surfaces are met by the quadric represent the two scrolls R^8 and ρ^8; these scrolls, like the Veronese surfaces, are symmetrically related to one another, and *any generator of either is a quadritangent line of the other.*

This note has, for its main purpose, to set on record the discovery of ρ^8 and of the two cubic complexes; there is no intention of giving an exhaustive account of their relations with one another and with R^8, but some of the more obvious of these are discussed in the third section (§§ 10–15).

I.

1. The locus of the vertices of cones which belong to a net of quadrics is the Jacobian curve ϑ, and each point of ϑ is associated with a conjugate trisecant of ϑ, this trisecant being common to the polar planes of the point with respect to all the quadrics of the net. ϑ is of order six and genus 3; its trisecants generate a scroll R^8, of order eight, which, since its generators are in (1, 1) correspondence with the points of ϑ, is also of genus 3.

Consider then a set of four points of ϑ such that the four cones of the net, whose vertices are at these points, belong to a pencil of quadrics; in other words, a canonical set* on ϑ. The tetrahedron of which these four points are the vertices is self-conjugate with respect to all the quadrics of the pencil. The four trisecants that are conjugate to the vertices of the tetrahedron therefore lie in the opposite faces, and are the lines in which these faces are met by the polar planes of the opposite vertices with respect to any quadric of the net that does *not* belong to the pencil. These four lines, being the intersections of the faces of a tetrahedron with the polar planes of its opposite vertices with respect to a quadric, belong to a regulus†.

Since there is a (1, 1) correspondence between the points and trisecants of ϑ we may speak of the four trisecants that are conjugate to the four

* That these sets of points belong to the canonical series on ϑ was pointed out in *Proc. London Math. Soc.* (2), 34 (1932), 517.

† This is a classical proposition in the geometry of quadric surfaces; see, for example, Chasles, *Aperçu historique* (3rd ed., Paris, 1889), 692; or Salmon, *Geometry of Three Dimensions* (4th edition, Dublin, 1882), 121.

points of a canonical set on ϑ as *a canonical set of generators* of R^8. We have therefore proved that *any canonical set of generators of* R^8 *consists of four lines belonging to a regulus.* Whereas a set of four generators of R^8 has, in general, two transversal lines, when the set is a canonical set it has an infinity of transversal lines.

The eight trisecants of ϑ that meet any line λ are conjugate to eight points of ϑ lying on the twisted cubic λ' which is conjugate (in the sense described* in Note I) to λ; λ' meets ϑ in eight points, and the quadrics through λ' cut out the canonical series on ϑ. When the quadrics of the net are represented, in the usual way†, by the points of a plane, a (1, 1) correspondence is thereby established between ϑ and a plane quartic δ, the twelve intersections of any quadric with ϑ corresponding to twelve points of δ which lie on a cubic curve; thus the eight points of δ that correspond to the intersections of λ' and ϑ must be such that they lie, together with any canonical set of δ, on a cubic curve. But a canonical set of δ consists of four collinear points; the set of eight points must therefore lie on a conic, and so it is a bicanonical set. Thus *the eight trisecants of* ϑ *which meet any line* λ *form a bicanonical set.*

Suppose now that, of the eight generators of R^8 which meet λ, four are the members of a canonical set; in other words we take λ to be a transversal of a canonical set of generators. The bicanonical set of generators of R^8 which meet λ then includes a canonical set, and so must consist of two canonical sets (correspondingly, when four of the eight intersections of δ with a conic are collinear the conic consists of two lines). Hence *the remaining four generators of* R^8 *that meet a transversal line of a canonical set of generators consist of a second canonical set of generators.*

2. Suppose that the quadrics of the net are given by the equation

$$\xi Q_0+\eta Q_1+\zeta Q_2=0, \tag{1}$$

where Q_0, Q_1, Q_2 are homogeneous quadratic functions of four coordinates (x_0, x_1, x_2, x_3), and ξ, η, ζ are parameters. The polar planes of (X_0, X_1, X_2, X_3) with respect to the two quadrics

$$Q' \equiv \xi' Q_0+\eta' Q_1+\zeta' Q_2=0,$$

$$Q'' \equiv \xi'' Q_0+\eta'' Q_1+\zeta'' Q_2=0,$$

* The first of this series of Notes was published in *Proc. London Math. Soc.* (2), 43 (1937), 302–315.

† Hesse, *Journal für Math.*, 49 (1855), 279–332; *Gesammelte Werke*, 345.

are $$x_0\frac{\partial Q'}{\partial X_0}+x_1\frac{\partial Q'}{\partial X_1}+x_2\frac{\partial Q'}{\partial X_2}+x_3\frac{\partial Q'}{\partial X_3}=0,$$

and $$x_0\frac{\partial Q''}{\partial X_0}+x_1\frac{\partial Q''}{\partial X_1}+x_2\frac{\partial Q''}{\partial X_2}+x_3\frac{\partial Q''}{\partial X_3}=0;$$

where (X_0, X_1, X_2, X_3) have been substituted for (x_0, x_1, x_2, x_3) in Q' and Q'' before differentiation. If, now, (X_0, X_1, X_2, X_3) lies on ϑ, these two polar planes intersect in the conjugate trisecant, whose six line coordinates are therefore the six Jacobians

$$\frac{\partial(Q', Q'')}{\partial(X_1, X_2)}, \quad \frac{\partial(Q', Q'')}{\partial(X_2, X_0)}, \quad \frac{\partial(Q', Q'')}{\partial(X_0, X_1)}; \quad \frac{\partial(Q', Q'')}{\partial(X_3, X_0)}, \quad \frac{\partial(Q', Q'')}{\partial(X_3, X_1)}, \quad \frac{\partial(Q', Q'')}{\partial(X_3, X_2)}.$$

The point (X_0, X_1, X_2, X_3) on ϑ corresponds, when the quadrics of the net are represented by the points (ξ, η, ζ) of a plane, to a point (ξ, η, ζ) on a quartic curve δ in this plane; and, as Hesse showed when he first introduced this representation*, the ten squares and products of the four coordinates of a point on ϑ are proportional to ten linearly independent cubic polynomials in (ξ, η, ζ). Now the six coordinates of a trisecant are homogeneous quadratic functions of (X_0, X_1, X_2, X_3)—the coordinates of the point on ϑ to which the trisecant is conjugate; hence they are proportional to six cubic polynomials in (ξ, η, ζ). We can show, however, that *these six cubic polynomials have a common linear factor.* For when the cubic polynomial which arises from $\partial(Q', Q'')/\partial(X_i, X_j)$ is equated to zero, the cubic curve so obtained meets δ in the twelve points which correspond to the twelve intersections of ϑ with the quadric $\partial(Q', Q'')/\partial(x_i, x_j)=0$. This quadric, however, passes through the vertices of the four cones which belong to the pencil of quadrics determined by $Q'=0$ and $Q''=0$; hence the intersections of δ with the cubic curve include the four collinear points in which δ is met by the line joining (ξ', η', ζ') and (ξ'', η'', ζ''). The cubic curve must therefore contain this line as a part, and this is so whichever of the six line coordinates of the trisecant of ϑ we take. It follows that the six line coordinates of the trisecant are proportional to homogeneous quadratic polynomials in (ξ, η, ζ). Moreover, since the removal of a common factor does not affect the linear independence of a set of polynomials, these six quadratic polynomials cannot be linearly dependent. Hence we have the following:

To any point of ϑ there correspond a conjugate trisecant of ϑ and a point (ξ, η, ζ) of δ; the six line coordinates of the trisecant are proportional to six linearly independent homogeneous quadratic polynomials in ξ, η, ζ.

* *Loc. cit.*, 288.

It follows immediately that those generators of R^8 which belong to any linear complex correspond to eight points of δ on a conic, and conversely; the conics of the plane are, through the curve δ and the net of quadrics, associated with the linear complexes of space. When, in particular, the linear complex is special we have the result, already proved otherwise, that the eight trisecants of ϑ which meet any line λ correspond to eight points of δ lying on a conic.

II. *The five-dimensional configuration.*

3. We now represent, in the usual way, the lines of [3] by the points of a quadric Ω in [5]. Suppose the equation of Ω, in matrix notation*, is

$$y'Ay = 0, \tag{2}$$

where y denotes a single *column* of six point-coordinates $y_0, y_1, y_2, y_3, y_4, y_5$, the dash signifies transposition, and A is a symmetric matrix of six rows and columns. Since Ω is not a cone, A is non-singular.

The point of Ω which represents the trisecant of ϑ that is associated with the point (ξ, η, ζ) of δ is given by

$$\begin{aligned} y &= M[\xi^2, \eta^2, \zeta^2, \tau\eta\zeta, \tau\zeta\xi, \tau\xi\eta]' \\ &= M\Xi, \quad \text{say,} \end{aligned} \tag{3}$$

where M is a matrix, of six rows and columns, which, since the six quadratic polynomials that are the line coordinates of the trisecant are linearly independent, is non-singular. The numerical multiplier τ is a square root of 2, and has been introduced in order that the algebra may the more consistently reflect the self-dual character of the geometrical configuration that we obtain.

The locus of the point given by (3), as (ξ, η, ζ) varies on δ, is a curve Γ, lying on Ω, whose points represent the generators of R^8. If, however, (ξ, η, ζ) is not restricted to lie on δ, but varies over the whole plane, the locus of the point given by (3) is a Veronese surface† v whose points, in virtue

* A reader who is unfamiliar with this use of the matrix notation may like to refer to Turnbull, *Proc. Camb. Phil. Soc.*, 31 (1935), 174.

† The classical properties of the Veronese surface, and the geometry of loci intimately connected with it, will be assumed here; for an account of them see, for example, Segre: *Atti della R. Accad. delle Scienze di Torino*, 20 (1885), 487–504; also Bertini, *Geometria proiettiva degli iperspazi* (Messina, 1923), Capitolo 16. The configuration of a quadric and a Veronese surface in [5] is discussed briefly at the end of Segre's paper; but the particular case when the quadric and surface are so related that their common curve represents the scroll of trisecants of the Jacobian curve of a net of quadrics is not alluded to; nor is there any hint of reciprocation. Bertini's use of reciprocation is not to the purpose here.

of (3), are in (1, 1) correspondence with those of the plane. The curve Γ is the intersection of Ω and v. When the value of y given by (3) is substituted in (2), the equation so arising, namely $\Xi' M'AM\Xi = 0$, must be that of the quartic curve δ.

In the (1, 1) correspondence between the points of v and the points (ξ, η, ζ) of the plane, lines in the plane correspond to conics on v, and conversely. Now any line in the plane meets δ in four points which form a canonical set; hence any conic on v meets Γ in four points which represent a canonical set of generators of R^8. Those points of the conic other than its four intersections with Γ do not lie on Ω, and so do not represent lines of [3]; but the plane of the conic meets Ω in a second conic, passing through the four intersections of the plane with Γ, whose points represent lines of [3] belonging to a regulus. Hence we have again the result that *a canonical set of generators of R^8 consists of four lines belonging to a regulus.*

Since any two conics on v together form one of its prime sections, *any two canonical sets of generators of R^8 belong to a linear complex*; in particular, given any canonical set of generators, there is a linear complex such that the eight generators of R^8 which belong to it consist of this canonical set counted twice. Again: any prime which contains the plane of a conic on v meets v further in a second conic which, like the first, has four intersections with Γ; hence, *if a linear complex contains a canonical set of generators of R^8, the remaining four generators of R^8 that belong to the complex consist of a second canonical set.* If, in particular, the linear complex is special, this result reduces to one previously obtained in § 1. Further, since a prime section of v corresponds to a conic in the plane, *those eight generators of R^8 which belong to a linear complex form a bicanonical set, and conversely.*

4. Any line in [3] has eight intersections with R^8, but the eight generators of R^8 passing through these points do not, in general, consist of two canonical sets. In order that they should do so the line must be subjected to some condition, and so belong to some complex; this complex of course bears an invariant relation to the net of quadrics. The equation of the complex is found by using the fact that a line of [3], which is met by the generators of each of two canonical sets, is represented by a point of Ω such that the tangent prime of Ω there meets v in a pair of conics.

Take, then, any prime

$$u_0y_0+u_1y_1+u_2y_2+u_3y_3+u_4y_4+u_5y_5 \equiv uy = 0,$$

where the matrix u is the single row of six prime coordinates u_i. The six

coordinates of the pole of this prime with respect to Ω are given by*

$$u = y'A. \tag{4}$$

The prime meets v in a pair of conics when

$$uM[\xi^2,\ \eta^2,\ \zeta^2,\ \tau\eta\zeta,\ \tau\zeta\xi,\ \tau\xi\eta]',$$

regarded as a quadratic form in ξ, η, ζ, is the product of two linear factors; the condition for this is $D = 0$, D being a symmetrical determinant of the third order whose elements are homogeneous linear functions of the six prime coordinates u_i. If we substitute for these six prime coordinates in D their values given by (4) we obtain the equation of a cubic primal C— the locus of the poles, with respect to Ω, of those primes which meet v in pairs of conics. Those lines of [3] which are transversals of pairs of canonical sets of generators of R^8 are therefore represented by those points in which Ω is met by C; hence the lines belong to a cubic complex U.

5. A cubic primal in [5], the left-hand side of whose equation may be expressed as a symmetrical determinant of the third order, is the locus of chords of a Veronese surface; this surface is a double surface on the primal, and the coordinates of any point on the surface cause all the first minors of the determinant to vanish. The primal C is therefore the locus of chords of a Veronese surface w, and w is a double surface on C. Moreover, as the coordinates of any point of w cause all the first minors of the determinant in the equation of C to vanish, the polar prime, with respect to Ω, of any point of w does not merely meet v in a pair of conics; it meets v in a single conic counted twice, touching v all along this conic. The prime is therefore a quadritangent prime of Γ, its eight intersections with Γ consisting of the four points, taken twice over, in which Γ meets the conic along which the prime touches v; thus it is associated with a linear complex such that the eight generators of R^8 belonging to the complex consist of a single canonical set, counted twice. The linear complexes which have this property are thus identified: they are associated with the polar primes of the points of w with respect to Ω. Now Ω meets w in a curve Δ which, like Γ, is of order eight and genus 3; a point of Δ represents a line of [3] such that the eight generators of R^8 which meet the line consist of a single canonical set, counted twice. Thus we have a scroll ρ^8, whose generators are represented in [5] by the points of Δ, generated by quadritangent lines of R^8; ρ^8 is, like R^8, of order eight and genus 3, and is a combinantal covariant of the net of quadrics. Since Δ is a double curve on the threefold

* *Cf.* Turnbull, *loc. cit.*

in which Ω meets C, the generators of ρ^8 are all double lines of the complex U.

A scroll in [3] has, in general, only a finite number of quadritangents, and R^8 is exceptional in having a singly-infinite set. The representation of the lines of [3] by the points of Ω has indicated this set of quadritangents very clearly; whereas, in general, a curve in [5] has ∞^1 quadritangent primes, of which a finite number touch a given quadric, the curve common to a quadric and a Veronese surface has ∞^2 quadritangent primes, of which ∞^1 touch the quadric. The scroll represented by the curve common to Ω and *any* Veronese surface, whether it is the trisecant scroll of the Jacobian curve of a net of quadric surfaces or not, has thus a scroll of quadritangents; both scrolls are of order eight and genus 3 and are, like R^8 and ρ^8, mutually related each to the other (*cf.* § 8 below).

6. Since the equation (2) of Ω, and the parametric equations (3) of v, are known, the parametric equations of w can be found. The tangent plane at the point (ξ, η, ζ) on v is the plane of the three points

$$y = M[\tau\xi,\ 0,\ 0,\ 0,\ \zeta,\ \eta]',$$

$$y = M[0,\ \tau\eta,\ 0,\ \zeta,\ 0,\ \xi]',$$

$$y = M[0,\ 0,\ \tau\zeta,\ \eta,\ \xi,\ 0]'.$$

Hence, if (ξ, η, ζ) varies subject to the condition $l\xi+m\eta+n\zeta=0$, this plane always lies in the prime

$$[l^2,\ m^2,\ n^2,\ \tau mn,\ \tau nl,\ \tau lm]M^{-1}y = 0,$$

or, as we shall write it, $\Lambda M^{-1}y=0$.

Such a prime, containing the tangent planes of v at all the points of a conic, we shall describe as a *tangent prime* of v. The different tangent primes of v are obtained by taking different values of $l:m:n$.

The coordinates of the pole of this prime with respect to Ω are given by

$$\Lambda M^{-1} = y'A,$$

$$y = (\Lambda M^{-1}A^{-1})' = A^{-1}(M^{-1})'\Lambda',$$

remembering that A is symmetric. Thus the coordinates of a point of w are given by

$$y = A^{-1}(M^{-1})'\,[l^2,\ m^2,\ n^2,\ \tau mn,\ \tau nl,\ \tau lm]' \qquad (5)$$

$$= (M'A)^{-1}\Lambda'.$$

This point describes the Veronese surface w as $l:m:n$ vary.

Here we may pause for a moment to emphasise the appropriateness of introducing the multiplier τ in § 3. Whatever multiplier, unity or any other, appears before the three products in the column Ξ, a corresponding multiplier must appear before the three products in the row Λ; since the product of the two multipliers must be 2, in order to make $\Lambda\Xi$ equal to $(l\xi+m\eta+n\zeta)^2$, the second multiplier is determined by the first. It is desirable for the two multipliers to be equal; they can only be so by both being one of the square roots of 2.

7. A point of the curve Δ in which Ω meets w represents a quadritangent line of R^8, the four generators of R^8 which pass through the points of contact belonging to a canonical set. There are thus ∞^1 canonical sets of generators which give rise to quadritangent lines; not every canonical set does so, and in order that it may do so it must be subjected to some condition. This condition is expressed simply by the fact that, for such a canonical set, the point given by (5) lies on Ω. Substituting the value for y given by (5) in the equation, $y'Ay=0$, of Ω, we obtain

$$\Lambda\{(M'A)^{-1}\}'A(M'A)^{-1}\Lambda'=0,$$

which reduces to

$$\Lambda(M'AM)^{-1}\Lambda'=0.$$

The scalar quantity on the left of this equation is a homogeneous quartic polynomial in l, m, n. Hence, in order that a canonical set of generators of R^8 should give rise to a quadritangent line, the corresponding canonical set of points on δ must lie on a line belonging to a certain envelope of the fourth class. Of the lines of this envelope, 48 touch δ; there thus arise 48 generators of ρ^8 which have two ordinary contacts and one four-point contact with R^8.

8. The Veronese surface w has been obtained from v by reciprocation with respect to Ω, w being the locus of the poles of the tangent primes of v. We can now repeat the process, taking w to play the part in a second reciprocation that v played in the first; and so we obtain a Veronese surface which is the locus of the poles of the tangent primes of w. This surface is the original surface v; the geometry of the configuration shows this, but it is also clearly and swiftly demonstrated by using the analytical expressions (3) and (5) for the coordinates of the points of v and w. Since v is given by $y=M\Xi$, and w by $y=(M'A)^{-1}\Lambda'$, the locus of the poles, with respect to Ω, of the tangent primes of w is the Veronese surface $y=N\Xi$, where N is that non-singular matrix which arises from $(M'A)^{-1}$ in the same way that

$(M'A)^{-1}$ arises from M. Hence, remembering that A is symmetric,

$$N = [\{(M'A)^{-1}\}' A]^{-1} = A^{-1}\{M' A\}' = A^{-1} A' M = M.$$

The two Veronese surfaces v and w are therefore symmetrically related to one another; each surface is the locus of the poles of the tangent primes of the other, and those tangent primes of either surface which touch Ω do so at the points of the curve of intersection of Ω with the other surface.

Since the two Veronese surfaces v and w are symmetrically related to one another, the two scrolls R^8 and ρ^8 are also symmetrically related to one another; not only is each generator of ρ^8 a quadritangent line of R^8, but each generator of R^8 is a quadritangent line of ρ^8; the two scrolls touch along a curve and are inscribed in a common developable. Moreover, those properties of canonical sets of generators of R^8, which were deduced from the fact that it is represented by the curve common to Ω and a Veronese surface, are also true of canonical sets of generators of ρ^8; a canonical set of generators of ρ^8 consists of four lines belonging to a regulus, any two canonical sets of generators of ρ^8 belong to a linear complex, and so on. Just as those lines which are transversals of pairs of canonical sets of generators of R^8 belong to the cubic complex U, represented by the intersection of Ω with the cubic primal C generated by the chords of w, so the lines which are transversals of pairs of canonical sets of generators of ρ^8 belong to a cubic complex T, represented by the intersection of Ω with the cubic primal B generated by the chords of v. The generators of R^8 are all double lines of T.

9. There are, associated with any Veronese surface, two doubly-infinite families of planes—its tangent planes and its secant planes, the latter being the planes which contain the conics of the surface; either of these two families of planes generates the cubic primal of which the surface is a double surface. Since the primal is generated by the secant planes it follows that the complex T is generated by the ∞^2 reguli determined by the canonical sets of generators of R^8, while U is similarly generated by those reguli determined by the canonical sets of generators of ρ^8.

The work of § 6 shows that the tangent plane of v, at the point given by (3), lies in the prime whose pole is the point of w given by (5), provided only that the relation $l\xi + m\eta + n\zeta = 0$ is satisfied. If, then, ξ, η, ζ are fixed, while l, m, n are allowed to vary subject to this relation, it is seen that the polar plane, with respect to Ω, of any tangent plane of v is a secant plane of w. The same is true with the surfaces v and w interchanged; *the polar plane of a tangent plane of either of the surfaces is a secant plane of the other*

surface. Thus T and U may also be generated by the reguli complementary respectively to those reguli by means of which we have already generated U and T; this generation, by transversals of canonical sets of generators, is the way in which the two complexes were obtained above.

III.

10. A plane which lies on the cubic primal B meets Ω in a conic whose points represent lines of T belonging to a regulus; since each of the two doubly-infinite families of planes which generate B contains a singly-infinite set of planes which meet Ω in pairs of lines, there are two singly-infinite sets of reguli, belonging to T, which degenerate into two plane pencils, the two pencils having in common the line which is represented by the intersection of the two coplanar lines on Ω. A few words may be said here concerning these degenerate reguli.

Through any two points of Γ there passes one secant plane of v, and it meets Γ in two further points. If the line joining the first two points lies on Ω, the secant plane meets Ω further in a second line, which must contain the other two points; the join of these other points therefore also lies on Ω. This gives the result, equivalent to one already proved*, that if two generators of R^8 intersect, so do the other two generators which belong to the canonical set determined by the first two.

Two points of Γ which are such that their join lies on Ω represent two intersecting trisecants of ϑ; since the points are both double points of B their join lies on B, so that the whole pencil of lines determined by two intersecting trisecants of ϑ belongs to T. Thus *the cubic cone which is generated by the lines of T passing through any point of ϑ breaks up into three planes*.

It appears now that if a secant plane of v meets Ω in two lines, the point O in which these lines intersect represents a chord PQ of ϑ; this chord lies in the plane of two of the trisecants through P and also in the plane of two of the trisecants through Q, while the two pencils of lines determined by these two pairs of intersecting trisecants together constitute one of the degenerate reguli of T.

Since the plane of two of the trisecants intersecting at P meets ϑ again in Q, Q lies on the trisecant conjugate to P. Hence the chord PQ is a generator of the scroll R^{18} which was obtained in § 13 of Note I. It follows

---

* *Journal London Math. Soc.*, 11 (1936), 198.

that the locus of the points of contact of Ω with those secant planes of v which touch it is a curve of order eighteen.

11. Suppose that a secant plane σ of v meets Ω in two lines λ_1, λ_2; the tangent prime of Ω at O, the intersection of these lines, contains that tangent plane of w which is the polar plane of σ with respect to Ω; moreover, since the tangent prime of Ω at O contains λ_1 and λ_2, and therefore the plane σ, this tangent plane of w passes through O, and so meets Ω in two lines μ_1, μ_2. It follows that the curve which is the locus of points of contact of Ω with secant planes of v is also the locus of points of contact of Ω with tangent planes of w; thus the scroll R^{18} belongs to both the complexes T and U. The four planes $\lambda_1\mu_1$, $\lambda_1\mu_2$, $\lambda_2\mu_1$, $\lambda_2\mu_2$ all lie on Ω; $\lambda_1\mu_1$ and $\lambda_2\mu_2$ belong to one system while $\lambda_2\mu_1$ and $\lambda_1\mu_2$ belong to the opposite system.

Associated with this five-dimensional figure is a figure in [3]. The point O represents a chord PQ of ϑ; λ_1 represents the pencil of lines determined by two trisecants t_1, t_2 which intersect at P, while λ_2 represents the pencil of lines determined by two trisecants u_1, u_2 which intersect at Q. Suppose now that the system of planes on Ω to which $\lambda_1\mu_1$ and $\lambda_2\mu_2$ belong is associated with the *points* of [3], so that the opposite system of planes on Ω is associated with the *planes* of [3]; then $\lambda_1\mu_1$ is associated with the point P and $\lambda_2\mu_2$ with Q, while $\lambda_1\mu_2$ is associated with the plane t_1t_2 and $\lambda_2\mu_1$ with u_1u_2. It follows that μ_1 represents the pencil of lines passing through P and lying in the plane u_1u_2. These lines all belong to U because μ_1, lying in a tangent plane of w, lies on C. If follows that *the cubic cone which is generated by the lines of U passing through any point of ϑ breaks up into three planes.* The three planes which arise in this way from the point P of ϑ are at once identified; for if the trisecant conjugate to P meets ϑ in Q_1, Q_2, Q_3, the plane of the remaining two trisecants through any one of these points passes through P and is one of the planes in question.

The lines of U passing through P could of course have been identified, once the complex U is defined as that consisting of transversals of canonical sets of generators of R^8, without further reference to the five-dimensional figure. For, since the eight generators of R^8 which meet a line of U consist of two canonical sets, a line of U passing through P must be such that one of the two canonical sets of which it is a transversal includes *two* of the trisecants through P; the remaining two trisecants belonging to this canonical set intersect, and the line of U must lie in the plane containing them. There arise in this way three canonical sets of generators of R^8, one set being determined by each pair of trisecants through P; thus the lines of U passing through P consist of three plane pencils.

12. The locus of points of contact of Ω with secant planes of w, which is the same as the locus of points of contact of Ω with tangent planes of v, is a second curve of order eighteen. The points of this curve represent the generators of a scroll ρ^{18} which, like R^{18}, belongs to both the complexes T and U.

13. The generators of R^8 are all double lines of T. If, however, a cubic complex does not contain all the generators of R^8, it will contain twenty-four of them; those generators of R^8 which belong to U are found immediately, for it was pointed out, in § 13 of Note I, that twenty-four generators of R^8 are also generators of R^{18}, and since all the generators of R^{18} belong to U these must be the twenty-four generators of R^8 in question; they consist of the eight sets of three generators which lie in the eight tritangent planes of R^8.

Consider now the surface S^{16} generated by the tangents of ϑ; a cubic complex contains, in general, forty-eight generators of this surface. Since twenty-four tangents meet ϑ again, and so are generators of R^8 and double lines of T, these, counted twice over, account for all the tangents of ϑ which belong to T.

Suppose now that EMN, FNL, GLM are three coplanar trisecants* of ϑ, lying in one of the eight tritangent planes of R^8; it can be shown that those tangents of ϑ which belong to U consist of the tangents at the eight sets of three points such as L, M, N, all these tangents being counted twice. In the first place, these tangents do actually belong to U. For consider the plane which joins FNL to the third trisecant through N; it is known that this plane must meet ϑ again in the point conjugate to the trisecant EMN, and this point† is L; the plane therefore touches ϑ at L. Since, in accordance with § 11, all the lines which pass through L and lie in this plane belong to U, the tangent at L does so. In the second place, no tangents of ϑ other than these twenty-four belong to U. For suppose that L is a point of ϑ at which the tangent belongs to U. The tangent of ϑ at L must lie in a plane containing two trisecants of ϑ meeting in a point M, L and M being such that the trisecant conjugate to either of them passes through the other. The six intersections of the plane with ϑ consist of L, M and four other points

* If two trisecants of $\mathfrak{d}$ meet, their intersection must lie on $\mathfrak{d}$; consequently, if three trisecants are coplanar the triangle formed by them has its vertices on $\mathfrak{d}$, each side meeting $\mathfrak{d}$ in one further point. Concerning these sets of three coplanar trisecants see Edge, *Proc. London Math. Soc.* (2), 34 (1932), 513–514.

† Edge, *loc. cit.*

—two on each of the two trisecants; since the plane touches ϑ at L one of these four other points must coincide with L, so that LM is itself a trisecant. Let N be that point of ϑ to which the trisecant LM is conjugate; then the trisecant conjugate to L must be MN and the trisecant conjugate to M must be NL. The plane LMN is therefore one of the eight tritangent planes of R^8, and the tangent at L one of the twenty-four which, we have seen, belong to U. These tangents, taken twice over, therefore account for all the tangents of ϑ which belong to U.

14. The chords of ϑ form a congruence of order 7 and class 15, so that those chords which belong to a cubic complex constitute a scroll of order $3(7+15) = 66$. The scroll generated by chords of ϑ which belong to T is identified immediately; for each trisecant counts three times as a chord of ϑ and is a double line of T, so that R^8 must occur six times as part of the scroll; the residual scroll is then of order $66-48 = 18$, and so is the scroll R^{18} which, we already know, belongs to T. The same result is obtained by using the fact that those lines of T which pass through any point P of ϑ are the lines through P which lie in the planes of the three pairs of trisecants which intersect at P; for the only chords of ϑ, other than the trisecants themselves, which satisfy these conditions are those which join P to the three intersections of ϑ with the trisecant conjugate to P, and these three chords are generators of R^{18}.

The scroll generated by the chords of ϑ which belong to U consists of R^{18} and a scroll G^{48}. The condition for a chord of ϑ to be a generator of G^{48} is found by considering the three planes which contain the lines of U passing through a point P of ϑ. If a chord of ϑ passes through P and lies in one of these three planes, it either joins P to one of the three intersections of ϑ with the trisecant conjugate to P, in which case it is a generator of R^{18}; or else it joins P to a point Q of ϑ such that one of the trisecants through Q has one of its remaining two intersections with ϑ on the trisecant conjugate to P, in which case it is a generator of G^{48}. There are twelve generators of G^{48} passing through P.

If PQ is a generator of G^{48} it must lie not only in one of the three planes formed by the lines of U which pass through P, but also in one of the three planes formed by the lines of U which pass through Q; thus the points P and Q must be symmetrically related, and, in fact, the trisecant conjugate to either point has one of its intersections with ϑ on a trisecant passing through the other point. Suppose then that PQ is a generator of G^{48}; the trisecant conjugate to P is such that one of its intersections with ϑ lies on a trisecant through Q; let Q' be the third intersection of this latter trisecant with ϑ; similarly we have a point P' such that PP' meets ϑ in a third point, the other two trisecants through this third point being those

conjugate to Q and Q'. Then, as has been shown previously* for this configuration, the four points P, P', Q, Q' form a canonical set on ϑ. The four chords PQ, PQ', $P'Q$, $P'Q'$ are all generators of G^{48}; since they consist of the two pairs PQ, $P'Q'$ and PQ', $P'Q$, each of which is a pair of conjugate chords in the sense of Note I, G^{48} is a self-conjugate scroll.

15. We conclude by showing that the intersections of ρ^8 and ϑ consist of the twenty-four points whose conjugate trisecants touch ϑ, ρ^8 touching ϑ at each of these points.

Let X be an intersection of ϑ and ρ^8; since every generator of ρ^8 touches R^8 in four points, the generators through which belong to a canonical set, the trisecants through X must all belong to the same canonical set; hence the trisecant conjugate to X must be such that its intersections with ϑ all belong to the same canonical set. It is, therefore, necessary that two of the three intersections of this trisecant with ϑ should coincide; for the canonical set which is determined by two points of ϑ that are joined by a trisecant cannot include the third intersection of the trisecant with ϑ when it is distinct from these two points, because the remaining two points of the set lie on the trisecant conjugate to this third intersection. The trisecant conjugate to X is therefore one of those which touch ϑ, and X is such that two of the three trisecants which pass through it coincide.

Mathematical Institute,
16 Chambers Street,
Edinburgh, 1.

BIBL.
LONDIN.
UNIV.

* *Journal London Math. Soc.*, 11 (1936), 198.